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QUASI-TOPOLOGICAL QUANTUM FIELD THEORIES AND $\mathbb{Z}_2$ LATTICE GAUGE THEORIES

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We consider a two-parameter family of $\mathbb{Z}_2$ gauge theories on a lattice discretization $T(\mathcal{M})$ of a three-manifold $\mathcal{M}$ and its relation to topological field theories. Familiar models such as the spin-gauge model are curves on a parameter space Γ. We show that there is a region $\Gamma_0 \subset \Gamma$ where the partition function and the expectation value $\langle W_R(\gamma)\rangle$ of the Wilson loop can be exactly computed. Depending on the point of Γ_0, the model behaves as topological or quasi-topological. The partition function is, up to a scaling factor, a topological number of $\mathcal{M}$. The Wilson loop on the other hand, does not depend on the topology of γ. However, for a subset of Γ_0, $\langle W_R(\gamma)\rangle$ depends on the size of γ and follows a discrete version of an area law. At the zero temperature limit, the spin-gauge model approaches the topological and the quasi-topological regions depending on the sign of the coupling constant.

Keywords: $\mathbb{Z}_2$ gauge theories; topological field theories; Wilson loops.

1. Introduction

A lattice gauge theory with gauge group $\mathbb{Z}_2$ is the simplest example of a gauge theory.[1] In dimension $d = 2$, the partition function for $\mathbb{Z}_2$ (as for any other compact gauge group) can be computed in various ways. In dimensions larger than two, however, the simplicity of $\mathbb{Z}_2$ does not help us to solve the model. Even without matter, the relevant models are nontrivial and exact solutions are not known. Such solutions would be a very important achievement. A $\mathbb{Z}_2$ gauge theory on a cubic lattice can be made dual to the 3D Ising model, an outstanding problem in statistical mechanics.[2,3] It is clear that in $d = 4$, the problem is at least as difficult as in $d = 3$.

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In this paper, we will not have much to say about $d = 4$ since the tools we use are peculiar to dimension three.

However difficult, 3D lattice models with local gauge symmetry are not always beyond the reach of exact solutions. That depends on the dynamics, i.e. the choice of an action for a lattice plaquette. Topological Quantum Field Theories (TQFTs) are examples where one can perform exact computations. Examples have been constructed on the lattice in dimensions $d = 3$[4,5] as well as $d = 4$.[6] Once the topology on the manifold in question is fixed, the partition function does not depend on the lattice size and can be trivially computed for a discretization with very small number of sites, links and plaquettes. Such models are very simple from the physical point of view. It follows from topological invariance that transfer matrices are trivial.

Despite being trivial dynamically for a fixed topology, TQFTs are quite relevant in physics. The reason being that TQFTs can come out as limits of ordinary field theories in the continuum as well as in the lattice. The most celebrated example is topological order in condensed matter physics[7] where the physics at large scales is described by a TQFT. Something of similar nature also happens for lattice theories in $d = 2$ that are quasi-topological.[8,9] They are reduced to a TQFT at the appropriate limits.

In order to understand the relationship between fully dynamical $d = 2$ models and their possible topological limits, one can first look at quasi-topological models. They are very easy to work with since we can compute all relevant quantities. Quasi-topological models in $d = 2$ are a nice set of toy models for this purpose. In particular, the relation between the original models and their topological limits is made very explicit. As for $d = 3$, the situation is more complicated. We do not have at our disposal toy models that are at the same time not topological and easily computable. We have no choice but to work with fully dynamical theories where no exact computations are available.

The focus of this paper is to investigate how lattice field theories with local $\mathbb{Z}_2$ gauge symmetry are related to topological theories. Before going any further we need to say what we mean by a lattice model being topological. Let $T(\mathcal{M})$ be a lattice triangulation of a fixed compact three-manifold $\mathcal{M}$. We say that such a model is topological if the partition function is the same for all triangulations $T(\mathcal{M})$. Actually, this is a weak definition since we may ask that not only the partition function but the expectation value of all observables to be of a topological nature. In any case, we have to go beyond the usual regular cubic lattices and take into account arbitrary lattices.

In Ref. 10, we investigated the spin-gauge model in $d = 3$. We showed that in the limit $\beta \to \infty$ the partition function Z is given by

$$Z(T(\mathcal{M}))|_{\beta\to\infty} = Z^{\text{top}}(\mathcal{M})\ 2^{N_F+N_L-N_T}\,, \qquad (1)$$

where N_T, N_F and N_L are the number of tetrahedra, faces and links of $T(\mathcal{M})$ and $Z^{\text{top}}(\mathcal{M})$ is a topological number and as such does not depend on the discretization $T(\mathcal{M})$. Equation (1) tells us that at the limit $\beta \to \infty$ the partition function is not strictly speaking topological since it depends on the triangulation. However, it does not depend on the details of the triangulation but only on its size. For this reason we say that the partition function is quasi-topological. It follows from (1) that the partition function can be computed for all triangulations. Let $T_0(\mathcal{M})$ be a lattice triangulation where the numbers N_{T_0}, N_{F_0} and N_{L_0} are very small such that $Z(T_0)$ can be written down explicitly. For an arbitrary lattice $T(\mathcal{M})$ we have

$$Z(T)|_{\beta\to\infty} = Z(T_0)|_{\beta\to\infty} 2^{(N_F - N_{F_0}) + (N_L - N_{L_0}) - (N_T - N_{T_0})} \,. \tag{2}$$

We will find it convenient to rewrite Z as a product of local Boltzmann weights, in the form

$$Z = \sum_{\{g_a\}} \prod_f W(f) \,, \tag{3}$$

where the product is over all faces of the triangulation and sum is over all configurations. Let a, b and c be the links of a face f and (g_a, g_b, g_c) a gauge configuration at f. The corresponding local Boltzmann weight for the spin-gauge model can be written as

$$W^{(1)}(g_a, g_b, g_c) = e^{\beta g_a g_b g_c} \,. \tag{4}$$

Another very common choice is to set the local Boltzmann weight to

$$W^{(2)}(g_a, g_b, g_c) = e^{-\beta(1 - g_a g_b g_c)} \,, \tag{5}$$

which corresponds to the usual gauge theories where flat holonomies will have the highest weight.

As we will see in this paper, the relationship between gauge models and TQFTs can be better understood if we depart from a specific example as in Ref. 10 and consider a more general class of gauge models. In order to have a gauge theory, the local weight should depend only on the product of the gauge variables around an oriented plaquette:

$$W(g_a, g_b, g_c) = M(g_a g_b g_c) \,. \tag{6}$$

Gauge invariance means that $M(g)$ is a class function or, in other words $M(hgh^{-1}) = M(g)$, $\forall h \in G$. The character expansion for the group $\mathbb{Z}_2$ is very simple and implies that

$$M(g_a, g_b, g_c) = m_1(g_a g_b g_c) + m_0 \,, \tag{7}$$

where $m_i \in \mathbb{R}$. The original spin-gauge model with one parameter β can be recovered by restricting the model to a curve $(m_0, m_1) = (\cosh(\beta), \sinh(\beta))$ in this two-dimensional parameter space. We will refer to the parameter space as Γ.

The first question to be addressed is the generalization of Eq. (2). We will show that there is a subset Γ_0 of the parameter space Γ such that the partition function $Z(m_0, m_1, T(\mathcal{M}))$ can be written as a product of a topological invariant $Z^{\rm top}(\mathcal{M})$ times a known function depending on the numbers N_T, N_F and N_L of tetrahedra, faces and links. Again, an easy consequence is that at Γ_0 the partition function $Z(m_0, m_1, T(\mathcal{M}))$ can be computed for any lattice $T(\mathcal{M})$. The subset Γ_0 is made of two pairs of lines, namely, $m_i = 0$, $i = 0, 1$ and $m_1 = \pm m_0$. It turns out that these two regions of Γ_0 have different properties. For instance, the topological invariant that appears in the first pair is trivial and $Z^{\rm top}(\mathcal{M}) = 1$ for all compact manifolds $\mathcal{M}$. As for second pair, $Z^{\rm top}(\mathcal{M})$ depends on the first group of co-homology of $\mathcal{M}$.[10] The two pairs of solutions are also related to high and low temperature limits as it will be clear from the discussion on Sec. 2.

