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Minimal Klein bottles in $\mathbb{R}^3$ with finite total curvature.

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MINIMAL KLEIN BOTTLES IN $\mathbb{R}^3$ WITH FINITE TOTAL CURVATURE

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In a recent work [3] we gave some examples of minimal Klein bottles in $\mathbb{R}^3$ and $\mathbb{R}^4$ with total curvature equal to the admissible upper bound.

These examples can be naturally generalized. We will prove that there exist minimal Klein bottles in $\mathbb{R}^3$ with one end and total curvature $-2\pi m$, for every m even and greater than 4. We can also construct a minimal Klein bottle with two ends in $\mathbb{R}^3$ with total curvature -12π .

1. Preliminaries

Let $X\colon M \to \mathbb{R}^3$ be a nonorientable regular minimal surface, and $\tilde{X} = X\circ\pi$ the associated double surface, with $\pi\colon \tilde{M} \to M$ the two-sheeted covering of M. We have an involution $\tilde{I}\colon \tilde{M} \to \tilde{M}$ and a conformal structure such that $\tilde{I}$ is anti-holomorphic. The double surface is minimal, regular, and admits an integral representation:

$$X(p) = \mathrm{Re}\int_{p_o}^{p} \phi(z)\,dz \ , \ p_o, p \in M \ ,$$

or else,

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$$(1.1)\quad \tilde{X}(p) = \mathrm{Re}\int_{p_0}^{p} \frac{f(z)}{2}(1-g^2(z)\ ,i(1+g^2(z)),2g(z))dz, p_0, p\in\tilde{M}.$$

The function g(z) and the form $w = f(z)dz$ of (1.1) satisfy the nonorientability conditions:

$$(1.2)\quad \begin{cases} g(\tilde{I}(p)) = -1/\overline{g(p)}\ ,\ \forall\ p \in \tilde{M}\ , \text{ and} \\ \tilde{I}^*w = -\ \overline{g^2(z)\ w}\ ; \end{cases}$$

these conditions can be obtained from $\tilde{I}^*(\emptyset(x)dz) = \overline{\emptyset(z)dz}$.

The double surfaces of minimal Klein bottles with ends will be punctured toruses. The total curvature of the double surface is $\tilde{c} = -4\pi.$(degree g) and $c = \tilde{c}/2$ is the corresponding total curvature of the Klein bottle. Details can be found in [2].

The functions f and g will be doubly periodic, defined on the universal recovering of the torus, that is, f and g will be elliptic functions with period lattice:

$$\Omega_{\lambda_1,\lambda_2} = \{m\ \lambda_1 + n\ \lambda_2,\ m,n\in\mathbb{Z}\ ,\ \lambda_1.\lambda_2\in\mathbb{C}/\mathrm{Im}(\lambda_2/\lambda_1) > 0\}.$$

In [3] we observe that minimal Klein bottles in $\mathbb{R}^3$ can be obtained considering elliptic functions f(z) and g(z) such that m = degree g is even and greater than 4.

We will work with the Weirstrass'function $p(z)$ with lattice of periods:

$$\Omega_{1,i} = \{m+ni, m, n \in \mathbb{Z}\}$$

as the universal recovering of a torus; the corresponding involution will be

$$\tilde{I}(z) \equiv \bar{z} + \frac{1}{2} = \overline{z + \frac{1}{2}} \pmod{\Omega_{1,i}}.$$

In this lattice, the Weirstrass'function $p(z)$ and its derivative $p'(z)$ have the properties ([1]):

$$\overline{p(\bar{z})} = p(z); \quad \overline{p'(\bar{z})} = p'(z);$$

$$(p'(z))^2 = 4\, p(z)(p(z)-e_1)(p(z)+e_1);$$

$$p(z+1/2) = e_1 \frac{p(z)+e_1}{p(z)-e_1}, \quad e_1 = p(1/2);$$

$$p(z+1/2)-\bar{a} = (e_1-\bar{a}) \frac{(p(z)-b)}{(p(z)-e_1)};$$

$$p(z+1/2)-\bar{b} = (e_1-\bar{b}) \frac{(p(z)-a)}{(p(z)-b)},$$

with $b = \dfrac{e_1(\bar{a}+e_1)}{\bar{a}-e_1}$, $a \in \mathbb{C} - \{e_1\}$;

