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Antonio C. Asperti and
Barbara C. Valério

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Abstract. Let M^2 be a ruled surface in $\mathbb{S}^3(1)$ or in $\mathbb{H}^3(-1)$, given locally by its natural parametrization $X(s,t)$. We show that such a ruled surface with non-constant curvature, is a Weingarten surface if and only if K and H are constant along s. This is a well known fact for ruled surfaces in $\mathbb{R}^3$.

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1. Introduction

Let $\mathbb{Q}^3(c)$ be a complete, simply connected Riemannian manifold of constant sectional curvature $c = \pm 1$. We can assume that $\mathbb{Q}^3(c)$ is the unity sphere $\mathbb{S}^3(1) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$, if $c = 1$, or the hyperbolic space $\mathbb{H}^3(-1) = \{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 : -x_1^2 + x_2^2 + x_3^2 + x_4^2 = -1, x_1 > 0\}$, if $c = -1$, where $\mathbb{L}^4$ denotes the 4-dimensional Lorentz-Minkowski space, that is,the Euclidean space $\mathbb{R}^4$ endowed with the pseudo-metric $\langle\ ,\ \rangle := -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$.

We say that a 2-dimensional submanifold M^2 of $\mathbb{Q}^3(c)$ is a ruled surface if there exists a foliation of M^2 by geodesics of $\mathbb{Q}^3(c)$. It is known that any ruled surface in $\mathbb{Q}^3(c)$ can be locally parametrized by

$$X(s,t) = exp_{e_0(s)}(t\ e_1(s))$$

where exp is the exponential map of the space forms $\mathbb{Q}^3(c)$ and $e_0(s)$, $e_1(s)$ are suitably chosen curves in $\mathbb{R}^4$ or $\mathbb{L}^4$, see the next section.

A surface is called a Weingarten surface or a W-surface, if the Gaussian curvature K and the mean curvature H satisfy a nontrivial relation $\phi(H, K) = 0$. For any local parameters (s,t) on M^2, this is equivalent to $H_s K_t - H_t K_s = 0$. Surfaces with H or K constant are clearly Weingarten surfaces. Also a helicoidal surface (that is, a surface $S \subset \mathbb{Q}^3(c)$ which is invariant by a helicoidal motions of $\mathbb{Q}^3(c)$), is also a Weingarten surface.

The ruled surfaces in $\mathbb{R}^3$ that are also W-surfaces are well understood and were classified independently by Beltrami [1] and Dini [2] around 1865. A short proof of their result is given in [3]. In the course of the proof it is shown that a non-developable ($K \neq 0$) ruled surface in $\mathbb{R}^3$, given locally by $X(s,t) = e_0(s) + te_1(s)$ where $e_0(s)$ is the striction line, is a Weingarten surface if and only if H and K are constant along the translational parameter s. In this work, we show that this is also true for ruled surfaces in $\mathbb{Q}^3(c)$, $c = \pm 1$.

2. Preliminaries

From now on we consider a ruled surface M^2 in $\mathbb{Q}^3(c)$. Then M^2 can be locally parametrized by

$$\begin{aligned} X(s,t) &= exp_{e_0(s)}(t\ e_1(s)) \\ &= e_0(s)\varphi_0(t) + e_1(s)\varphi_1(t), \end{aligned} \tag{1}$$

where

$$\begin{cases} \varphi_0(t) = cosht, & \varphi_1(t) = sinht, \quad if \quad c = -1, \\ \varphi_0(t) = cost, & \varphi_1(t) = sint, \quad if \quad c = 1, \end{cases}$$

$e_0(s)$ is a curve in M^2 and $e_1(s)$ is a unit vector tangent to M^2 along $e_0(s)$ such that $\langle e_0, e_1\rangle = 0$, $\langle e_1, e_1\rangle = 1$ and $\langle \dot{e}_0, \dot{e}_1\rangle = 0$. Moreover, we may choose the parameter s such that $\langle \dot{e}_1, \dot{e}_1\rangle = 1$, where the dot means the derivative with respect to s.