As in any gauge theory, one would be interested in more observables than just the partition function. In particular it is important to calculate the expectation value $\langle W_R(\gamma)\rangle$ of Wilson loops for arbitrary representations R of the gauge group and closed curves γ. In our previous work,[10] we considered only the partition function. In the present paper, we would like to go further and ask whether $\langle W_R(\gamma)\rangle$ can be computed for some points of Γ. As it happens for the partition function, such computation can be performed for all points of Γ_0. Note that, in a truly topological gauge theory such as Chern–Simons, $\langle W_R(\gamma)\rangle$ is a topological invariant of γ. That is not true for all points of Γ_0. It turns out that $\langle W_R(\gamma)\rangle$ depends on the size of γ for points of Γ_0 of the form $(m_0, m_1) = (\lambda, -\lambda)$. Since there is no dependence on the details of γ we say that the observables $\langle W_R(\gamma)\rangle$ are quasi-topological.

Our approach is based on the fact that a large class of lattice models, topological or otherwise, can be described by a set of algebraic data on a vector space V. These data comprises of a multiplication m, a co-multiplication Δ and an endomorphism S such that $S^2 = 1$. It is also assumed that there is a unity e and a co-unity ϵ. It has been shown in Refs. 4 and 5 that when the data $(m, \Delta, S, e, \epsilon)$ defines a Hopf algebra, one can construct a lattice topological field theory. We observed in Ref. 10 that the same data can be used to describe an ordinary $\mathbb{Z}_2$ gauge theory. In Ref. 10, however, $(m, \Delta, S, e, \epsilon)$ is not a Hopf algebra. In particular, the co-multiplication is not an algebra morphism as it happens for Hopf algebras. Another important difference is that instead of a fixed algebraic data, we had a one parameter family $m(\beta)$ of multiplications where β is the coupling constant of the model. It turns out that a Hopf algebra is recovered in the limit $\beta \to \infty$ and the model becomes quasi-topological. It is also possible to have examples with matter fields via a family of co-multiplications $\Delta(\lambda)$ where λ is the corresponding coupling constant.[11] In this paper, however, we will be limited to pure gauge theories. As stated before, we will consider gauge theories with two coupling constants given by (7). If we were to follow the formalism of Ref. 10, that would be encoded in a two parameter family of multiplications $m(m_0, m_1)$. The model considered in Refs. 4 and 5 corresponds to the unique point $(m_0, m_1) = (1, 1)$ where $m(1, 1)$ together with Δ, S, e and ϵ define a Hopf algebra. For the present paper, however, the Hopf structure is less important.

What matter are the points where the model is quasi-topological. That happens for (m_0, m_1) belonging to the region Γ_0 described above. The data $m(m_0, m_1), \Delta, S$ and e defines a Hopf algebra only at a single point of Γ_0. Just as the topological model of Refs. 4 and 5, the limit $\beta \to \infty$ of Eq. (1) also corresponds to a point in the set Γ_0.

The organization of the paper goes as follows: In Sec. 2, we explain how the algebraic data can be used to encode the model and how familiar models, such as the spin-gauge model, fit into the parameter space Γ. In Sec. 3, we determine the subset Γ_0 where the model is quasi-topological. The computation of the expectation value for Wilson loops is investigated on Sec. 4. We show that $\langle W_R(\gamma)\rangle$ can be computed for Γ_0. The model is not topological for all Γ_0. We show that for a particular region of Γ_0, $\langle W_R(\gamma)\rangle$ depends on the size of γ and follows a discrete version of an area law.

We close the paper with some final remarks on Sec. 5.

2. The Partition Function

In this section, we introduce the formalism we will use to describe the partition function of a $\mathbb{Z}_2$ gauge theories. We will follow Refs. 4, 5 and 10, but we will make a few modifications to suit our purpose.

Let $T(\mathcal{M})$ be a triangulation of a compact three-dimensional manifold $\mathcal{M}$. For the description of a pure gauge theory, what is relevant in a discretization is the set of faces and how they are interconnected. To encode the connectivity information we will split $T(\mathcal{M})$ into individual faces and record the information on how they should be put back together. This process can be described as follows. For each face $f_k \in T(\mathcal{M})$, we associate a disjoint face F_k and for each link $l_j \in T(\mathcal{M})$, we associate a hinge object H_j with n_j flaps as illustrated in Fig. 1. The number n_j

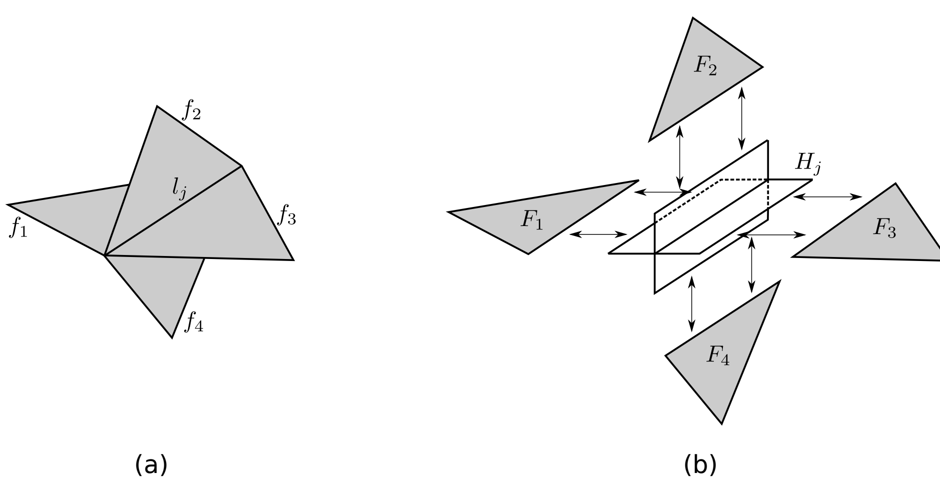

Fig. 1. In (a) we have a small piece of a triangulation. Figure (b) shows its corresponding decomposition into flaps and faces.

of flaps is equal to the number $I(l_j)$ of faces of $T(\mathcal{M})$ that share the link l_j. To reconstruct $T(\mathcal{M})$ from the disjoint faces F_k, we can use the hinges H_j to determine which faces are to be joint together. This is illustrated by Fig. 1. The faces F_k and the hinges H_j are to be given an extra structure called orientation. For the case of $\mathbb{Z}_2$ gauge theories, this orientations are not relevant but we will mention them as they help to organize the model. We will call the set $\{F_k, H_j\}$ and its interconnections a decomposition of $T(\mathcal{M})$.

We will use the decomposition $\{F_k, H_m\}$, plus some extra data, to define a partition function. The first step is to choose a vector space V of dimension n. The edges of a face F_k have to be enumerated from 1 to 3. That amounts to a choice of orientation of the face and a choice of the starting point [see Fig. 2(a)]. The edges of $l_m \in F_k$ carry configurations (a_1^k, a_2^k, a_3^k), with $a_i^k \in \{1, \ldots, n\}$. A statistical weight $M_{a_1^k a_2^k a_3^k}$ will be associated to the face F_k. Note that $M_{a_1^k b_2^k c_3^k}$ can be viewed as the components of a tensor $M \in V \otimes V \otimes V$. Furthermore, it should be invariant by cyclic permutations of its indices since we do not care which edge is to be numbered as the first one. On the other hand, a change in orientation can affect the corresponding weight since M_{abc} may not be the same as M_{cba}. In a similar fashion, the flaps of a hinge H_m can be cyclically numbered from 1 to $q = I(l_m)$. Once more, this is equivalent to give H_m an orientation, as illustrated by Fig. 2(b). The flaps of H_m carry configurations $(a_1^m, \ldots, a_q^m)$, $a_i^m \in \{1, \ldots, n\}$ just like the edges of a face. That will correspond to a statistical weight $\Delta^{a_1^m \cdots a_q^m}$.

Such numbers can be interpreted as the components of a tensor $\Delta \in V^* \otimes \cdots \otimes V^*$ and have to be invariant by cyclic permutations of the indices. As before, change in orientation of H_m will change the statistical weight to $\Delta^{a_q^m a_{q-1}^m \cdots a_1^m}$.

