A Note on Lattices with a Symmetric Difference Like Operation

Francisco Miraglia[*]
University of São Paulo, São Paulo, Brazil
Alejandro Petrovich[†]
University of Buenos Aires, Buenos Aires, Argentina

The purpose of this note is to discuss the existence of a binary operation in lattices with properties analogous to that of symmetric difference in a Boolean algebra (BA).

The original intent of the authors was to

(1) Investigate the existence of reduced special groups (cf. [3]) inside distributive lattices.

(2) That this question might also be relevant regarding the coding of logical validity via polynomials (cf. [1], [2]) and the references therein).

We show that that under very mild hypothesis, the existence of such an operation entails the original lattice to be a BA and the proposed operation to be classical symmetric difference.

This result constituted motivation to approach the question from a different perspective: define a binary operation, $*$, which would be simulate the properties of symmetric difference in a Boolean algebra in a general distributive lattice, via an adjunction, leading us to examine the structure and basic properties of **$*$-lattices**.

After developing some of the basic properties of $*$-lattices, we give a number of equivalence conditions for a $*$-lattice to be a Boolean algebra.

We then prove that the class of $*$-lattices coincides with Brouwer (or Brouwerian lattices) as defined in [4]. We also give an explicit equational characterization of $*$-lattices (or Brouwer lattices) and show that the join operation can be obtained from the operations $*$ and meet, generalizing a well-known result in Boolean algebras.

It is explicitly stated in [4] that the notion of ideal in a Brouwer algebra will not be discussed therein. We go on to describe the basic properties of ideals and the congruence they generate, the $*$-algebra nature of quotients by ideals and then present a $*$-lattice version of Glivenko's Theorem.

It is then proven that there is an anti-equivalence (Duality) between the categories of $*$-Lattices and their morphisms with a certain category of Spectral Spaces and their morphisms (dubbed $*$-Spectral Spaces).

References

[1] Carnielli, W.A., Matulovic, M. The method of polynomial ring calculus and its potentialities. *Theoretical Computer Science*, 606:42-56, 2015.

[2] Carnielli, W.A., Mariano, H.L. Matulovic, M. Reconciling First-Order Logic to Algebra., *Trends in Logic*, 47:273–305, Springer-Verlag, 2018.

[3] Dickmann, M., Miraglia, F. Special Groups : Boolean-Theoretic Methods in the Theory of Quadratic Forms, *Memoirs Amer. Math. Soc.*, Providence, R.I., 2000.

[*] miraglia@ime.usp.br
[†] apetrov@dm.uba.ar

[4] McKinsey, J.C.C., Tarski, A. On Closed Elements in Closure Algebras, *Annals of Math.* 47:122–162, 1946.