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SEMIREGULAR SURFACES WITH TWO TRIPLE POINTS AND TEN CROSS CAPS

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ABSTRACT

This work contains the classification of singular surfaces with two triple points, ten Whitney umbrellas (cross-caps or pinch points) and simply connected self-intersection curve. There are 89 topological types of such surfaces, one of which is orientable.

Key words:
singular surfaces
triple points
cross-caps

RESUMO

Este trabalho contém a classificação das superfícies singulares com dois pontos triplos, dez guarda-chuvas de Whitney (cross-caps ou pinch points) e auto-intersecção simplesmente conexa. Existem 89 tipos topológicos de tais superfícies, uma das quais é orientável.

Palavras chaves:

superfícies singulares
pontos triplos
cross-caps

Semiregular surfaces with two triple points and ten cross caps

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1 Introduction

In [9] Whitney introduces the concept of semiregular mappings, that is, mappings $f : M^n \to \mathbb{R}^{2n-1}$ from an n-dimensional closed manifold M into $\mathbb{R}^{2n-1}$ which are immersions with normal crossings except at a finite number of singular points called cross caps or Whitney umbrellas. When $n = 2$ the image of f is locally homeomorphic to one of the following:

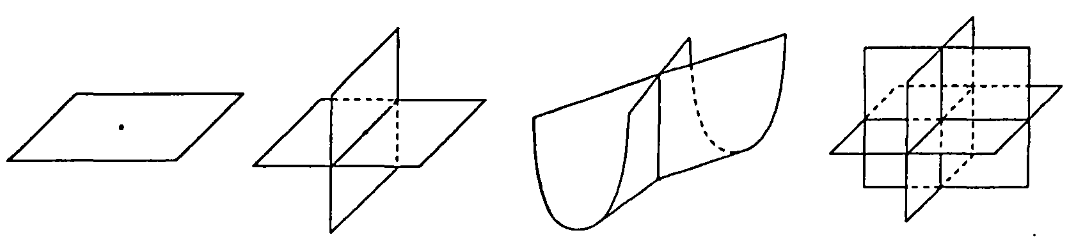

Figure 1

A semiregular surface is the image of a 2-manifold M by a semiregular mapping $f : M \to \mathbb{R}^3$. We call M the abstract surface of $f(M)$. These objects have been studied in many contexts (see [1],[8]). In particular, in Singularity Theory when studying stable perturbations of map germs from the plane to 3-space [6].

The semiregular surfaces with one triple point and six cross caps have been classified in [3]. There are seven topologically distinct surfaces with self intersection curve simply connected as in Figure 2.(a). Among them we find Steiner's roman surface which is the image of the projective plane $\mathbb{P}^2$ in $\mathbb{R}^3$ [4]. Here we classify semiregular surfaces with two triple points and ten cross caps whose self intersection curve is simply connected as in Figure 2.(b).

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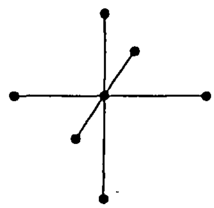


Figure 2.(a)

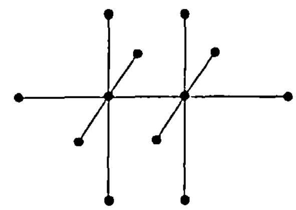


Figure 2.(b)

We show that there are only 89 topological types of such surfaces. They can be built by means of a surgery of two of the seven types with one triple point. Our method, as in [3] (see also [2]), is based on the study of the symmetries of all possible surfaces of the type prescribed and for such we make use of Mathematica.

2 Combinatorial classification

As it happens with the Roman surface, which has been modelled on an octahedron, the surfaces here will be modelled on the polyhedron H with vertices $a_1, \ldots, a_5, b_1, \ldots, b_5$ as in Figure 3. This polyhedron H is the convex hull of the two hexagons corresponding to $a_1, b_1, b_5, b_3, a_3, a_5$ and $a_2, b_2, b_5, b_4, a_4, a_5$, and the two squares corresponding to a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4.

