PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

# UNIFORM HOMEOMORPHISMS BETWEEN SPHERES INDUCED BY INTERPOLATION METHODS

WILLIAN CORRÊA

(Communicated by Stephen Dilworth)

ABSTRACT. M. Daher [8] showed that if $(X_0, X_1)$ is a regular couple of uniformly convex spaces then the unit spheres of the complex interpolation spaces $X_\theta$ and $X_\eta$ are uniformly homeomorphic for every $0 < \theta, \eta < 1$. We show that this is a rather general phenomenon of the interpolation methods described by the discrete framework of interpolation of [13].

## 1. Introduction

M. Daher showed in [8] that in many natural situations complex interpolation generates uniform homeomorphisms between the unit spheres of the interpolated spaces (this result was obtained by Kalton independently [2, page 216]). More precisely:

**Theorem 1.1** (M. Daher)**.** *Let $(X_0, X_1)$ be a regular compatible couple of uniformly convex Banach spaces. Then for any $\theta, \eta \in (0, 1)$ the spheres of the complex interpolation spaces $X_\theta$ and $X_\eta$ are uniformly homeomorphic.*

Daher's Theorem is not exclusive to the complex method. The reiteration theorem between the real and complex methods [3, Theorem 4.7.2] shows that the interior of real interpolation scales are preserved by complex interpolation (at least up to equivalence of norms). Therefore, if one starts with a regular compatible couple $(X_0, X_1)$ of uniformly convex spaces and takes $0 < \theta_0 < \theta_1 < 1$ and $1 < p_0 < p_1 < \infty$, a uniform homeomorphism between the spheres of $X_{\theta_0,p_0}$ and $X_{\theta_1,p_1}$ may be found by considering complex interpolation between $X_{\theta_0',p_0'}$ and $X_{\theta_1',p_1'}$ with $0 < \theta_0' < \theta_0 < \theta_1 < \theta_1' < 1$ and $0 < p_0' < p_0 < p_1 < p_1' < \infty$. By [13, Example 6.6], we have similar results for the Rademacher, $\gamma$ and (sometimes) the $\alpha$ methods.

General interpolation frameworks serve as unifying umbrellas under which various interpolation methods can be comprehensively described. This perspective is valuable for demonstrating that results obtained in one method can be applied to other ones. For example, using the Cwikel-Kalton-Milman-Rochberg framework of pseudolattices [7], Ivtsan showed in [9] that Stafney's Lemma holds in many interpolation methods.

2020 *Mathematics Subject Classification.* Primary 46B70, 46B80.

The author was supported by São Paulo Research Foundation (FAPESP), grants 2016/25574-8 and 2021/13401-0.

Our goal is to show that one does not need to pass through complex interpolation to obtain Daher's Theorem, but, instead, that it is a general feature embedded into the discrete framework of interpolation of Lindemulder and Lorist presented in [13]. So in principle one does not need a reiteration theorem with the complex method to have uniform homeomorphisms between spheres of interpolation spaces.

The classical example is the couple $(L_{p_0}, L_{p_1})$ where $1 < p_0 \neq p_1 < \infty$, for which the induced uniform homeomorphisms are the Mazur maps [15]. Extending a result of Odell and Schlumprecht [16], Chaatit proved that the unit sphere of a Banach lattice $X$ with a weak unit which does not contain uniform copies of $\ell_\infty^n$ is uniformly homeomorphic to the unit sphere of a Hilbert space [5]. Daher's Theorem may be used to give a proof of Chaatit's result: an extrapolation theorem of Pisier [17] (extended by Kalton in [10]) ensures the existence of a regular compatible couple $(X_0, X_1)$ of uniformly convex spaces such that for some $\theta, \eta \in (0,1)$ we have $X_\theta = X^{(2)}$ (the 2-convexification of $X$) and $X_\eta$ is a Hilbert space. Daher's Theorem then gives a uniform homeomorphism between the spheres of $X^{(2)}$ and $X_\eta$, and all that is left to do is showing that the spheres of $X^{(2)}$ and $X$ are uniformly homeomorphic. See an exposition of these results in [2, Chapter 9]. See also [1] for a finite-dimensional quantitative version of Daher's Theorem with an application to the Approximation Near Neighbor search.