It is convenient to introduce the following notation

$$\begin{cases} A(s) &= |\ \dot{e}_0\ |^2 - \langle \dot{e}_0, e_1\rangle^2 \\ B(s) &= 1 - c\,\langle \dot{e}_0, e_1\rangle^2 \\ P(s,t) &= A(s)\varphi_0^2(t) + B(s)\varphi_1^2(t) \\ Q(s,t) &= \langle \dot{e}_0, e_1 \wedge \dot{e}_1 \wedge e_0\rangle \\ h(s,t) &= F(s)\varphi_0^2(t) + D(s)\varphi_1^2(t) + (L(s) + G(s))\varphi_0(t)\varphi_1(t), \end{cases} \tag{2}$$

where $F(s)$, $D(s)$, $L(s)$ and $G(s)$ are given by

$$\begin{cases} F(s) &= \langle \ddot{e}_0, e_1 \wedge \dot{e}_0 \wedge e_0\rangle\,, \\ D(s) &= \langle \ddot{e}_1, e_1 \wedge \dot{e}_1 \wedge e_0\rangle\,, \\ G(s) &= \langle \ddot{e}_1, e_1 \wedge \dot{e}_0 \wedge e_0\rangle\,, \\ L(s) &= \langle \ddot{e}_0, e_1 \wedge \dot{e}_1 \wedge e_0\rangle\,, \end{cases} \tag{3}$$

and $u \wedge v \wedge w$ denotes the vector in $\mathbb{R}^4$ or $\mathbb{L}^4$ characterized by $\langle z, u \wedge v \wedge w\rangle = c\ det(z,u,v,w)$. We observe that $Q^2 = cAB$.

Let us take $\tilde{X}_s$ and X_t tangents to M^2 and $\xi = X_t \wedge \frac{\tilde{X}_s}{|\tilde{X}_s|} \wedge X$ normal to M^2, where $\tilde{X}_s = X_s - < X_t, X_s > X_t$. The coefficients of the second fundamental form of the immersion are given by

$$e = 0, \qquad f = -\frac{Q}{|\tilde{X}_s|}, \qquad g = \frac{h + 2 < \dot{e}_0, e_1 > Q}{|\tilde{X}_s|}, \tag{4}$$

and it follows that the Gaussian and mean curvature of the surface are

$$K = c - \frac{Q^2}{P^2}, \qquad H = \frac{(h + 2Q\langle \dot{e}_0, \dot{e}_1\rangle)}{2P^{\frac{3}{2}}}. \tag{5}$$

From the Codazzi's equation we get that

$$\bar{\nabla}_{X_t}\left(fX_t + g\frac{\bar{X}_s}{|X_s|^2}\right) = \bar{\nabla}_{X_s}\left(f\frac{\bar{X}_s}{|X_s|^2}\right) - < X_s, X_t > \bar{\nabla}_{X_t}\left(f\frac{\bar{X}_s}{|X_s|^2}\right).$$

Then

$$\begin{aligned} f_tX_t & + f\bar{\nabla}_{X_t}X_t + g_t\frac{\bar{X}_s}{|X_s|^2} + g\bar{\nabla}_{X_t}\frac{\bar{X}_s}{|X_s|^2} = f_s\frac{\bar{X}_s}{|X_s|^2} + \\ & + f\bar{\nabla}_{X_s}\frac{\bar{X}_s}{|X_s|^2} - < X_s, X_t > \left(f_t\frac{\bar{X}_s}{|X_s|^2} + f\bar{\nabla}_{X_t}\frac{\bar{X}_s}{|X_s|^2}\right) \end{aligned}$$

or equivalently

$$\begin{aligned} f_t & = -f\frac{\langle X_s, X_{st}\rangle}{|\bar{X}_s|^2}, \\ g_t - f_s & = f\frac{1}{|\bar{X}_s|^2}(- < X_{ss}, X_s > + < X_s, X_t > \\ & (< X_{ss}, X_t > + 3 < X_s, X_{st} >)) + g\frac{\langle X_s, X_{st}\rangle}{|\bar{X}_s|^2}. \end{aligned} \tag{6}$$