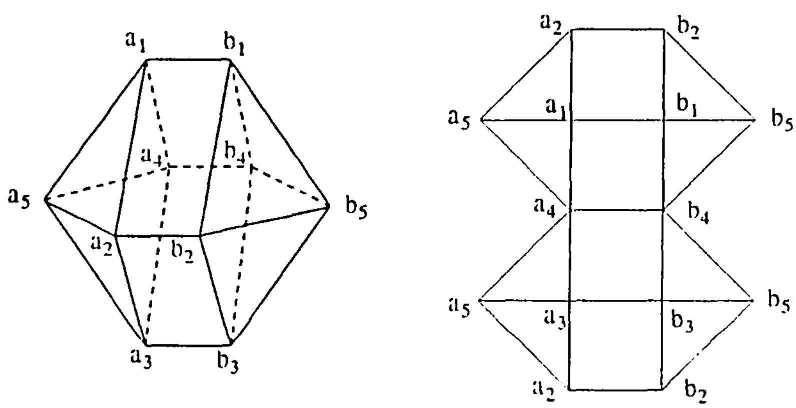


Figure 3

To represent the cross caps at each of the vertices of the polyhedron H we add a pair of small triangles, so that the faces of H are among the shaded patterns as in Figure 4. If we consider the two hexagons and the two squares inscribed in H and the ten pairs of small triangles we get a semiregular surface with boundary, which will be called a partial surface. The boundary of this partial surface is given by the union of the basis of the small triangles. Then, we can obtain a polyhedral model for the closed surface by attaching a disc to each boundary component.

As an alternative method to construct these semiregular surfaces, we consider the dual D of the polyhedron H as in Figure 5.(c). Each face of D corresponds

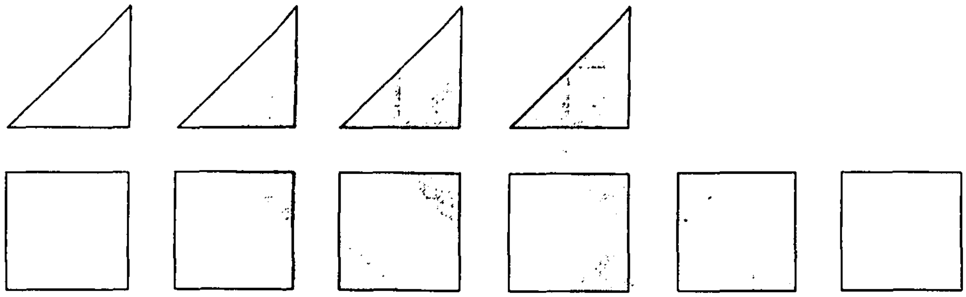
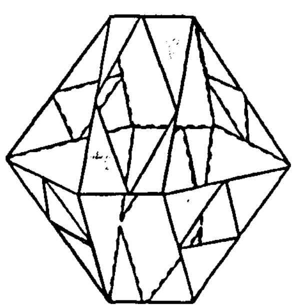
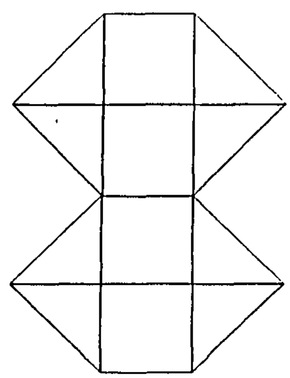


Figure 4

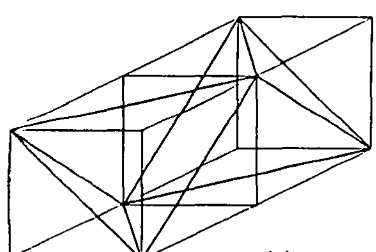
to a vertex of H and we denote that face with the same letter assigned to the vertex. Now, the cross caps will be represented by choosing a diagonal for each of the ten faces of D. For instance, in Figure 5.(a), (b) and (c) we represent the two constructions for a connected sum of two roman surfaces.



(a)



(b)



(c)

Figure 5

From this point of view it is easy to see that enumerating the surfaces with two triple points and ten pinch points is equivalent to the combinatorial problem of enumerating the dual model with particular diagonals on its ten faces.