$$p(z) = \frac{1}{z^2} + \frac{e_1^2}{5} z^2 + \frac{e_1^4}{75} z^4 + \dots ;$$

If $\alpha(t) = i/3 + t$ and $\beta(t) = 1/3 + ti$,

$0 \leq t \leq 1$, we also have:

$$\int_{\alpha} p(z)dz = -\pi \quad \text{and} \quad \int_{\beta} p(z)dz = \pi i \; ;$$

$$\int_{\alpha} p^{n}(z)dz = e^{2}\,\frac{(2n-3)}{(2n-1)}\int_{\alpha} p^{n-2}(z)dz \; , \; n \in \mathbb{Z}.$$

Denoting γ_1 and γ_2 simple closed curves around $t = 0$ and $z = i/2$, respectivelly, and taking $\bar{I}(\gamma_1), \bar{I}(\gamma_2)$, $\bar{I}(\alpha)$, $\bar{I}(\beta)$, from the non-orientability conditions (see [3]) we have:

$$2.\mathrm{Re}\int_{\beta}\phi(z)dz = \int_{\gamma_1}\phi(z)dz + \int_{\gamma_2}\phi(z)dz$$

and

$$\mathrm{Im}\int_{\alpha}\phi(z)dz = -\,\mathrm{Im}\int_{\gamma_1}\phi(z)dz \; .$$

Using these basic facts, we can construct the double surfaces associated to minimal Klein bottles.

2. The existence of minimal Klein bottles:

Initially we will study minimal Klein bottles with one end. We have:

<u>Theorem</u> 2.1.: There exists a regular complete minimal Klein bottle in $\mathbb{R}^3$ with one end and total curvature $-2\pi m$, for m even, $m \geq 4$.

<u>Proof</u>: We will consider two cases:

(i) $m = 2k$, k even, $k \geq 2$;

(ii) $m = 2\ell$, ℓ odd, $\ell \geq 3$.

In the first case, we can choose $f(z)$ and $g(z)$ given by:

$$g(z) = \mu \left(\frac{p'(z)}{p(z)+e_1}\right)^{k-1} \frac{p(z)-a}{p(z)-b}, \text{ with}$$

$$b = \frac{e_1(\bar{a}+e_1)}{\bar{a}-e_1}, \; a \neq 0, \; a \neq e_1, \text{ and}$$

$$f(z) = \lambda \frac{(p(z)-b)^2}{(p(z)-e_1)} (p(z+1/2))^{k-1}.$$

These functions satisfy the regularity condition and the double surface will be complete, with ends at $z = 0$ and $z = 1/2$. The nonorientability conditions (1.2) are equivalent to:

$$(2.1) \qquad |\mu|^2 = \frac{(e_1-\bar{b})}{2^{2k-2} e_1^{k-1} (e_1-\bar{a})},$$

and

$$\bar{\lambda}\,\bar{\mu}(e_1-\bar{b})(e_1-\bar{a}) = -2e_1^2\lambda\mu.$$

We can easily observe that $\phi_3(z)dz = g(z).w$ is exact; thus, we don't have periods and residues for $\int \phi_3(z)dz$.

Comparing and calculating:

$$\int_\alpha f(z)dz = \int_\alpha \lambda \frac{(p(z+1/2))^{k-1}}{p(z)-e_1}(p(z)-b)^2 dz$$

and

$$\int_\alpha -g^2(z)f(z)dz = \int_\alpha \overline{f(\bar{I}(z))dz} = \int_\alpha \bar{\lambda}\frac{(p(z+1/2))^{k-1}}{(p(z)-e_1)}(p(z)-\bar{b})^2 dz\ ,$$

we conclude that these two integrals are given by second degree polynomials in (e_1-b) or $(e_1-\bar{b})$, respectively, with the same real coefficients. So, if (e_1-b) is such that $\int_\alpha f(z)dz=0$, $(e_1-\bar{b})$ anihilates $\int_\alpha \overline{f(\bar{I}(z)dz}$. We also have $\int_{\gamma_j} \phi_i(z)dz=0$, j, i = 1,2, then, $\mathrm{Re}\int_\beta \phi_i(z)dz = 0$, i=1,2.