In light of the facts of the previous paragraph, an extension of Daher's Theorem to other interpolation methods might be useful in the uniform classification of spheres of Banach spaces. In particular, it would be interesting to obtain extrapolation results akin to those obtained by Pisier's, that could work for other methods (or, for that matter, to obtain a Kalton Calculus [10] for other methods).

Our proof highlights an often neglected feature of the complex method of interpolation, namely, that associated to a couple $(X_0, X_1)$ one does not have simply an interpolation scale $(X_\theta)_{0<\theta<1}$, but an interpolation family $(X_z)_{0<\mathrm{Re}(z)<1}$. Most of the time the family is overlooked because $X_z = X_{\mathrm{Re}(z)}$ isometrically, but that need not be true for other methods. Our proof of Daher's Theorem attests that the following comment from [6] does not only applies to the complex method:

"*The current theory of interpolation involves the intermediate spaces between* two *given Banach spaces (the 'boundary' spaces). It is our claim that the natural setting for the complex method of interpolation involves a* family *of 'boundary' Banach spaces distributed on the boundary, $\partial D$, of a domain in $\mathbb{C}$.*"

It is worth noting that the Lindemulder-Lorist framework for interpolation of [13] is at the same time more general and more restrictive than the Cwikel-Kalton-Milman-Rochberg framework of [7]: more general because it does not necessarily define interpolation functors; and more restrictive because all the sequence structures admit differentiation, in the language of [7]. Our results may be adapted to the framework of [7].

## 2. The discrete framework for interpolation

For background on interpolation spaces we refer the reader to [3]. The authors of [13] provided a general framework that encompasses many interpolation methods. We describe it now with some adaptations: first, instead of restricting our interpolation parameter $\theta$ to $(0,1)$ we allow it to be any complex number with real part in $(0,1)$. Second, instead of taking a maximum for the norm of intersection spaces we take an $\ell_2$-norm. It is clear that this last change will give the same interpolation spaces up to equivalence of norms, which in turn will not impact the existence of uniform homeomorphisms with respect to the original norms.

Let $X$ be a complex Banach space, and let $\ell^0(\mathbb{Z};X)$ be the space of $X$-valued sequences. A *sequence structure* on $X$ is Banach space $\mathfrak{S}$ contained in $\ell^0(\mathbb{Z};X)$ which is translation invariant and for which we have norm 1 inclusions

$$\ell^1(\mathbb{Z};X) \subset \mathfrak{S} \subset \ell^\infty(\mathbb{Z};X)$$

The couple $\mathcal{X} = [X, \mathfrak{S}]$ is called a *sequentially structured Banach space.* If $a \in \mathbb{C}^*$ we let $\mathfrak{S}(a)$ be the space of all $\vec{x} = (x_k) \in \ell^0(\mathbb{Z}, X)$ such that

$$\|\vec{x}\|_{\mathfrak{S}(a)} = \|(a^k x_k)\|_{\mathfrak{S}} < \infty$$

A couple $\overline{X} = (X_0, X_1)$ of Banach spaces is called *compatible* if we are given a Hausdorff topological vector space $V$ and continuous linear injections $i_0 : X_0 \to V$ and $i_1 : X_1 \to V$. Notice that there are many ways one can see a given couple as compatible, but the terminology means that we have fixed one such choice of $V$, $i_0$ and $i_1$. Usually one replaces $V$ by the so called sum space

$$\Sigma(\overline{X}) = X_0 + X_1 = \{i_0(x_0) + i_1(x_1) : x_0 \in X_0, x_1 \in X_1\}$$

endowed with the complete norm

$$\|x\|_\Sigma = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = i_0(x_0) + i_1(x_1)\}$$

and treat $i_0$ and $i_1$ as inclusions. Eventually we will need to consider the equivalent norm

$$\|x\|_{\Sigma_2} = \inf\{(\|x_0\|^2_{X_0} + \|x_1\|^2_{X_1})^{\frac{1}{2}} : x = i_0(x_0) + i_1(x_1)\}$$

and we denote $X_0 +_2 X_1 = (X_0 + X_1, \|\cdot\|_{\Sigma_2})$.