Once we fix an orientation for each F_k and an orientation for each H_m, we produce a tensor $M_{a_1 a_2 a_3}(F_k)$ for each face and a tensor $\Delta^{b_1 \cdots b_q}(H_m)$, $q = I(l_m)$

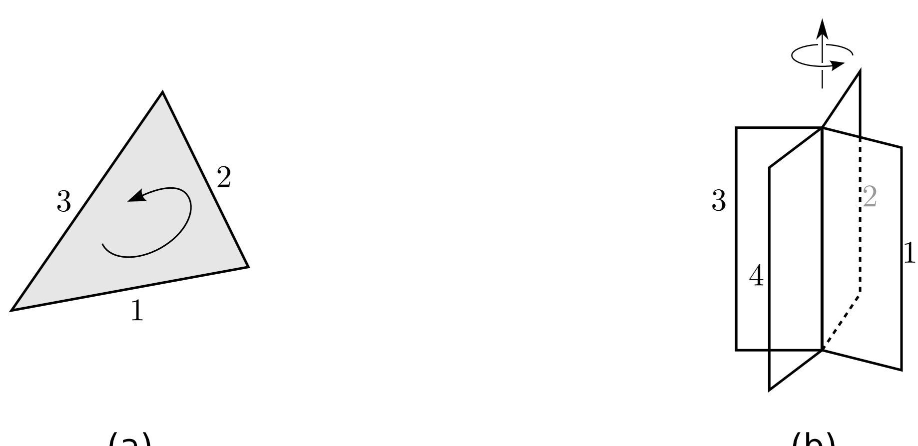

Fig. 2. The links on a triangular face are numbered from 1 to 3. The orientation is indicated in (a). Figure (b) shows a hinge with four flaps and its orientation.

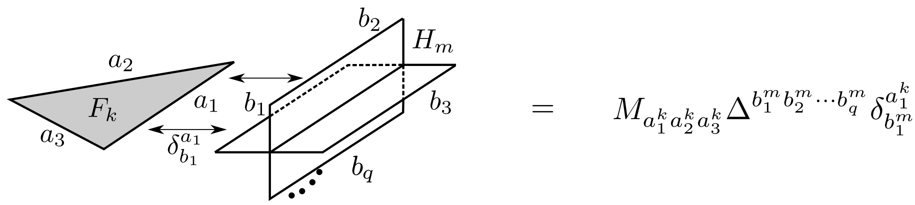

Fig. 3. In terms of tensor, we interpret the gluings as contractions of the corresponding indices.

for each hinge. One can see that the product

$$\prod_k M_{a_1^k a_2^k a_3^k}(F_k) \prod_m \Delta^{b_1^m \cdots b_q^m}(H_m)\,, \tag{8}$$

has one covariant index for each edge and one contra-variant index for each flap of $\{F_k, H_m\}$. The partition function will be the scalar constructed by contracting all indices. A covariant index a_i^k is to be contracted to a contra-variant index b_j^m whenever the corresponding edge F_k and flap H_m are to be glued together. In other words, we define the scalar Z as

$$Z = \prod_k M_{a_1^k a_2^k a_3^k}(F_k) \prod_m \Delta^{b_1^m \cdots b_q^m}(H_m) \prod_{b \in \{\text{gluings}\}} \delta^{a_i^r}_{b_j^s}(b)\,, \tag{9}$$

where the last product is responsible for contracting indices. There will be a $\delta^{a_i^r}_{b_j^s}(b)$ for each paring edge-flap (F_r, H_s) that are glued together, as illustrated by Fig. 3. We can simplify the notation of (9) by eliminating the Kronecker deltas and writing

$$Z = \prod_{f \in \{F\}} M_{abc}(f) \prod_{l \in \{L\}} \Delta^{b_1 \cdots b_{N_L}}(l)\,, \tag{10}$$

where $\{F\}$ and $\{L\}$ are the set of faces and links of the triangulation. A contraction on the indices corresponding to gluings is understood.

As of now, the partition function (9) is not very useful. We need to be more precise about the weights $M_{a_1 a_2 a_3}(F_k)$ and $\Delta^{b_1 \cdots b_n}(H_m)$ if we want Z to be related to physical models like the spin-gauge model. Note that Z depends on the choice of orientations of individual faces and hinges. This dependence on the orientation should not be present in the final model. Furthermore, the weight function should be the same for all faces F_k and hinges H_m. To go any further we need to constraint the tensors $M_{abc}(F_k)$ and $\Delta^{b_1 \cdots b_n}(H_m)$. That can be done with the help of some algebraic data that we will now introduce.

The first algebraic structure we need is a product on V defined by

$$\phi_a \cdot \phi_b = M_{ab}{}^c \phi_c\,, \tag{11}$$

where we are using a tensorial notation with the usual convention of sum over repeated indices. We will use the symbol A to refer to the vector space V to emphasize that we are now working with an algebra.

In this paper, we will choose A to be the group algebra of $\mathbb{Z}_2$. The group elements are written as $(-1)^a$, $a = 0, 1$ and a basis for A is $\{\phi_0, \phi_1\}$. The product is defined by

$$\phi_a \cdot \phi_b = \phi_{a+b} \,. \tag{12}$$

Whenever a sum $a + b$ of indices appear, as in (12), it will always denote sum module 2. This product can also be given in terms of the tensor ${M_{ab}}^c$ as

$${M_{ab}}^c = \delta(a + b, c) \,. \tag{13}$$

It is also convenient to define the dual vector space A^*, the dual base $\{\psi^i\}$ and the usual pairing

$$\langle \psi^a, \phi_b \rangle = \delta^a_b \,. \tag{14}$$

We then define the trace $T \in A^*$ as

$$T = {M_{ab}}^b \psi^a \,. \tag{15}$$

Given a face F_k, as in Fig. 3, we define the associated weight $M_{a_1a_2a_3}$ as

$$M_{a_1a_2a_3}(z) = \langle T, \phi_{a_1} \cdot \phi_{a_2} \cdot \phi_{a_3} \cdot z \rangle \,, \tag{16}$$

where $z = \alpha^0\phi_0 + \alpha^1\phi_1$ is a generic element of A. Since the underlining group is Abelian, the weight $M_{a_1a_2a_3}(z)$ is automatically cyclic and gauge invariant. Furthermore, $M_{a_1a_2a_3}(z)$ does not depend on the orientation of F_k.

The choice of z in (16) will determine the model we are describing. For example, consider the curve $(\alpha^0, \alpha^1) = \left(\frac{1}{2}e^{\beta}, \frac{1}{2}e^{-\beta}\right)$ parametrized by β. The corresponding weight

$$M_{a_1a_2a_3}\left(\frac{1}{2}e^{\beta}, \frac{1}{2}e^{-\beta}\right) = e^{\beta(-1)^{a_1+a_2+a_3}} \,, \tag{17}$$

corresponds to the spin-gauge model. We can also choose the curve $(\alpha^0, \alpha^1) = \left(\frac{1}{2}, \frac{1}{2}e^{-2\beta}\right)$ and that will give

$$M_{a_1a_2a_3}\left(\frac{1}{2}, \frac{1}{2}e^{-2\beta}\right) = e^{-\beta[1-(-1)^{a_1+a_2+a_3}]} \,, \tag{18}$$

describing yet another model.

The second algebraic information is a co-product $\Delta : A \to A \otimes A$ defined by

$$\Delta(\phi_c) = {\Delta_c}^{ab} \phi_a \otimes \phi_b \,. \tag{19}$$

The tensor ${\Delta_c}^{ab}$ defines also a product on the dual space A^*. Using the dual basis $\{\psi^a\}$ we define

$$\psi^a \cdot \psi^b = {\Delta_c}^{ab} \psi^c \,. \tag{20}$$

In analogy with (15) and (16) we define the co-trace $T^* \in A$ as

$$T^* = {\Delta_b}^{ba} \phi_a \tag{21}$$

and the tensor $\Delta^{b_1\cdots b_n}$ as

$$\Delta^{b_1\cdots b_n} = \left\langle \psi^{a_1}\cdot\psi^{a_2}\cdots\psi^{a_n}, T^*\right\rangle . \tag{22}$$

It turns out that the co-product that we will need is very simple. We will set

$${\Delta_c}^{ab} = \delta_c^a\delta_c^b . \tag{23}$$

Therefore,

$$\Delta^{b_1\cdots b_n} = \begin{cases} 1 & \text{if all indices are equal},\\ 0 & \text{otherwise}. \end{cases} \tag{24}$$

Note that the orientations of hinges do not affect the corresponding statistical weight.