Given a partial surface S, we introduce a labelling by comparing the assignement of diagonals of S with the corresponding diagonals of the reference model R given in Figure 5.(c). We assign the label 0 if the diagonals agree and 1 otherwise. This defines a correspondence α from the ordered set of faces of D,

$a_1, \ldots, a_5, b_1, \ldots, b_5$, into the set $\{0,1\}$. The 2^{10} labellings of D correspond to 1024 possible partial surfaces.

The group of symmetries G_2 of the reference model R is generated by the following: 1) a mirror symmetry with respect to the plane which separates the a_i faces from the b_i faces; 2) a single axis of 2-fold rotation; 3) a mirror symmetry with respect to the plane that contains the diagonal of the face a_5 and the axis of 2-fold rotation.

For the dual model D, the group of symmetries G_1 includes one more generator: the mirror symmetry with respect to the plane perpendicular to the plane described in 3) above and containing the axis of 2-fold rotation.

Any symmetry of G_2 acting on a labelled dual model permutes the faces but leaves their corresponding labels unchanged. The other symmetries of G_1 permute the faces swapping the labels 0 and 1.

Definition 2.1 Two labellings α_1 and α_2 of the faces of the dual model D are equivalent if there is some $\sigma \in G_1$ wich carries α_1 to α_2 such that for every face f of D we have

$$\alpha_2(f) = \alpha_1(\sigma(f)) = \begin{cases} \alpha_1(f), & \text{if } \sigma \in G_2, \\ 1 - \alpha_1(f), & \text{if } \sigma \notin G_2. \end{cases}$$

Under this equivalence relation the 1024 partial surfaces fall into 89 orbits. In order to simplify the notation, we shall write a labelling in decimal form. For instance, the reference labelling will be written with the number 0.

Proposition 2.2 *The eighty-nine partial surfaces are generated by the labellings:*

0							
1	30	46	74	85	103	117	216
2	31	47	75	86	105	118	217
3	33	62	76	87	106	120	231
6	34	66	77	88	107	121	234
7	35	67	78	89	108	124	235
10	38	68	79	90	109	198	236
11	39	69	80	92	110	199	249
12	42	70	81	93	113	202	330
13	43	71	82	99	114	203	331
14	44	72	83	101	115	204	340
15	45	73	84	102	116	205	341

3 Topological classification

Theorem 3.1 *There are eighty-nine topologically distinct semiregular closed surfaces having two triple points and ten cross caps. Only one of them is orientable.*

Proof. Given a semiregular surface with two triple points and ten cross caps S, we describe a well defined way to associate a labelled dual model.

Looking only at the singular part of S (given by the double point curve, the two triple points and the ten cross caps), we find a graph structure in which five cross caps are adjacent with one of the triple points and the other five cross caps with the other triple point. Then we reserve the vertices $a_1, \dots, a_5$ for one group of cross caps and the rest $b_1, \dots, b_5$ for the other group. Note that the choice of which group we assign the a_i's or the b_i's is not important, because different choices would give equivalent labellings through the first symmetry of the group G_2.

However, the graph structure of the singular part of S is not enough for looking at the relative position of the cross caps in each group. We need more information in order to assign each cross cap to one vertex a_i or b_i. We remove the cross caps from the singular graph and then we take a small "tubular" neighbourhood in S. This neighbourhood is composed by strips that would connect the removed cross caps (Figure 6). This process introduces a new adjacency relation among the cross caps of S: two cross caps are adjacent if they can be connected by one of these strips. Thus, we find that in each group there is a special cross cap, namely the one that is adjacent exactly to the other four cross caps of the same group. We assign to it the vertex a_5 and b_5 respectively. The adjacency for the rest of the cross caps is similar: each cross cap is adjacent exactly to one cross cap of the other group (called the *homologous cross cap*), to the special cross cap a_5 or b_5 of the same group and to two more cross caps of the same group. There is only one cross cap in the same group that is not adjacent to it, called the *opposite cross cap*. Then we will choose the assignement of the vertices so that: a_i is the homologous cross cap of b_i for $i = 1, 2, 3, 4$ and the opposite cross cap of a_1 is a_3, the opposite cross cap of a_2 is a_4 and the same for the b_i.