Suppose we have a compatible couple $(X_0, X_1)$ of Banach spaces such that each $X_j$ is a sequentially structured Banach space, i.e., we have sequence structures $\mathcal{X}_j = [X_j, \mathfrak{S}_j]$. The couple $(\mathcal{X}_0, \mathcal{X}_1)$ is called a *compatible couple of sequentially structured Banach spaces.*

For a compatible couple $(X_0, X_1)$ we denote by $X_0 \cap_2 X_1$ the space $X_0 \cap X_1$ with the norm $\|x\|_{X_0 \cap_2 X_1} = (\|x\|^2_{X_0} + \|x\|^2_{X_1})^{\frac{1}{2}}$. The couple $(X_0, X_1)$ is said to be *regular* if $X_0 \cap X_1$ is dense in $X_0$ and in $X_1$.

Let $\mathbb{S} = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and let $z \in \mathbb{S}$. For $x \in X_0 + X_1$ we let

$$\|x\|_z = \inf \|\vec{x}\|_{\mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z})} = \inf\ (\|\vec{x}\|^2_{\mathfrak{S}_0(e^{-z})} + \|\vec{x}\|^2_{\mathfrak{S}_1(e^{1-z})})^{\frac{1}{2}}$$

where the infimum is over all sequences $\vec{x} = (x_k) \in \mathfrak{S}_0(e^{-z}) \cap \mathfrak{S}_1(e^{1-z})$ such that $x = \sum_{k \in \mathbb{Z}} x_k$ in $X_0 + X_1$. Define

$$(\mathcal{X}_0, \mathcal{X}_1)_z = \mathcal{X}_z = \{x \in X_0 + X_1 : \|x\|_z < \infty\}$$

This definition recovers the spaces $(\mathcal{X}_0, \mathcal{X}_1)_\theta$ of [13] for $0 < \theta < 1$ (with an equivalent norm). One may check that the spaces $(\mathcal{X}_0, \mathcal{X}_1)_z$ satisfy an interpolation estimate like the one of [13, Theorem 5.2], substituting $e^\theta$ by $e^{\mathrm{Re}(z)}$.

## 3. Optimal representations

One important step in the proof of Daher's Theorem for the complex method is the following result on optimal representations [8, Proposition 3]:

**Theorem 3.1.** *Let $(X_0, X_1)$ be a regular compatible couple of reflexive Banach spaces. Let $\theta \in (0,1)$.*

1. *If $x \in S_{X_\theta}$ then there is $g$ in the Calderón space $\mathcal{H}^\infty(X_0, X_1)$ such that $g(\theta) = x$ and $\|g(j+it)\|_{X_j} = 1$ for almost every $t$, $j = 0, 1$. In particular, $\|g\| = \|x\|_\theta$.*
2. *If $X_0$ is strictly convex then $g$ is unique with the property that $g(\theta) = x$ and $\|g\| = \|x\|_\theta$.*

Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a compatible couple of sequentially structured Banach spaces, $z \in \mathbb{S}$, and consider the map

$$\Sigma : \mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z}) \to X_0 + X_1$$

given by $\Sigma(\vec{x}) = \sum_{k\in\mathbb{Z}} x_k$. Following [13, Remark 3.2], it is possible to prove that $\Sigma$ is well-defined and bounded. It follows directly from the definitions that we have an induced isometry

$$\tilde{\Sigma} : \mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z}) /_{\ker\ \Sigma} \to (\mathcal{X}_0, \mathcal{X}_1)_z$$

**Definition 3.2.** A sequentially structured Banach space $\mathcal{X} = [X, \mathfrak{S}]$ will be called *reflexive* (resp. *strictly convex*, *uniformly convex*) if so is $\mathfrak{S}$. A compatible couple $(\mathcal{X}_0, \mathcal{X}_1)$ of sequentially structured Banach spaces is called *regular* if so is $(X_0, X_1)$.

