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A note on affine quotients

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A NOTE ON AFFINE QUOTIENTS.

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Abstract. The purpose of this note is to present an elementary and self-contained proof of the following result (see [2]): If G is an affine algebraic group and K a closed subgroup, then if K is exact in G, the quotient variety G/K is affine.

Introduction: Let G be an affine algebraic group and K a closed subgroup. In [2] the induction representation functor from K-modules to G-modules was defined, as well as the concept of exact subgroup. The subgroup K is said to be exact in G, if the induction representation functor from K to G, is exact. In the mentioned paper the authors prove that K is exact in G if and only if the quotient variety G/K is affine. The proof that G/K affine implies that K is exact in G is based on the fact that if G/K is affine, the algebra of polynomial functions on G is faithfully flat when considered as a module over the algebra of the K-invariant polynomials on G. Here we present a proof of the other implication, (Theorem 2.1), that goes as follows. First we use a criterion developed in [1] in order to prove that if K is exact in G, then the quotient variety G/K is quasi-affine. Second we observe that if G/K is quasi-affine it can be covered by a finite number of principal affine open sets $(G/K)_f$, with f in ${}^{K}P(G)$. Here P(G) represents the algebra of polynomial functions on G and ${}^{K}P(G)$ the algebra of K-fixed elements of P(G). Then, we use the exactness again to guarantee that the ideal generated by the f's is all of ${}^{K}P(G)$. The rest is an easy exercise in algebraic geometry.

SECTION 1. We start by fixing some notations. All algebraic groups will be affine and defined over an algebraically closed field F. If G is such a group we indicate by $\underline{M}(G)$ the category of all rational G-modules. If M is a G-module we denote by ${}^{G}M=\{m\in M \mid x.m=m \ \forall\ x\in G\}$. If G is an affine group and K a closed subgroup we think of K acting on G on the right by multiplication. The orbit space of this action will be denoted by G/K. It is well known that G/K has a natural structure of quasi-projective variety. The action of K on G, induces an action of K on P(G) on the left as follows: if $x\in K$ and $f\in P(G)$, x.f is the element of P(G) that takes at an element y of G the value $(x.f)(y) = f(yx)$. The algebra of regular functions on G/K can be identified with ${}^{K}P(G)$ where the K-fixed part is taken with respect to the natural action defined above. The restriction map from P(G) to P(K) will be denoted by Π.

If M is an arbitrary object of $\underline{M}(K)$ we endow $P(G)\otimes M$ with the diagonal K-action and define $M|^{G}$ as $M|^{G} = {}^{K}(P(G)\otimes M)$. Note that $P(G)\otimes M$ has a natural structure of G-module as follows: if $x\in G, f\in P(G)$ and $m\in M$, $x.(f\otimes m)=f.x^{-1}\otimes m$, where $f.x^{-1}$ is the element of P(G) that at the point y of G takes the value $f(x^{-1}y)$. With the G-module structure defined above $M|^{G}$, is a G-submodule of $P(G)\otimes M$. Moreover if: $\alpha: M\to N$ is a K-module homomorphism, $id\otimes\alpha$ from $P(G)\otimes M$ into $P(G)\otimes N$, is a G-map and sends $M|^{G}$ into $N|^{G}$.

<u>Defn. 1.1.</u> The induction representation functor, that we denote as $\mathrm{Ind}_{K,G}$, or simply as Ind, from $\underline{M}(K)$ to $\underline{M}(G)$, is the functor that sends M into $M|^{G}$, and α into the restriction of $id\otimes\alpha$ to $M|^{G}$, for any K-module M and for any K-homomorphism α.

<u>Defn. 1.2.</u> In the situation above, we say that K is exact in G if the functor $\mathrm{Ind}_{K,G}$ is exact.

Note that Ind, is automatically left exact. Thus, K is exact in G if and only if for every morphism of K-modules $\alpha:N\to M$ such that $\alpha(N)=M$, we have that $(id\otimes\alpha)(N|^G)=M|^G$.

Defn. 1.3. If G is an affine algebraic group and K a closed subgroup, a rational multiplicative character γ of K is said to be extendible to G, if there is a non zero polynomial $f\in P(G)$ such that for every $x\in K$, $x.f=\gamma(x)f$. Such an f is called an extension of γ.

It is easy to show that if γ is extendible to G, there is an extension f of γ such that $\Pi(f)=\gamma$, or equivalently such that $f(1)=1$

If γ is a rational character of K we denote by γ^* the character $\gamma^*(x)=\gamma(x)^{-1}$.

It was proved in [1] that if for every character γ that is extendible to G, the character γ^* is also extendible to G, then the quotient G/K is quasi-affine.

Theorem 1.4. Let G be an affine algebraic group and K a closed subgroup. If K is exact in G then G/K is quasi-affine.