The tensors ${M_{ab}}^c$ and ${\Delta_a}^{bc}$ given in (13) and (23) together with the weights M_{abc} and $\Delta^{a_1\cdots a_n}$ defined by (16) and (22) completely specify our model. It is a simple exercise to show that the partition function (9) reduces to

$$Z(\alpha^0,\alpha^1) = \sum_{\{\sigma_l\}}\prod_{f\in T(\mathcal{M})} M_{a_1a_2a_3}(f) , \tag{25}$$

where the sum $\{\sigma_l\}$ is over the configurations on the links $l\in T(\mathcal{M})$ and $M_{abc}(f)$ is the weight of a configuration (a_1,a_2,a_3) on the face $f\in T(\mathcal{M})$. Notice that

$$Z(\lambda\alpha^0,\lambda\alpha^1) = \lambda^{N_F} Z\big(\alpha^0,\alpha^1\big) . \tag{26}$$

As $M_{a_1a_2a_3}$ depends on $z=\alpha^0\phi_0+\alpha^1\phi_1\in A$, the model depends on two parameters (α^0,α^1). We will denote the parameter space by Γ. The parameters (α^0,α^1) are related to the parameters (m_0,m_1) of (7) as

$$\begin{aligned} m_0 &= \alpha^0+\alpha^1 ,\\ m_1 &= \alpha^0-\alpha^1 . \end{aligned} \tag{27}$$

Let us recall that the algebra defined in (12) is a group algebra and as such it is also a Hopf algebra with co-product coming from ${\Delta_a}^{bc}$. The maps antipode $S: A\to A$, unity $e:\mathbb{C}\to A$ and co-unity $\epsilon: A\to\mathbb{C}$ can be described in terms of tensors as $S(\phi_a)=S_a^b\phi_b$, $e(1)=e^a\phi_a$ and $\epsilon(\phi_a)=\epsilon_a$. For the case of $\mathbb{Z}_2$ we have

$$S_a^b=\delta_a^b , \quad e_a=\delta_a^0 \quad\text{and}\quad \epsilon_a = 1 . \tag{28}$$

Notice that these tensors are essentially trivial and will not show up explicitly in the calculations. For a non-Abelian case, for example, S_a^b is related to the orientation but that will play no role in the $\mathbb{Z}_2$ case.

3. Quasi-Topological Limits

We would like to explore the model (25) and look for points of the parameter space (α^0, α^1) where the model has a topological or quasi-topological behavior. The simplest case is the one considered in Refs. 4 and 5. It corresponds the point $(\alpha^0, \alpha^1) = (1, 0)$ in the parameter space Γ or, equivalently, to the choice in (16) of z equals to the identity ϕ_0 of the algebra. For this particular point of the parameter space we can bring in the Hopf structure of A, follow[4,5] and conclude that

$$Z\big(T(\mathcal{M}), \alpha^0 = 1, \alpha^1 = 0\big) = 2^{N_F+N_L-N_T} Z^{\text{top}}(\mathcal{M}) \,, \tag{29}$$

where N_T, N_F and N_L are the number of tetrahedra, faces and links of $T(\mathcal{M})$. However, $(\alpha^0, \alpha^1) = (1, 0)$ is not the only quasi-topological point. In this section, we will show that there is a subset $\Gamma_0 \subset \Gamma$ with dimension one such that the partition function is quasi-topological. In Fig. 4, we have the parameter space Γ where the set Γ_0 consists of four straight lines: the diagonals and the axis. We also have included the models with weights $W^{(1)}$ and $W^{(2)}$ defined in (4) and (5). They are curves parametrized by a single parameter β.

Let us consider $(\alpha^0, \alpha^1) = (\lambda, 0)$. It corresponds to $z = \lambda\phi_0 \in A$. The new weight is simple $M_{abc}(z) = \lambda M_{abc}(\phi_0)$. Therefore, the partition function is the same as for $(\alpha^0, \alpha^1) = (1, 0)$ multiplied by a factor. In other words

$$Z(T(\mathcal{M}), \lambda, 0) = \lambda^{N_F} 2^{N_F+N_L-N_T} Z^{\text{top}}(\mathcal{M}) \,. \tag{30}$$

The tensor components $M_{abc}(z)$ and $\Delta^{b_1\cdots b_n}$ depend on the choice of a basis of the algebra A. Therefore, different choices of basis will lead to different weights and therefore different models. However, the partition function $Z(T(\mathcal{M}), \alpha^0, \alpha^1)$ has been written as a scalar and therefore is invariant under a change of basis.

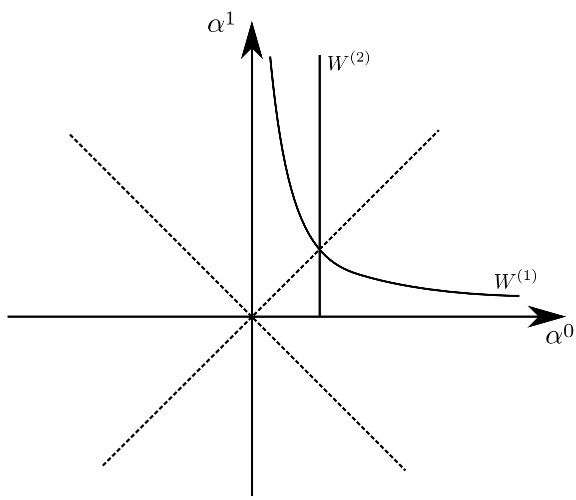

Fig. 4. The set Γ_0 consists of the two axes plus the dashed diagonal lines. The models $W^{(1)}$ and $W^{(2)}$ are curves on Γ.

This large invariance of $Z(T(\mathcal{M}), \alpha^0, \alpha^1)$ has been interpreted as dualities between different models. This fact has been explored by us in Ref. 11 to show that the classical Kramers and Wannier duality relations are special cases of these more general dualities. In what follows, we will show that there is a duality relation between the model at $(\alpha^0, \alpha^1) = (\lambda, 0)$ and $(\alpha^0, \alpha^1) = (0, \lambda)$. We will show that

$$Z(T(\mathcal{M}), 0, \lambda) = Z(T(\mathcal{M}), \lambda, 0)\,, \tag{31}$$

which allow us to compute $Z(T(\mathcal{M}), 0, \lambda)$.

Let us start by recalling that at the topological point $(\alpha^0, \alpha^1) = (1, 0)$ the weights read

$$M_{abc}(\phi_0) = 2\delta(a + b + c, 0)\,, \tag{32}$$

$$\Delta^{b_1\cdots b_n} = \delta(b_1, b_n)\delta(b_2, b_n)\cdots\delta(b_{n-1}, b_n)\,. \tag{33}$$

Consider the matrix

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{34}$$

and a new basis $\{\phi'_0, \phi'_1\}$ defined as $\phi'_a = E^b_a \phi_b$. This transformation simply changes an index a by $\bar{a}$, where $\bar{0} = 1$ and $\bar{1} = 0$. In the new basis we have

$$M'_{abc}(1, 0) = M_{\bar{a}\bar{b}\bar{c}}(1, 0) = 2\delta\big(\bar{a} + \bar{b} + \bar{c}, 0\big) = 2\delta(a + b + c, 1)\,, \tag{35}$$

$$\begin{aligned} \Delta'^{b_1\cdots b_n} &= \delta\big(\bar{b}_1, \bar{b}_n\big)\delta\big(\bar{b}_2, \bar{b}_n\big)\cdots\delta\big(\bar{b}_{n-1}, \bar{b}_n\big) \\ &= \delta\big(b_1, b_n\big)\delta\big(b_2, b_n\big)\cdots\delta\big(b_{n-1}, b_n\big)\,. \end{aligned} \tag{36}$$

Note that

$$\Delta'^{b_1\cdots b_n} = \Delta^{b_1\cdots b_n}\,. \tag{37}$$

On the other hand, the tensor components $M_{abc}(0, 1)$ on the original basis is $M_{abc}(0, 1) = \delta(a + b + c, 1)$. Therefore

$$M'_{abc}(1, 0) = M_{abc}(0, 1)\,. \tag{38}$$

Equations (37) and (38) imply that $Z(T(\mathcal{M}), 0, \lambda) = Z(T(\mathcal{M}), \lambda, 0)$ as announced.

The partition function corresponding to the diagonal line $\alpha^0 = \alpha^1$ in Fig. 4 can be obtained by choosing $z_{\text{int}} = \phi_0 + \phi_1$ in (16). One can see that z_{int} is such that $\phi_g \cdot z_{\text{int}} = z_{\text{int}}$. In a Hopf algebra, such element is called a co-integral.[12] Therefore

$$M_{abc}(1, 1) = \langle T, z_{\text{int}}\rangle = 2\,. \tag{39}$$

The fact that the weight $M_{abc}(1, 1)$ is independent of the configurations turns the computation of $Z(T(\mathcal{M}), \lambda, \lambda)$ completely trivial. One can immediately see that

$$Z(T(\mathcal{M}), \lambda, \lambda) = 2^{N_F + N_L}\lambda^{N_F}\,. \tag{40}$$

Note that $Z(T(\mathcal{M}), \lambda, \lambda)$ depends only on the size of the lattice $T(\mathcal{M})$. In contrast with (30), the topological invariant is trivial since the partition function does not depend on the topology of the underlining manifold $\mathcal{M}$.