Notice that if $\mathcal{X} = [X, \mathfrak{S}]$ is a sequentially structured Banach space then $X$ is a subspace of $\mathfrak{S}$, and therefore if $\mathcal{X}$ is reflexive (resp. strictly convex, uniformly convex) then so is $X$. We at once get the following result:

**Theorem 3.3.** *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a compatible couple of sequentially structured Banach spaces and let $z \in \mathbb{S}$.*

1. *If $\mathcal{X}_0$ and $\mathcal{X}_1$ are reflexive then given $x \in S_{\mathcal{X}_z}$ there is $\vec{x} \in \mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z})$ of norm 1 such that $\Sigma(\vec{x}) = x$.*
2. *If $\mathcal{X}_0$ and $\mathcal{X}_1$ are strictly convex then the previous element $\vec{x}$ is unique.*

To continue, Daher uses properties of the function $g$ of Theorem 3.1. We shall therefore use the complex description of the discrete framework of interpolation. If $X$ is a Banach space let $\mathcal{H}(\mathbb{S}, X)$ be the space of analytic $X$-valued functions on $\mathbb{S}$. We will say that $f \in \mathcal{H}(\mathbb{S}, X)$ is $2\pi$-periodic if $f(z + 2\pi i) = f(z)$ for every $z \in \mathbb{S}$ and let $f_z(t) = f(z + it)$ be defined for $t \in \mathbb{R}$..

Let us consider the space $\mathcal{H}_\pi(\mathbb{S}, X)$ of $2\pi$-periodic functions in $\mathcal{H}(\mathbb{S}, X)$. For those functions it makes sense to take Fourier coefficients:

$$\hat{f}_z(k) = \frac{1}{2\pi} \int_0^{2\pi} f(z + it) e^{-ikt} dt$$

for $k \in \mathbb{Z}$. According to [13, Lemma 4.1], the sequence $(e^{-ks}\hat{f}_s(k))_{k\in\mathbb{Z}}$ is independent of $s \in (0,1)$. Similarly, we have:

**Lemma 3.4.** *If $z, w \in \mathbb{S}$ then the sequences $(e^{-kz}\hat{f}_z(k))_{k\in\mathbb{Z}}$ and $(e^{-kw}\hat{f}_w(k))_{k\in\mathbb{Z}}$ are equal, i.e., $(e^{-kz}\hat{f}_z(k))_{k\in\mathbb{Z}}$ is independent of $z \in \mathbb{S}$.*

The lemma above implies that the following definition is independent of $z_0 \in \mathbb{S}$. Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a compatible couple of sequentially structured Banach spaces and define $\mathcal{H}^2_\pi(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1)$ as the space of all $f \in \mathcal{H}_\pi(\mathbb{S}, X_0 + X_1)$ such that

$$\|f\|_{\mathcal{H}^2_\pi} := (\|\hat{f}_{z_0}\|^2_{\mathfrak{S}_0(e^{-z_0})} + \|\hat{f}_{z_0}\|^2_{\mathfrak{S}_1(e^{1-z_0})})^{\frac{1}{2}} < \infty$$

As in [13, Lemma 4.2], the map $f \mapsto \hat{f}_{z_0}$ is an isometric isomorphism from $\mathcal{H}^2_\pi(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1)$ onto $\mathfrak{S}_0(e^{-z_0}) \cap_2 \mathfrak{S}_1(e^{1-z_0})$. Its inverse is given by

$$\vec{x} \mapsto f(z) = \sum_{k \in \mathbb{Z}} e^{k(z-z_0)} x_k$$

It follows at once that

$$\|x\|_{z_0} = \inf\{\|f\|_{\mathcal{H}^2_\pi} : f \in \mathcal{H}^2_\pi(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1), f(z_0) = x\} \tag{3.1}$$

Let $\delta_{z_0} : \mathcal{H}^2_\pi(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1) \to X_0 + X_1$ be given by $\delta_{z_0}(f) = f(z_0)$.

**Lemma 3.5.** *$\delta_{z_0}$ is bounded and $\mathcal{X}_{z_0} = \mathcal{H}^2_\pi(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1) \big/ \ker\, \delta_{z_0}$ isometrically.*

*Proof.* The first part follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^2_\pi(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1) & \longrightarrow & \mathfrak{S}_0(e^{-z_0}) \cap_2 \mathbb{S}_1(e^{1-z_0}) \\ \big\downarrow{\scriptstyle \delta_{z_0}} & & \big\downarrow{\scriptstyle \Sigma} \\ X_0 + X_1 & = = = & X_0 + X_1 \end{array}$$

where the horizontal arrow is the isometry described above. The second part of the result is simply (3.1). □