Choosing μ real, $\mu = |e_1-b|/\sqrt{2^{2k-1}e_1^{k+1}}$ and $\lambda=(e_1-\bar{b})i$, we have the double surface given by $\tilde{X}(p) = \mathrm{Re}\int_{p_o}^{p} \phi(z)dz$.

For the second case, let

$$g(z) = k.\left(\frac{p'(z)}{p(z)}\right)^{\ell-2}\frac{(p(z)-e_1)(p(z)-a)}{(p(z)-b)}$$

and

$$f(z) = \lambda\left(\frac{p(z)}{p(z)-e_1}\right)^{\ell}\left(\frac{p(z)-b}{p(z)}\right)^2 ,$$

be the functions of (1.2), with

(2.2) $$|k|^2 = \frac{(e_1-\bar{b})}{(e_1-\bar{a})}\frac{1}{2^{3\ell-5}e_1^{\ell}}$$

and

$$\bar{\lambda}(e_1-\bar{b})^2 = -\lambda\, k^2 e_1^{\ell+2}2^{3\ell-4}\ .$$

As in the first case, we can verify that the real periods and residues vanish if we choose (e -b) root of a second degree polynomial with real coefficients. Thus if k is real, $k = |e_1-b|/\sqrt{2^{3\ell-4}e_1^{\ell+2}}$ and $\lambda = (e_1-\bar{b})i$, (2.2) is satisfied and we have the double surface.

□

With the same kind of calculations, we can exhibit a minimal Klein bottle with two ends:

<u>Theorem</u> 2.2.: There exists a regular complete minimal Klein bottle in $\mathbb{R}^3$ with two ends and total curvature -12π.

<u>Proof</u>: For the construction of the double surface, we choose the functions:

$$g(z) = \frac{k.p'(z)\ (p(z)+e_1)}{(p(z))^2}\ \frac{(p(z)-a)}{(p(z)-b)}$$

and

$$f(z) = \lambda\ \frac{(p(z))^2}{(p(z)-e_1)^2}\ \frac{(p(z)-b)^2}{(p(z)+e_1)},\ b = \frac{e_1(\bar{a}+e_1)}{\bar{a}-e_1},$$

$a \neq 0$, $a \neq e$, with

$$(2.3)\qquad |k|^2 = \frac{e_1-\bar{b}}{16e_1(e_1-\bar{a})} \text{ and } \bar{k}\bar{\lambda} = \frac{-2e_1^2}{(e_1-\bar{a})(e_1-\bar{b})}\lambda k\ .$$

The form $\emptyset_3(z)dz$ is exact and regular at $z = i/2$; so it doesn't have residues or periods. We can easily verify that $\int_{\gamma_1}\emptyset_j(z)dz$ and $\int_{\gamma_2}\emptyset_j(z)dz$ vanish for $j = 1,2$.

Then for j=1,2, $\mathrm{Re}\int_{\beta}\phi_j(z)dz = 0$.

As in the first theorem, $\int_{\alpha}f(z)dz$ and $\int_{\alpha}f(z)g^2(z)dz$ are given by second degree polynomials in e_1-b and $e_1-\bar{b}$, respectively. Thus if e_1-b is a root of the first polynomial, $e_1-\bar{b}$ anihilate the second, and $\int_{\alpha}\phi_j(z)dz = 0$ for $j = 1,2$. Choosing k and λ we have the double surface.

It is easy to see that the double surface is regular and complete, with ends at $z = 0$, $z = 1/2$, $z = i/2$, $z = (1+i)/2$.

□

Finally, we can ask about the existence of further examples, or about a classification of minimal Klein bottles in $\mathbb{R}^3$.

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