To compute the partition function for $\alpha^1 = -\alpha^0$, it is enough to consider the point $(\alpha^0, \alpha^1) = (1, -1)$. That is the same as setting $z = \phi_0 - \phi_1$ on (16). The weight $M_{abc}(1, -1)$ reads

$$M_{abc}(1, -1) = 2(-1)^{a+b+c} . \tag{41}$$

Instead of computing $Z(T(\mathcal{M}), 1, -1)$ directly, we will make use of the duality related to change of basis. Let us choose another basis by applying the transformation matrix $E_a^b = (-1)^a \delta_a^b$. In the new basis the weights are

$$\begin{aligned} M'_{abc} &= 2 , \\ (\Delta')^{a_1 a_2 \cdots a_n} &= (-1)^{a_1 + a_2 + \cdots + a_n} \delta_{a_n}^{a_1} \delta_{a_n}^{a_2} \cdots \delta_{a_n}^{a_{n-1}} . \end{aligned} \tag{42}$$

The resulting model has weights associated to the links only and the partition function can be easily computed. After plugging (42) in (9) and taking into account the scaling factor we have

$$Z(T(\mathcal{M}), \lambda, -\lambda) = \lambda^{N_F} \prod_l \left(1 + (-1)^{I(l)}\right) , \tag{43}$$

where the product runs over the links of the lattice and $I(l)$ denotes the number of faces that share the link l. Note that $I(l)$ is not of topological nature. Furthermore, the partition function vanishes whenever there is a link that is shared by a odd number of faces. This is an indication that $(\alpha^0, \alpha^1) = (\lambda, -\lambda)$ is a very peculiar model.

It is clear from the computation of the partition function that the lines that make up Γ_0 are not all equivalent. Actually they are all different from each other. It is true that $Z(T(\mathcal{M}), \lambda, 0)$ is the same as $Z(T(\mathcal{M}), 0, \lambda)$. However, the expectation value of Wilson loops are not the same for these two models. As we will show in the next section, only $(\lambda, 0)$ is in fact a topological theory.

Before we conclude this section, we would like to point out the relation between Γ_0 and familiar models such as the ones defined by (4) (spin-gauge model) and (5) (gauge theories). As we have discussed before, the spin-gauge model given in (17) corresponds to the curve $(\alpha^0, \alpha^1) = (1/2e^{\beta}, 1/2e^{-\beta})$. It is clear from Fig. 4 that this curve approaches Γ_0 as β goes to $+\infty$ and $-\infty$. Another point of contact with Γ_0 is when $\beta \to 0$. As for the model (5), only the limits $\beta \to 0$ and $\beta \to \pm\infty$ are part of Γ_0.

4. Wilson Loops

The two-parameter gauge model (7) of last section have numerical quantities $\langle W_R(\gamma) \rangle$ that are the natural generalization of the expectation value of Wilson loops for a closed curve γ and irreducible representation R. The definition of $\langle W_R(\gamma) \rangle$

reduces to the familiar expression when restricted to the usual gauge models. In this section, we will define and compute these observables for Γ_0.

For simplicity, we will start by considering the loop γ to be unknotted. Knotted loops will be considered in the last part of this section.

Let γ be a loop made of a set of links $\omega(\gamma) = \{\omega_1, \ldots, \omega_p\}$. For such a loop we introduce the tensor $W_{a_1 a_2 \cdots a_p}$ with p indices given by

$$W^R_{a_1 \cdots a_p} = \big\langle T, \phi_{a_1} \cdots \phi_{a_p} \cdot z_R \big\rangle\,, \tag{44}$$

where a_k is the configuration at link l_k and z_R is the unique element in the center of A such that

$$\big\langle T, \phi_g \cdot z_R \big\rangle = \mathrm{Tr}_R(g)\,, \tag{45}$$

where R denotes an irreducible representation of the group.

The group $\mathbb{Z}_2$ has only two irreducible representations labeled $R = 0$ and $R = 1$, such that

$$\mathrm{Tr}_0(\phi_a) = 1\,,$$

$$\mathrm{Tr}_1(\phi_a) = (-1)^a\,.$$

We only need to consider the nontrivial representation. In other words, we will set $z_R = \frac{1}{2}(\phi_0 - \phi_1)$ since that will give us $\langle T, \phi_a \cdot z_R \rangle = (-1)^a$. For now on we will omit the index R indicating the representation simply write (44) as

$$W_{a_1 \cdots a_p} = (-1)^{a_1 + \cdots + a_p}\,. \tag{46}$$

We would like to construct $\langle W(\gamma) \rangle$ as a scalar in the same way it has been done for the partition function (10). As before, we make use of the contra-variant tensors $\Delta^{b_1 \cdots b_n}$ associated to the links. Consider a link l shared by $I(l)$ faces. If l does not belong to the loop γ, the corresponding tensor is the same as for the partition function and will be written as $\Delta^{b_1 \cdots b_{I(l)}}$. If, however, the link in question is one of the links of γ ($l = \omega_j \in \omega(\gamma)$), the corresponding tensor will be $\Delta^{b_1 \cdots b_{I(l)} a_j}$. Note that the new tensor has an extra contra-variant index a_j. The expectation value of the Wilson loop is defined to be

$$\langle W(\gamma) \rangle = \frac{1}{Z} \prod_{f \in \{F\}} M_{abc}(f) \prod_{l \notin \omega(\gamma)} \Delta^{b_1 \cdots b_{I(l)}}(l) \prod_{\omega_j \in \omega(\gamma)} \Delta^{b_1 \cdots b_{I(l)} a_j} W_{a_1 \cdots a_p}\,, \tag{47}$$

where we are using the simplified notation of (10). The contraction of indices are as follows: each covariant index of $W_{a_1 \cdots a_p}$ is to be contracted with the extra index a_j in $\Delta^{b_1 \cdots b_{I(l)} a_j}$ as explicitly indicated. The indexes from the product $\prod_{f \in \{F\}} M_{abc}$ follows the same rule as for the partition function when contracting with the b_k indexes in $\Delta^{b_1 \cdots b_{I(l)}}$ and $\Delta^{b_1 \cdots b_{I(l)} a_j}$. An interpretation of (47) in terms of gluing of hinges (H_m) and faces (F_k) that is analogue to the partition function can also be given. To each link l we associate a hinge H_l. If the link is not part of the loop γ, the corresponding hinge has exactly $I(l)$ flaps that will connect to $I(l)$ faces in the usual way (see Fig. 3). For links $\omega_j \in \omega(\gamma)$, the corresponding hinge H_{ω_j} has

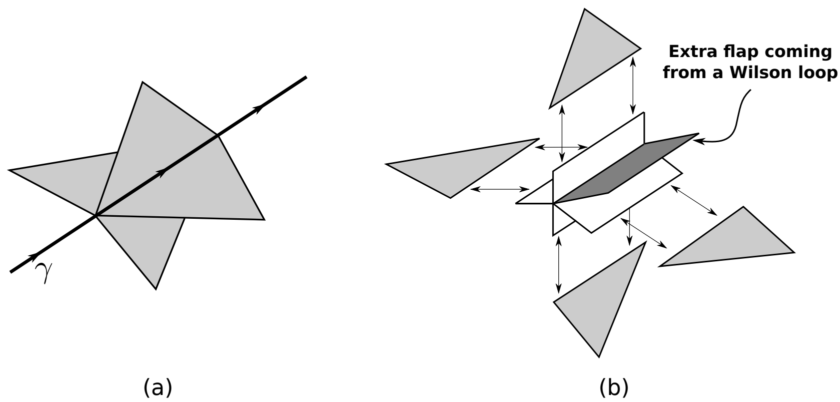

Fig. 5. A small region of the lattice with a Wilson loop γ is illustrated in (a). Figure (b) shows the corresponding decomposition. For a link of γ, the corresponding hinge has an extra flap as indicated.