We get at once:

**Theorem 3.6.** *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a compatible couple of sequentially structured Banach spaces and $z_0 \in \mathbb{S}$.*

1. *If $\mathcal{X}_0$ and $\mathcal{X}_1$ are reflexive, then given $x \in S_{\mathcal{X}_{z_0}}$ there is $f \in \mathcal{H}^2_\pi(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1)$ of norm 1 such that $f(z_0) = x$.*
2. *If $\mathcal{X}_0$ and $\mathcal{X}_1$ are strictly convex then the previous element $f$ is unique, and we denote it by $\Gamma_{z_0}(x)$ and call it* the optimal function associated to *$x$.*

Our goal is to show that, for any $z \in \mathbb{S}$, $x \mapsto \Gamma_{z_0}(x)(z)$ is a uniform homeomorphism between the unit spheres of $\mathcal{X}_{z_0}$ and $\mathcal{X}_z$. For that we need to show that $\|\Gamma_{z_0}(x)(z)\|_{\mathcal{X}_z} = 1$, what will be done via duality.

## 4. Duality

In [13] the dual of the interpolation spaces is described up to equivalence of norms. We will need an isometric description. With that purpose in mind, we will have consider more properties of sequence structures. A sequentially structured Banach space $\mathcal{X} = [X, \mathfrak{S}]$ is called *reflection invariant* if for every $(x_k)_{k \in \mathbb{Z}} \in \mathfrak{S}$ we have $\|(x_k)_{k \in \mathbb{Z}}\|_{\mathfrak{S}} = \|(x_{-k})_{k \in \mathbb{Z}}\|_{\mathfrak{S}}$. If $\lim_{n \to \infty} C_n \vec{x} = \vec{x}$ for every $\vec{x} \in \mathfrak{S}$, where $C_n$ is the Cesàro operator

$$C_n \vec{x} = \frac{1}{n+1} \sum_{m=0}^{n} (\cdots, 0, x_{-m}, \cdots, x_m, 0, \cdots)$$

and $\sup_n \|C_n\vec{x}\|_{\mathfrak{S}} \leq \|\vec{x}\|_{\mathfrak{S}}$ then $\mathcal{X}$ is called a *$c_0$-sequentially structured Banach space*. If $\mathcal{X}$ is a $c_0$-sequentially structured Banach space and $a \in \mathbb{C}$ then $\mathcal{X}(a)^* = \mathcal{X}^*(a^{-1})$ isometrically (see the commentary after [13, Lemma 3.13]).

**Definition 4.1.** A regular couple $(\mathcal{X}_0, \mathcal{X}_1)$ of $c_0$-sequentially structured Banach spaces which are reflection invariant will be called a *star couple*. A star couple of reflexive sequentially structured Banach spaces is called a *reflexive star couple*.

Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a star couple. Recall that we have

$$\mathcal{X}_z = \mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z}) / \ker\ \Sigma$$

Notice that $\mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z})$ is a closed subspace of $\mathfrak{S}_0(e^{-z}) \oplus_2 \mathfrak{S}_1(e^{1-z})$. Under this identification,

$$\ker\ \Sigma = \{(\vec{x}, \vec{x}) \in \mathfrak{S}_0(e^{-z}) \oplus_2 \mathfrak{S}_1(e^{1-z}) : \Sigma\vec{x} = 0\}$$

and from the continuity of $\Sigma$ on $\mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z})$ we have that $\ker \Sigma$ is closed in $\mathfrak{S}_0(e^{-z}) \oplus_2 \mathfrak{S}_1(e^{1-z})$. It follows that

$$\mathcal{X}_z^* = (\ker\ \Sigma)^\perp \Big/ (\mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z}))^\perp$$

isometrically, where the annihilators are taken inside $(\mathfrak{S}_0(e^{-z}) \oplus_2 \mathfrak{S}_1(e^{1-z}))^*$. We have:

$$(\mathfrak{S}_0(e^{-z}) \oplus_2 \mathfrak{S}_1(e^{1-z}))^* = \mathfrak{S}_0^*(e^{z}) \oplus_2 \mathfrak{S}_1^*(e^{z-1})$$

and by reflection invariance

$$\mathfrak{S}_0^*(e^{z}) \oplus_2 \mathfrak{S}_1^*(e^{z-1}) = \mathfrak{S}_0^*(e^{-z}) \oplus_2 \mathfrak{S}_1^*(e^{1-z})$$

The sum operator $\Sigma$ does not care about reflection, therefore $(\ker\ \Sigma)^\perp$ is still the same.