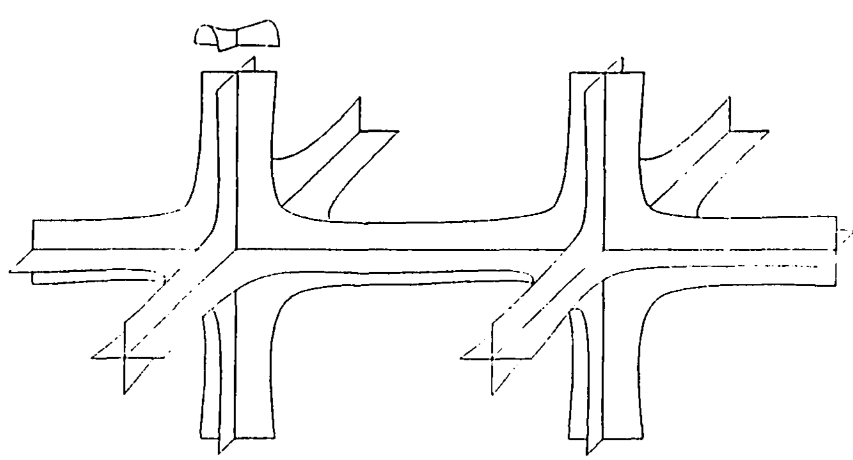


Figure 6

In this way, the position of one of the vertices a_1, a_3, b_1, b_3 determines automatically the position of the other three and the same thing can be said for a_2, a_4, b_2, b_4. Note that there are four possibilities for the position of one of the vertices a_1, a_3, b_1, b_3 and, once this position is fixed there are only two possibilities for the position of one of the vertices a_2, a_4, b_2, b_4. These eight possibilities together with the exchange between a_i and b_i respectively gives sixteen possibilities for locating all the vertices.

Now, to construct the labelled model it is enough to look at the relative position of each cross cap in S with respect to the reference model R (Figure 5). Each one of the sixteen possibilities for locating the vertices gives an equivalent labelled model under the group action.

It is clear from the above that this construction does not depend on the topological type of S and so defines a one-to-one correspondence between the topological classes of semiregular surfaces S and equivalence classes of labelled dual models.

To show that only one surface is orientable, we can assume without loss of generality that we have our basic polyhedron oriented as follows: we fix an orientation for the pair of hexagons and squares (Figure 7.(a), (b)).

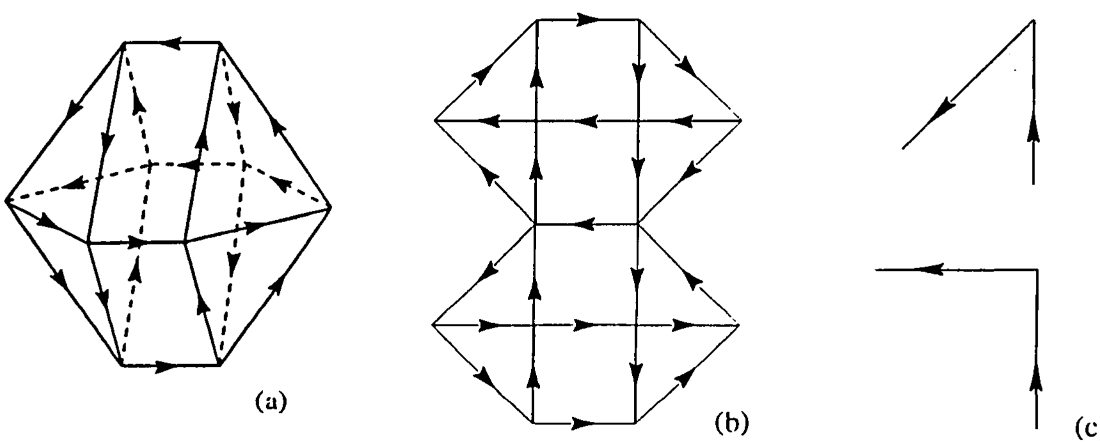


Figure 7

When we add to the polyhedron a pair of small triangles representing the cross cap, the resulting surface can be orientable only if the orientation are as shown in Figure 7.(c).