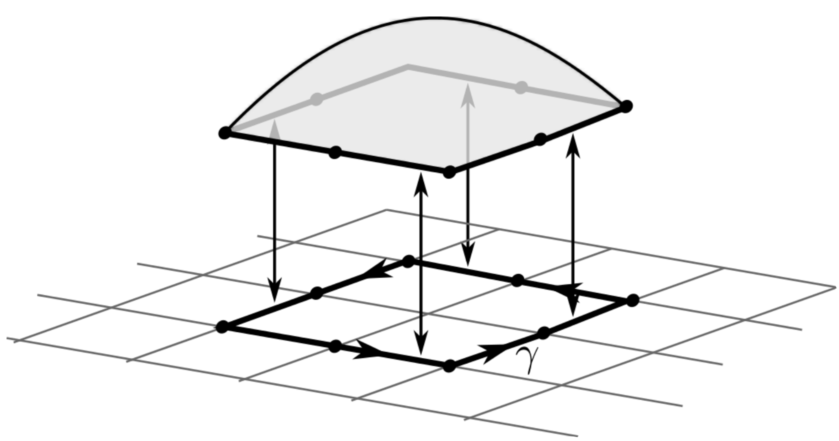

Fig. 6. Wilson loops can be thought as an extra face which is not part of the lattice.

$(I(\omega_j) + 1)$ flaps. After gluing the $I(\omega_j)$ faces to H_{ω_j}, we are left with extra flaps, one for each link of $\omega(\gamma)$ as illustrated in Fig. 5. The contraction of indices a_j in (47) can be seen as the attachment of a polyhedral face W_γ with p edges. An example of such attachment is shown in Fig. 6. This special face W_γ has weight $W_{a_1 \cdots a_p}$ and is not to be thought as part of the lattice. In general, it will not be possible to embed W_γ in 3D space.

The expectation value of the Wilson loop will be a function $\langle W(\gamma)\rangle(\alpha^0, \alpha^1)$. As a consequence of (26), one can see that

$$\langle W(\gamma)\rangle\big(\lambda\alpha^0, \lambda\alpha^1\big) = \langle W(\gamma)\rangle\big(\alpha^0, \alpha^1\big)\,. \tag{48}$$

Therefore, we only need to compute $\langle W(\gamma)\rangle(\alpha^0, \alpha^1)$ for one point on each straight line of Γ_0.

We now proceed with the computation of $\langle W(\gamma)\rangle$ when the parameters (α^0, α^1) belongs to Γ_0. That can be divided in three cases as follows.

Case 1 $(z = \phi_0 + \phi_1)$:

The first case correspond to $(\alpha^0, \alpha^1) = (1, 1)$. As we have seen, $M_{abc} = 2$. All the sums on b indices in (47) are straightforward due to the Kronecker deltas in $\Delta^{b_1\cdots b_{I(l)}}$ and $\Delta^{b_1\cdots b_{I(l)}a_j}$. We are then left with

$$\langle W(\gamma)\rangle \propto \prod_{i=1}^{p}\sum_{a_i}(-1)^{a_i} = 0\,. \tag{49}$$

This result does not depend on the loop γ.

Case 2 $(z = \phi_0 - \phi_1)$:

For this particular case we have $W_{a_1\cdots a_p} = (-1)^{a_1+\cdots+a_p}$. Notice that, this is the same function as the weights $M_{abc} = 2(-1)^{a+b+c}$ for the faces. Therefore, the numerator in (47) is the same thing as a partition function with an extra face determined by the loop γ (see Fig. 6). Using (43) we can write

$$\langle W(\gamma)\rangle = \frac{1}{\prod_l\left[1+(-1)^{I(l)}\right]}\prod_{l\notin\omega(\gamma)}\left[1+(-1)^{I(l)}\right]\prod_{l\in\omega(\gamma)}\left[1+(-1)^{I(l)+1}\right]. \tag{50}$$

This region of the parameter space is quite peculiar. Notice that the denominator of $\langle W(\gamma)\rangle$ vanish if $I(l)$ is odd for some l and $\langle W(\gamma)\rangle$ is not well defined. When $I(l)$ is even for all links, $\langle W(\gamma)\rangle$ is well defined but it is equal to zero due to the factors $[1 + (-1)^{I(l)+1}]$.

Case 3 $(z = \phi_g, g = 0, 1)$:

The points with coordinates (α^0, α^1) given by $(1, 0)$ and $(0, 1)$ in Γ_0 correspond to $z = \phi_g$ with $g = 0$ and $g = 1$, respectively. We know from previews sections that the partition function is the same for these two cases. However, this is not true for $\langle W(\gamma)\rangle$.

We will show that $\langle W(\gamma)\rangle$ is quasi-topological in the sense that it does not depend strongly on the geometry of γ. The idea is to investigate the behavior of $\langle W(\gamma)\rangle$ under small deformations. In a triangulation, it is natural to define a small deformation of a loop γ as follows. Let $\omega(\gamma)$ be the set of links of γ. A small deformation of γ is a local move that replaces one link $\omega_1 \in \omega(\gamma)$ by a pair of links ω_2, ω_3 such that ω_1, ω_2 and ω_3 belong to the same face. There is also the reverse move where a pair of links ω_1 and ω_2 is replaced by ω_3 provided they belong to the same face. A small deformation is shown in Fig. 7. With this definition of a small deformation we now establish how $\langle W(\gamma)\rangle$ changes under small deformations for $z = \phi_g$ and $g = 0, 1$.

Consider Fig. 8(a) where we have single out a small part of the partition (F_k, H_j). Only the elements connected to the link ω_1 are relevant. In Fig. 8(b)

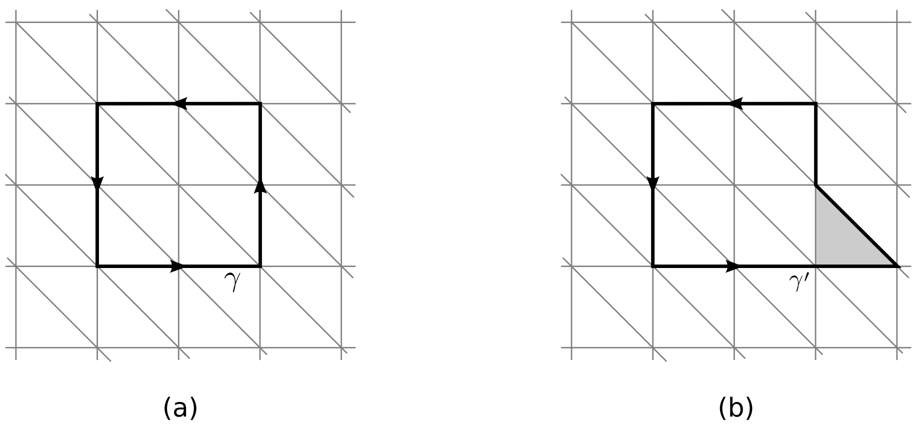

Fig. 7. The loop γ' in (b) is a small deformation of the loop γ in (a).

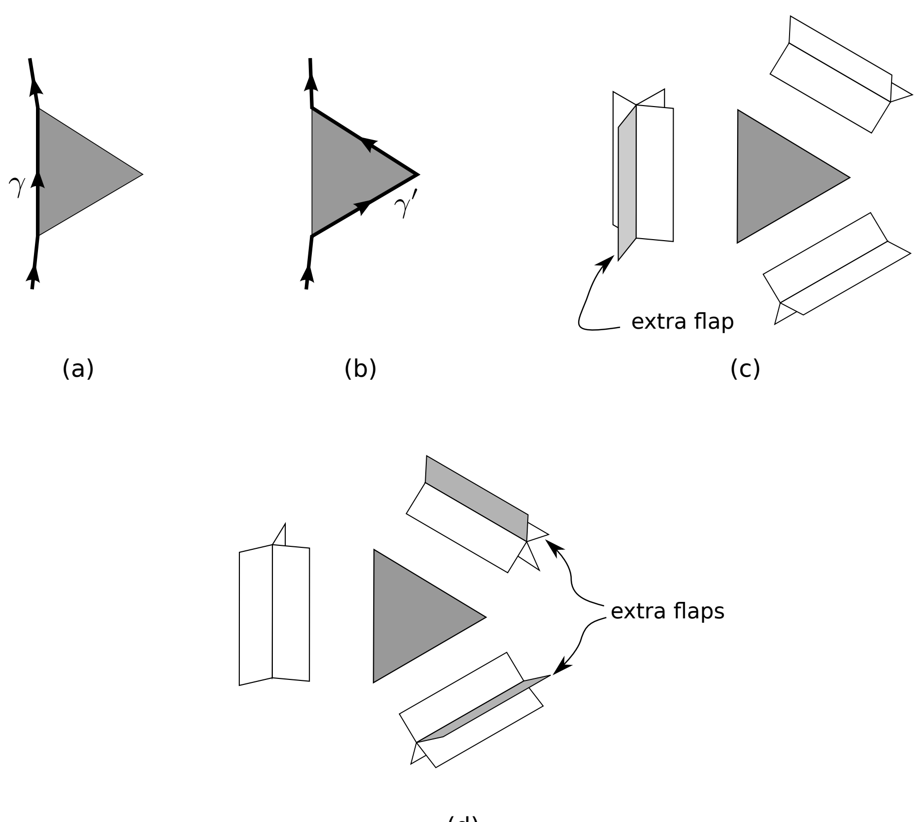

Fig. 8. In (a)–(b) we represent a small deformation of a loop γ in a lattice and in (c)–(d) we represent the same deformation in a lattice decomposition.