**Lemma 4.2.** $(\ker\ \Sigma)^\perp = \{(\vec{x^*}, \vec{y^*}) : \exists\, x^* \in X_0^* + X_1^* : x_k^* + y_k^* = x^*\ \forall k\}$.

*Proof.* Let $(\vec{x^*}, \vec{y^*}) \in (\ker\ \Sigma)^\perp$, and let $x \in X_0 \cap X_1$. Let $j \in \mathbb{Z}^*$. Consider the sequence $\vec{z} = (z_k)_{k\in\mathbb{Z}}$ such that $z_0 = x$, $z_{-j} = -x$, and $z_k = 0$ otherwise. Then $\vec{z} \in \ker\ \Sigma$, and $0 = (\vec{x^*}, \vec{y^*})(\vec{z}) = (x_0^* + y_0^* - x_j^* - y_j^*)(x)$. Since $X_0 \cap X_1$ is dense in $X_0$ and $X_1$, it follows that $x_0^* + y_0^* = x_j^* + y_j^*$ for every $j$. The other inclusion is clear. □

**Lemma 4.3.** $(\mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z}))^\perp = \{(\vec{x^*}, -\vec{x^*}) : \vec{x^*} \in \mathfrak{S}_0^*(e^{-z})^* \cap \mathfrak{S}_1^*(e^{1-z})\}$.

*Proof.* Let $(\vec{x^*}, \vec{y^*}) \in (\mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z}))^\perp$. Given any $x \in X_0 \cap X_1$ and $j \in \mathbb{Z}$, let $\vec{z}$ be the sequence such that $z_{-j} = x$, and $z_k = 0$ otherwise. Then $0 = (\vec{x^*}, \vec{y^*})(\vec{z}) = (x_j^* + y_j^*)(x)$. It follows that $\vec{x^*} = -\vec{y^*}$. The other inclusion is clear. □

Motivated by the previous results, for $z \in \mathbb{S}$ we let

$$\mathcal{X}_z^m = \{x \in X_0 + X_1 : \|(\cdots, x, x, x, \cdots)\|_{\mathfrak{S}_0(e^{-z}) +_2 \mathfrak{S}_1(e^{1-z})} < \infty\}$$

We notice that $\mathcal{X}_z^m$ already appears in [13, Section 3.3] for $z \in (0, 1)$, with an equivalent norm.

**Theorem 4.4.** *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a star couple. Then $\mathcal{X}_z^* = (\mathcal{X}^*)_z^m$ isometrically.*

*Proof.* We have seen that

$$\mathcal{X}_z^* \quad = \quad (\ker\ \Sigma)^\perp \Big/ (\mathfrak{S}_0(e^{-z}) \cap_2 \mathfrak{S}_1(e^{1-z}))^\perp$$

By Lemmas 4.2 and 4.3, $\mathcal{X}_z^*$ is the quotient of

$$\{(\vec{x^*}, \vec{y^*}) : \exists\, x^* \in X_0^* + X_1^* : x_k^* + y_k^* = x^*\ \forall k\} \subset \mathfrak{S}_0^*(e^{-z}) \oplus_2 \mathfrak{S}_1^*(e^{1-z})$$

by

$$\{(\vec{x^*}, -\vec{x^*}) : \vec{x^*} \in \mathfrak{S}_0(e^{-z})^* \cap \mathfrak{S}_1(e^{1-z})^*\}$$