Since $P(s,t) = |\bar{X}_s|^2$, by (4) it follows that

$$f_t = \frac{QP_t}{2P^{3/2}},$$

$$f_s = -\left(\frac{2\dot{Q}P - P_sQ}{2P^{3/2}}\right),$$

$$g_t = \frac{2h_tP - P_t(h + 2 < \dot{e}_0, e_1 > Q)}{2P^{3/2}}.$$

Using that

$$P_t = 2 < X_{st}, X_s >$$

and

$$P_s = 2(< X_{ss}, X_s > - < X_s, X_t > (< X_{ss}, X_t > + < X_s, X_{st} >)$$

we have that (6) can be written as

$$h_tP - P_t(h + < \dot{e}_0, e_1 > Q) + \dot{Q}P - P_sQ = 0. \tag{7}$$

It follows by (3) and (7) that M^2 verifies

$$\begin{cases} -Q\dot{A} + A(G + L + \dot{Q}) = 0 \\ -Q\dot{B} - B(G + L - \dot{Q}) = 0 \\ -B(F + < \dot{e}_0, e_1 > Q) + A(D + c < \dot{e}_0, e_1 > Q) = 0 \end{cases} \tag{8}$$

3. Main result

Lemma 3.1. *Let $M^2 \subset \mathbb{Q}^3(c)$, $c = \pm 1$, be a ruled connected surface parametrized as in (1) and such that the Gaussian curvature is not constant. If $K_t H_s - K_s H_t = 0$, then $K_s \equiv 0$ and $H_s \equiv 0$.*

Proof. By hypothesis, there exists a point $p \in M^2$ such that $Q(s_0) \neq 0$, where $p = (s_0, t_0)$. Therefore there exists an interval $\tilde{V} \subset \mathbb{R}$ with $s_0 \in \tilde{V}$, such that $Q(s) \neq 0 \; \forall s \in \tilde{V}$, that is, there exists a neighbourhood $V \subset M^2$ of p such that the Gaussian curvature of M^2 it is not constant. In this neighbourhood it holds that

$$\begin{aligned} H_s K_t - H_t K_s = 0 \quad &\Leftrightarrow \tfrac{1}{2} Q P \mathcal{J}(s,t) = 0 \\ &\Leftrightarrow \mathcal{J}(s,t) = 0, \end{aligned}$$

where

$$\mathcal{J}(s,t) = g_1(s)\varphi_1(2t) + g_2(s)\varphi_1(4t) + g_3(s)\varphi_0(2t) + g_4(s)\varphi_0(4t) + g_5(s),$$

with

$$\begin{aligned} g_1(s) &= 2Q(\,(F - cD)(\dot{A} + c\dot{B}) - (A - cB)(4Q < \ddot{e}_0, e_1 > + \dot{F} + c\dot{D})) + \\ &+ \; (-cB\,(5F + cD + 4 < \dot{e}_0, e_1 > Q) + A(F + 5cD + 4 < \dot{e}_0, e_1 > Q))\dot{Q}, \end{aligned}$$

$$\begin{aligned} g_2(s) &= 2Q(\,-c(F - cD)(\dot{A} - c\dot{B}) - (A - cB)(\dot{F} - c\dot{D})) + \\ &+ \; (A - cB)(-cF + D)\dot{Q}, \end{aligned}$$

$$g_3(s) = 4(G + L)(Q(\dot{A} + c\dot{B}) - c(cA + B)\dot{Q}),$$

$$g_4(s) = 2Q((G + L)(\dot{A} - c\dot{B}) - (A - cB)(\dot{G} + \dot{L})) + (A - cB)(G + L)\dot{Q},$$

$$g_5(s) = 2Q(A - cB)(\dot{G} + \dot{L}) + (G + L)(2Q(\dot{A} - c\dot{B}) + 5c(-cA + B)\dot{Q}).$$

Therefore

$$\begin{aligned} H_s K_t - H_t K_s = 0 \quad &\Leftrightarrow \mathcal{J}(s,t) = 0 \\ &\Leftrightarrow g_1(s) = g_2(s) = g_3(s) = g_4(s) = g_5(s) = 0. \end{aligned}$$