we have performed a small deformation of γ by replacing ω_2 and ω_3 by ω_1. The weights associated to Fig. 8(c) and 8(d) are the tensors A and B given by

$$\begin{aligned} A^{a_2\cdots a_p b_2\cdots b_q c_2\cdots c_r}_{\alpha_1\cdots\alpha_s} &= M_{a_1b_1c_1}\Delta^{a_1\cdots a_p\omega_1}\Delta^{b_1\cdots b_q}\Delta^{c_1\cdots c_r}W_{\omega_1\alpha_1\cdots\alpha_s} \\ &= \sum_{a_1,b_1,c_1} M_{a_1b_1c_1}\Delta^{a_1\cdots a_p}\Delta^{b_1\cdots b_q}\Delta^{c_1\cdots c_r}W_{a_1\alpha_1\cdots\alpha_s}\,, \end{aligned} \tag{51}$$

and

$$\begin{aligned} B^{a_2\cdots a_p b_2\cdots b_q c_2\cdots c_r}_{\alpha_1\cdots\alpha_s} &= M_{a_1b_1c_1}\Delta^{a_1\cdots a_p}\Delta^{b_1\cdots b_q\omega_2}\Delta^{c_1\cdots c_r\omega_3}W_{\omega_2\omega_3\alpha_1\cdots\alpha_s} \\ &= \sum_{a_1,b_1,c_1} M_{a_1b_1c_1}\Delta^{a_1\cdots a_p}\Delta^{b_1\cdots b_q}\Delta^{c_1\cdots c_r}W_{b_1c_1\alpha_1\cdots\alpha_s}\,, \end{aligned} \tag{52}$$

where we have performed the sums on ω_1, ω_2 and ω_3. We also have explicitly written the sums on a_1, b_1 and c_1. Note that, $W_{a_1\alpha_1\cdots\alpha_s} = (-1)^{a_1}W_{\alpha_1\cdots\alpha_s}$, $W_{b_1c_1\alpha_1\cdots\alpha_s} = (-1)^{b_1+c_1}W_{\alpha_1\cdots\alpha_s}$ and $M_{a_1b_1c_1} = 2\delta(a_1+b_1+c_1,g)$. That is enough to show that

$$M_{a_1b_1c_1}W_{b_1c_1\alpha_1\cdots\alpha_s} = (-1)^g M_{a_1b_1c_1}W_{a_1\alpha_1\cdots\alpha_s}\,, \tag{53}$$

therefore

$$A^{a_2\cdots a_p b_2\cdots b_q c_2\cdots c_r}_{\alpha_1\cdots\alpha_s} = (-1)^g B^{a_2\cdots a_p b_2\cdots b_q c_2\cdots c_r}_{\alpha_1\cdots\alpha_s}. \tag{54}$$

We see that $\langle W(\gamma)\rangle$ is invariant under small deformations for $g=0$ that correspond to $(\alpha^0,\alpha^1)=(1,0)$. As for $g=1$ or $(\alpha^0,\alpha^1)=(0,1)$, the number $\langle W(\gamma)\rangle$ flips sign each time we perform a small deformation on γ.

The simplest loop γ_0 is made of a single triangular face with links $\omega(\gamma) = \{\omega_1,\omega_2,\omega_3\}$. If we recall that $M_{abc} = 2\delta(a+b+c,g)$ and $W_{\omega_1\omega_2\omega_3} = (-1)^{\omega_1+\omega_2+\omega_3}$, it becomes a straightforward computation to show that $\langle W(\gamma_0)\rangle = (-1)^g$. We can now deform the smallest loop γ_0 by adding $(N-1)$ triangles and arriving at a planar loop γ_N. For such a loop we get

$$\langle W(\gamma_N)\rangle = (-1)^{gN}\,;\quad g=0,1\,. \tag{55}$$

Equation (55) for $g=1$ shows $\langle W(\gamma_N)\rangle$ as a function of the number of triangles swept in the process of stretching γ_0 into γ_N. This function is a very simple "area law." It depends only on the parity of N. Notice that, each time we add a triangle to γ_0, the number of links of the loop also changes by one unity. Therefore, we could have written

$$\langle W(\gamma_N)\rangle = (-1)^{gN_L}\,, \tag{56}$$

where N_L is the number of links of the loop γ. We could interpret this formula as a "perimeter law." The fact that there is no distinction between area or perimeter law is peculiar to the gauge group $\mathbb{Z}_2$ and the fact that we are using triangular lattices. For square lattices the variation in the number of links is even and (56) does not hold, but (55) is still true.

So far we have considered only planar loops. For knotted loops, the expectation value $\langle W(\gamma_N)\rangle$ for $g=0$ may depends on the class of isotopy of the loop γ.

Equation (55) is valid for loops γ that can be deformed to the trivial knot. If γ is a nontrivial knot, Eq. (55) may not hold. We will show that $\langle W(\gamma_N)\rangle$ actually does not depend on the class of isotopy of γ and therefore (55) is still correct.

Let γ be a knotted loop and its knot diagram as illustrated in Fig. 9(a). We can produce a new knot diagram by flipping under-crossing into over-crossings and vice versa. This flips are local moves that only affect the knot in a small region. It is a well known result from knot theory that any γ can be made into the trivial knot if we perform a number of flips, as we can see in Fig. 9(b). We will show that $\langle W(\gamma_N)\rangle$ is invariant by flips and therefore does not depend on the isotopy class of γ.

Let us consider a 3D ball B around a crossing in a knot K. That will give us curves γ_{ab} and γ_{cd} connecting the points (a, b) and (c, d) at the surface $\partial B = S^2$ of B as in Fig. 10(a). After a flip move, we have a new knot $\tilde{K}$ and new curves are $\tilde{\gamma}_{ab}$ and $\tilde{\gamma}_{cd}$ as illustrated in Fig. 10(b). Before analyzing the general case, we will

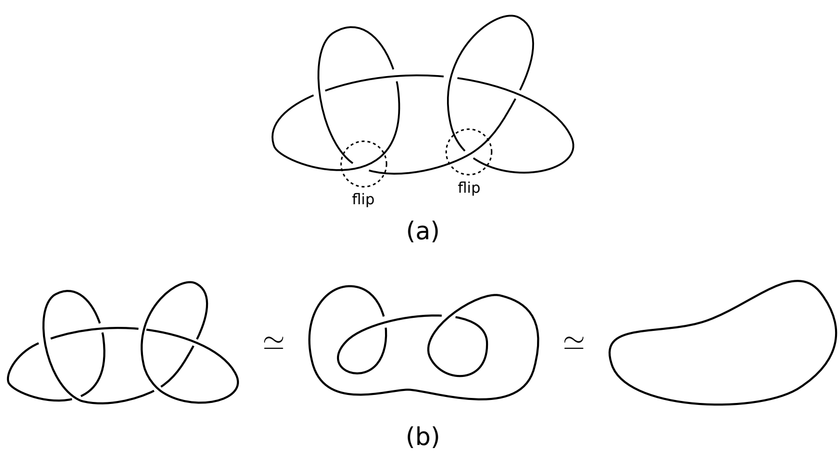

Fig. 9. In (a) we have a knotted loop that can be made trivial by two flip moves, as illustrated in (b).

Fig. 10. The situation shown in (a) represents a flip move of one shown in (b) and vice versa.