Using the planar model of the oriented polyhedron and the process above, we abtain only one possible orientable surface (Figure 8), being the one indexed by the number **231** (see Proposition 2.2). There are three more elements in the equivalence class of the element **231** which can be obtained by choosing the other orientations of H. □

The following two lemmas will be useful to decide the topological type of the abstract surfaces.

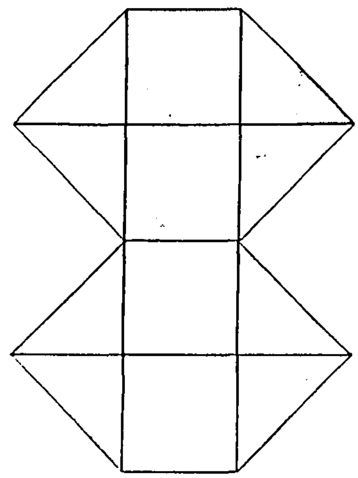


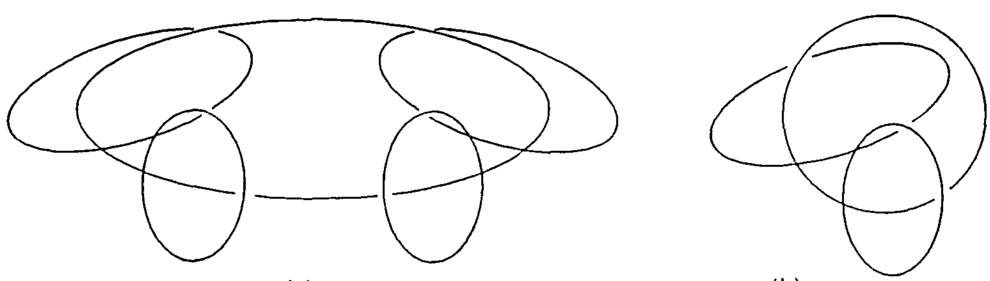
Figure 8

Lemma 3.2 *[5] Let $f : M \to \mathbb{R}^3$ be a semiregular map with $T(f)$ triple points and $C(f)$ cross caps. Then $\chi(f(M)) = \chi(M) + T(f) + C(f)/2$, where χ denotes the Euler characteristic.*

Lemma 3.3 *[7] Let $f : M \to \mathbb{R}^3$ be a semiregular map and let $A \subset M$ be the double point curve of f. Then the number of connected components of $\mathbb{R}^3 \setminus f(M)$ is equal to $2 + \dim_{\mathbb{Z}_2} ker(i_*) \cap ker(f|_A)_*$, where $i_* : H_1(A, \mathbb{Z}_2) \to H_1(M, \mathbb{Z}_2)$ and $(f|_A)_* : H_1(A, \mathbb{Z}_2) \to H_1(f(A), \mathbb{Z}_2)$ are the induced maps in homology.*

Proposition 3.4 *If M is an abstract closed surface associated to a semiregular closed surfaces having two triple points and ten cross caps, then M is homeomorphic to $T^2 \# T^2$ in the orientable case, or $-5 \le \chi(M) \le 0$ otherwise.*

Proof. Note that the semiregular surface $f(M)$ is homotopy equivalent to a wedge of w spheres, therefore $w + 1$ is the number of connected components of $\mathbb{R}^3 \setminus f(M)$. Since in this case $f(A)$ is contractible, by the previous lemma it is enough to show that $\dim_{\mathbb{Z}_2} ker(i_*) \le 5$, where $i_* : H_1(A, \mathbb{Z}_2) \to H_1(M, \mathbb{Z}_2)$ is the induced map.



(a) (b)

Figure 9

The double point curve A looks like the one in Figure 9.(a) and can be obtained as a connected sum of two copies of the corresponding double point curve for a semiregular surface with a single triple point (Figure 9.(b)). Indeed, the whole surface M can be obtained as a connected sum of two surfaces M_1 and M_2. This corresponds to obtaining $f(M)$ as a surgery of $f(M_1)$ and $f(M_2)$ as described in [3]). Let A_1 and A_2 denote the double point curves of M_1 and M_2 respectively. In Figure 10, we have the connected sum of two projective planes.