Proof. It is easy to reduce the theorem to the case where G is irreducible. In that case we prove that if γ is an extendible character, then γ^* is also extendible. Take g an extension of γ such that $g(1)=1$, and define $\tilde{g}$ in P(G) as $\tilde{g}(x)=g(x^{-1})$. Note that $\Pi(\tilde{g})=\gamma^*$. Call $\langle K\tilde{g}\rangle$ the K-submodule of P(G) generated by $\tilde{g}$. Consider the map α from $Fg\otimes\langle K\tilde{g}\rangle$ into F, defined as the composition $\alpha=j(\Pi\otimes\Pi)$, where j is the multiplication map from $F\gamma\otimes F\gamma^*$ into F that sends $\gamma\otimes\gamma^*$ into 1. It is clear that if we endow $Fg\otimes\langle K\tilde{g}\rangle$ with the diagonal K-module structure, and if we endow F with the trivial K-module structure, α is a surjective K-module homomorphism. By exactness we deduce that

$id\otimes\alpha:(Fg\otimes\langle K\tilde{g}\rangle)|^G \to F|^G={}^KP(G)$ is also surjective.

...sequently there is a $t\varepsilon^K(P(G)\otimes Fg\otimes <K\tilde{g}>)$ such that $(id\otimes\alpha)(t)=1$.

Write $t=\Sigma f_i\otimes g\otimes h_i$, and define f as $f=\Sigma f_ih_i$. From the fact that t is K-fixed, we deduce that gf is also K-fixed. Thus, for any x, K we have, $x.(gf)=gf$, and then, as $x.g=\gamma(x)g$, we deduce that $x.f=\gamma^*(x)f$.

Now, if we define $\lambda_i\varepsilon F$ by the equality $\Pi(h_i)=\lambda_i\gamma^*$, the equality $(id\otimes\alpha)(t)=1$ gives us that $\Sigma\lambda_if_i=1$. But $\lambda_i=h_i(1)$, thus, $f(1)=\Sigma f_i(1)h_i(1)=\Sigma \lambda_if_i(1)=1$. Consequently, f is an extension of γ^* to G.

Q.E.D.

<u>SECTION 2</u>. In this section we complete the proof of the main result by proving that if G/K is quasi-affine and K is exact in G, then the quotient variety G/K is affine.

<u>Theorem 2.1</u>. (Cline, Parshall and Scott). If K is an exact closed subgroup of the affine algebraic group G, then the quotient variety G/K is affine.

<u>Proof</u>. A standard argument reduces the proof to the case where G is irreducible. From Theorem 1.4, we deduce that G/K is quasi-affine. Then, there is a non zero element f in ${}^KP(G)$ such that the principal open set $(G/K)_f=\{xK\varepsilon G/K \mid f(x)\neq 0\}$, is affine. Consider the ideal I of ${}^KP(G)$ generated by the set $\{f.y \mid y\varepsilon G\}$. It is clear that the ideal $IP(G)$ cannot have any zero in G, so that $IP(G)=P(G)$. Then, we can find a finite number of elements $f_1,f_2,\ldots,f_n$ in ${}^KP(G)$, and $g_1,\ldots,g_n$ in P(G) such that $\Sigma f_ig_i=1$. Consider the K-module homomorphism α, from $\overset{n}{\underset{1}{\oplus}} P(G)$ into P(G), defined as $\alpha(a_1,\ldots,a_n) = \Sigma f_ia_i$. As α sends $(g_1,\ldots,g_n)$ into 1, we deduce that α is surjective. Using the exactness we deduce that the map $id\otimes\alpha: {}^K(P(G)\otimes \overset{n}{\underset{i=1}{\oplus}} P(G)) \rightarrow {}^K(P(G)\otimes P(G))$ is also surjective. The multi-

as a G-map. Thus, the map $\hat{\mu}$ defined as the restriction of μ to ${}^{K}(P(G)\otimes P(G))$, is also surjective. Consider now the map $\hat{\alpha}:\oplus_{1}^{n}{}^{K}P(G)\to {}^{K}P(G)$ defined as $\hat{\alpha}(b_1,\ldots,b_n)=\Sigma f_i b_i$. It is easy to see from the definitions above that: $\hat{\mu}(id\otimes\alpha)=\hat{\alpha}(\oplus\hat{\mu})$ and consequently that $\hat{\alpha}$ is surjective. That implies that there exist functions $h_1,\ldots,h_n$ in ${}^{K}P(G)$ such that $\Sigma f_i h_i=1$. Thus, $I={}^{K}P(G)$. Using [3], exercise 2.17, page 81, we deduce that G/K is affine.

Q.E.D.

The result used at the very end of the proof above, reads as follows. Let X be an algebraic variety defined over F with structure sheaf $\underline{0}$. Suppose there are elements $f_1,\ldots,f_n \in \underline{0}(X)$ such that: a) For every i the principal open set X_{f_i} is affine. b) The ideal generated in $\underline{0}(X)$ by the set $\{f_i : i=1,\ldots,n\}$ is the unit ideal. Then the variety X is affine.

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