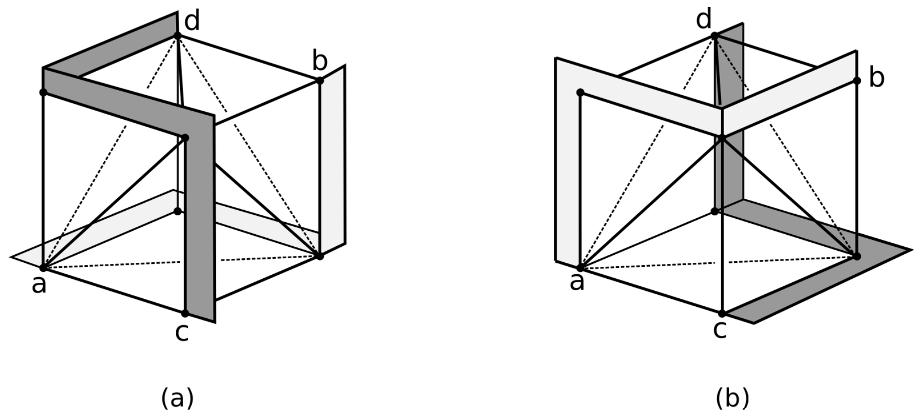

Fig. 11. Here we can see how a flip move acts on curves in a surface $\partial B = S^2$ of a ball.

look at a simple example where B is a cube inside the triangulation and the curves, before and after the flip move, are the ones given in Figs. 11(a) and 11(b). In this figure, we used the interpretation of the expectation value of the Wilson loop as an extra flap as was explained in Sec. 4.

One can see that the sequence of extra flaps can be deformed into the sequences given by Figs. 12(a) and 12(b). After we perform these deformations, the computation of $\langle W(K)\rangle$ and of $\langle W(\tilde{K})\rangle$ differ only at a single link. The difference is that the flaps coming from the two curves are swaped. In one case, the curves will contribute to (47) with a factor

$$\Delta^{a_1\cdots a_n\omega_1\omega_2} W^{(1)}_{\omega_1\alpha_1\cdots\alpha_s} W^{(2)}_{\omega_2\beta_1\cdots\beta_r}\,, \tag{57}$$

and in the other case the factor is

$$\Delta^{a_1\cdots a_n\omega_2\omega_1} W^{(1)}_{\omega_1\alpha_1\cdots\alpha_s} W^{(2)}_{\omega_2\beta_1\cdots\beta_r}\,. \tag{58}$$

The tensor $\Delta^{a_1\cdots a_n\omega_2\omega_1}$ is invariant by permutation of the indices ω_1 and ω_2 and these two factors are the same. Therefore, when comparing $\langle W(K)\rangle$ and $\langle W(\tilde{K})\rangle$ we can use a sequence of small deformations to arrive at the configuration on Figs. 10(a) and 10(b). The flip itself will not give any contribution and we can use the result for planar loops given in (55).

As for the generic case, we can proceed as follows. Choose curves γ_1, γ_2 and γ_3 connecting the pairs (a,d), (d,b) and (c,b) as in Fig. 13. Since the three-ball is simple connected, γ_{ab} is isotopic to $\gamma_1\circ\gamma_2$ and γ_{cd} can be deformed to $\gamma_3\circ\gamma_2^{-1}$. In a similar way, we have $\tilde{\gamma}_{ab}$ is isotopic to $\gamma_1\circ\gamma_2$ and $\tilde{\gamma}_{cd}$ can be deformed to $\gamma_3\circ\gamma_2^{-1}$. In a similar fashion as for the particular case of Fig. 10(a), there will be two sequence of flaps along γ_2. One comes from the deformation of γ_{ab} and the other comes from the deformation γ_{cd}. That has to be compared with a similar sequence of flaps coming from the deformation of $\tilde{\gamma}_{ab}$ and $\tilde{\gamma}_{cd}$. As in the case of Figs. 12(c) and 12(d), these sequences of pair of flaps along γ_2 can only differ by a permutation.

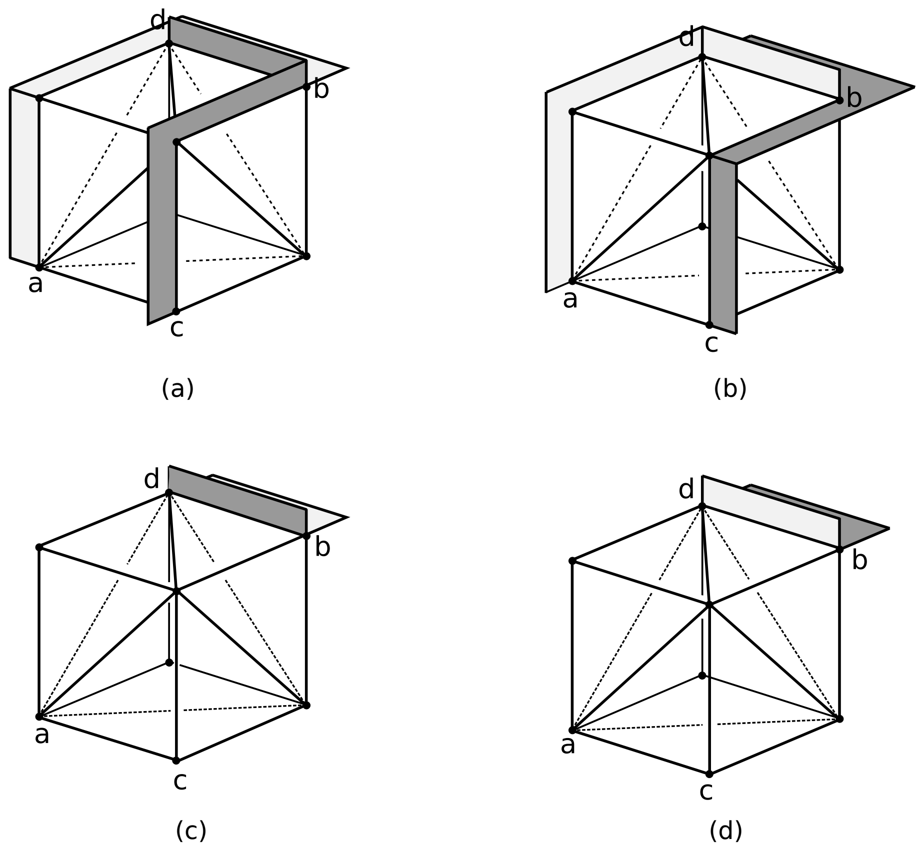

Fig. 12. The pictures (a) and (b) differs from each other only by the link (d, b), as we can see in (c) and (d).

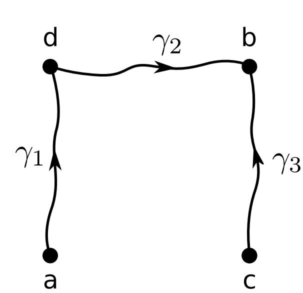

Fig. 13. The paths γ_1, γ_2 and γ_3 are some paths that connect the vertices (a, d), (d, b) and (b, c).

Since $\Delta^{a_1 \cdots a_n}$ is symmetric by permutation of indices we can conclude that flips do not give any contribution and we can use the result for the planar loops given in (55) for any knot K.

5. Final Remarks

Topological field theories are among the simplest lattice models we can have. From the physics point of view they are peculiar models. Partition function and correlations can be computed but the dynamics is too simple. Rather than considering TQFTs in isolation, we have looked at the problem from a broad perspective and investigated a two-parameter family of models where TQFTs can arise at certain points of the parameter space. We have considered gauge theories with $\mathbb{Z}_2$ symmetry since it is the simplest gauge group but can still accommodate nontrivial models, such as the 3D spin-gauge model. These more familiar models appear as one parameter curves in the 2D parameter space Γ.

We have found several limits that we can loosely call topological or quasi-topological comprising a subset Γ_0 of Γ. On Γ_0 both partition function and expectation value of Wilson loops were computed. The partition function points on Γ_0 are topological numbers up to an overall scale factor. One could think that Γ_0 contain only topological models but the expectation value of the Wilson reveals something else. First of all, $\langle W_R(\gamma)\rangle$ does not depend on the isotopy class of the curve γ. Furthermore, for a subset of Γ_0, $\langle W_R(\gamma)\rangle$ depends on the size of γ and follows a discrete version of an area law.

In the parametrization (α^0, α^1) of Γ used in the paper, the subset Γ_0 is made of four straight lines passing through $(0, 0)$. By looking at the gauge Ising model, we can see that it approaches three of these lines for $\beta \to \pm\infty$ and $\beta \to 0$. There is an extra line given by $\alpha^1 = -\alpha^0$ that, as far as we know, does not relate directly to any physical model.

The existence of a set Γ_0 in the parameter space where the model behaves in a topological way can be seen as an Euclidean version of topological order. It seems that, rather than a special case, the same phenomena will happen for gauge theories with any compact gauge group G. For $\mathbb{Z}_N$ and non-Abelian groups, the analysis is much more involved and it will be reported in a separated paper.

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