It is not difficulty to verify that these equations satisfy the matricial relation $[X].[Y] = [W]$ where

$$[X] = \begin{bmatrix} -2Q(A - cB) & -2cQ(A - cB) & 0 & 2Q(F - cD) & a \\ 2cQ(A - cB) & -2Q(A - cB) & 0 & -2cQ(F - cD) & 0 \\ 0 & 0 & 0 & Q(G + L) & b \\ 0 & 0 & -2Q(A - cB) & 2Q(G + L) & 0 \\ 0 & 0 & 2Q(A - cB) & 2Q(G + L) & 0 \end{bmatrix},$$

$$[Y] = \begin{bmatrix} \dot F \\ \dot D \\ \dot G + \dot L \\ |\dot{e_0}|^2 \\ < e_1, \dot{e_0} > \end{bmatrix} \qquad [W] = \begin{bmatrix} d \\ -\dot Q\ ((A-cB)(D-cF) \\ c\dot Q\ (G+L)(cA+B) \\ -\dot Q\ (G+L)(A-cB) \\ -5c\dot Q(G+L)(-cA+B) \end{bmatrix},$$

with $a = -8Q(< \dot e_0, e_1 > (F-cD) + Q(A-cB))$, $b = -4Q < \dot e_0, e_1 > (G+L)$ and $d = -\dot Q\ (4 < \dot e_0, e_1 > Q(A-cB) + A(F+5cD) - B(5cF+D))$.

Lets $[X](s_j)$ be the matrix $[X]$ evaluated in the point s_j. Suppose that there exists a point $s_1 \in \tilde V$ such that $\det[X](s_1) \neq 0$. Then there exists an interval $\tilde U \subset \tilde V$ with $s_1 \in \tilde U$ such that $\det[X](s) \neq 0$, $\forall s \in \tilde U$. In $\tilde U$, $[X]^{-1}$ is well defined, and then from $[Y] = [X]^{-1}.\ [W]$ it follows that

$$\begin{cases} \dot F & = \frac{cB\dot Q}{<\dot e_0, e_1>} + < \dot e_0, e_1 > \dot Q + \frac{3F\dot Q}{2Q}, \\ \dot D & = \frac{B\dot Q}{<\dot e_0, e_1>} + c < \dot e_0, e_1 > \dot Q + \frac{3D\dot Q}{2Q}, \\ (\dot G + \dot L) & = \frac{3(G+L)\dot Q}{2Q}, \\ |\dot{e_0}|^2 & = \frac{(A-cB)\dot Q}{Q}, \\ < e_1, \dot{e_0} > & = \frac{-cB\dot Q}{2Q<\dot e_0, e_1>}. \end{cases} \tag{9}$$

In order to determine which conditions M^2 have to satisfy such that the equation $K_t H_s - K_s H_t = 0$ is verified, we just have to solve the system (9). The last equation of the system is equivalent to $\frac{B}{Q}$ be constant, but in this case, its follows by (8) that $G + L = 0$, which is a contradiction, since $\det[X] = 256\ Q^5\ \langle \dot e_0, e_1\rangle\ (A + -cB)^3(G+L)^2$. Then $\det[X] \equiv 0$ in $\tilde V$.

Using (8) it is not difficult to verify that $\det[X] \equiv 0 \Leftrightarrow G + L = 0$, and consequently, $\frac{A}{Q}$ and $\frac{B}{Q}$ are constant. By (5) and (2) it follows that $K_s \equiv 0$ in V. Now using this and the equation $K_t H_s - K_s H_t = 0$ it follows easily that $H_s \equiv 0$ in V. □

We have proved the following result

Theorem 3.2. *Let $M^2 \subset \mathbb{Q}^3(c)$ be a ruled connected surface parametrized as in (1) and such that the Gaussian curvature is not constant. Then the following are equivalent:*

i) *M^2 is a Weingarten surface;*

ii) *The mean and Gaussian curvatures are constant along the parameter s.*

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Antonio C. Asperti
Universidade de São Paulo
Instituto de Matemática e Estatística - IME
Departamento de Matemática
Caixa Postal 66281
CEP 05311-970
São Paulo, Brazil
e-mail: `asperti@ime.usp.br`

Barbara C. Valério
Universidade de São Paulo
Escola de Artes, Ciências e Humanidades - EACH
Av. Arlindo Bettio, 1000. Ermelino Matarazzo
CEP 03828-000
São Paulo, Brazil
e-mail: `barbarav@usp.br`

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