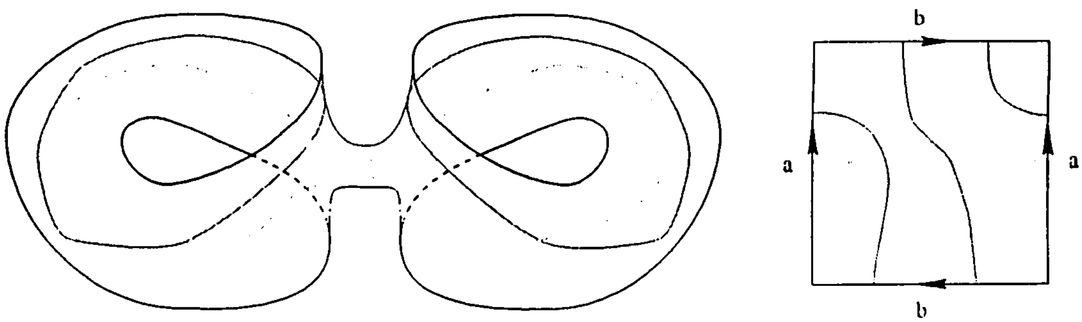


Figure 10

Then the map $i_* : H_1(A, \mathbb{Z}_2) \to H_1(M, \mathbb{Z}_2)$ is the same as $i_* : H_1(A_1, \mathbb{Z}_2) \oplus H_1(A_2, \mathbb{Z}_2) \to H_1(M_1, \mathbb{Z}_2) \oplus H_1(M_2, \mathbb{Z}_2)$. So, to show that $\dim_{\mathbb{Z}_2} ker(i_*) \leq 5$ is the same as to show that $\dim_{\mathbb{Z}_2} ker(j_*) \leq 3$, where $j_* : H_1(A_1, \mathbb{Z}_2) \to H_1(M_1, \mathbb{Z}_2)$ is the induced map for the surface with a single triple point. To see that, first observe that $H_1(A_1, \mathbb{Z}_2)$ has rank 4. Now, we cannot have the four generators homologous to zero, since that would imply to be able to have the Figure 9.(b) in a two-cell, with only three self-intersection points (pre-image of the triple point), which is impossible. Finally, three of the generators of $H_1(A_1, \mathbb{Z}_2)$ can be homologous to zero, as it is the case in the projective plane (Figure 11).

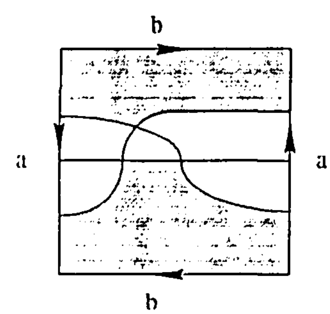
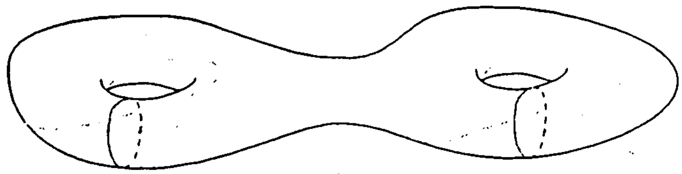


Figure 11

Therefore, the number w of spheres composing the semiregular surface $f(M)$ satisfies $1 \leq w \leq 6$, which means $2 \leq \chi(f(M)) \leq 7$. Now, the result follows from Lemma 3.2. Note that in the orientable case M is homeomorphic to $T^2 \# T^2$, since it is the surgery of two copies of T^2, the only orientable semiregular surface with a single triple point and six cross caps [3] (Figure 12). □



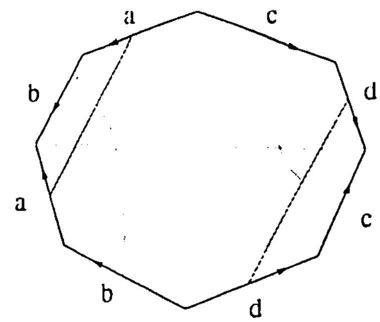


Figure 12

Remark 3.5 Semiregular surfaces with several triple points can be classified in an analogous way. For instance the reader may find 2235 topological types of surfaces with three triple points and 14 cross caps so that the self-intersection set is simply connected. Again, only one of them is orientable and its abstract type is homeomorphic to a three torus.

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