The pairs $(\vec{x^*}, \vec{y^*})$ and $(\vec{r^*}, \vec{t^*})$ are equivalent if and only if there is $x^* \in X_0^* + X_1^*$ such that $x^* = x_k^* + y_k^* = r_k^* + t_k^*$ for every $k$. Therefore

$$\|x^*\|_{\mathcal{X}_z^*} = \inf(\|\vec{x^*}\|^2_{\mathfrak{S}_0^*(e^{-z})} + \|\vec{y^*}\|^2_{\mathfrak{S}_1^*(e^{1-z})})^{\frac{1}{2}}$$

where the infimum is over all $\vec{x^*}, \vec{y^*}$ in the indicated spaces such that $x_k^* + y_k^* = x^*$ for every $k$. That is the norm of $x^*$ in $(\mathcal{X}^*)_z^m$. □

We at once get existence of an optimal representation for elements of $\mathcal{X}_z^*$:

**Lemma 4.5.** *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a reflexive star couple and let $z \in \mathbb{S}$. Then every element $x$ of $\mathcal{X}_z^*$ admits a representation $(\vec{x^*}, \vec{y^*})$ such that $x^* = x_k^* + y_k^*$ for every $k$ and*

$$\|x^*\|_{\mathcal{X}_z^*} = (\|\vec{x^*}\|^2_{\mathfrak{S}_0^*(e^{-z})} + \|\vec{y^*}\|^2_{\mathfrak{S}_1^*(e^{1-z})})^{\frac{1}{2}}$$

## 5. Daher's Extended Theorem

To prove Daher's Extended Theorem, we will use optimal representations to associate to an element $x^* \in \mathcal{X}_{z_0}^*$ an analytic function $g$ on the strip such that $g(z_0) = x^*$ and $\|g(z)\|_{\mathcal{X}_z^*} \le \|x^*\|_{\mathcal{X}_{z_0}^*}$ for every $z \in \mathbb{S}$. That is, $g$ will play the role of the optimal function of Theorem 3.6.

**Lemma 5.1.** *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a reflexive star couple. Let $x^* \in \mathcal{X}_{z_0}^*$ and take $\vec{x^*}, \vec{y^*}$ such that $x^* = x_k^* + y_k^*$ for every $k$. Define $Ax^*$ by $(Ax^*)_k = x_k^* - x_{k-1}^* = -(y_k^* - y_{k-1}^*) \in X_0^* \cap X_1^*$. Then $\Sigma A x^* = x^*$ and $Ax^* \in \mathfrak{S}_0^*(e^{-z_0}) \cap \mathfrak{S}_1^*(e^{1-z_0})$.*

*Proof.* It is similar to the case $z_0 \in (0,1)$, which is in the proof of [13, Theorem 3.12]. □

**Lemma 5.2.** *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a reflexive star couple. Let $x^* \in \mathcal{X}_{z_0}^*$ and take $\vec{x^*}, \vec{y^*}$ such that $x^* = x_k^* + y_k^*$ for every $k$ and $\|x^*\|_{\mathcal{X}_{z_0}^*} = (\|\vec{x^*}\|^2_{\mathfrak{S}_0^*(e^{-z_0})} + \|\vec{y^*}\|^2_{\mathfrak{S}_1^*(e^{1-z_0})})^{\frac{1}{2}}$. Define $Ax^*$ as in Lemma 5.1 and*

$$g(z) = \sum_{k\in\mathbb{Z}} e^{k(z-z_0)} (Ax^*)_k$$

*Then for every $z \in \mathbb{S}$ we have*

$$\|g(z)\|_{\mathcal{X}_z^*} \le \|x^*\|_{\mathcal{X}_{z_0}^*}$$

*Proof.* We have

$$g(z) = \sum_{k\in\mathbb{Z}} e^{k(z-z_0)}(Ax^*)_k = \sum_{k\in\mathbb{Z}} e^{k(z-z_0)}(x_k^* - x_{k-1}^*) = -\sum_{k\in\mathbb{Z}} e^{k(z-z_0)}(y_k^* - y_{k-1}^*)$$

Let $a_n = \sum\limits_{k=-\infty}^{n} e^{k(z-z_0)}(x_k^* - x_{k-1}^*)$ and $b_n = \sum\limits_{k=n+1}^{\infty} e^{k(z-z_0)}(x_k^* - x_{k-1}^*)$. Then $a_n + b_n = g(z)$ for every $n$ and

$$\begin{aligned} \|\vec{a}\|_{\mathfrak{S}_0^*(e^{-z})} &= \|(\sum_{k=-\infty}^{n} e^{-kz_0}(x_k^* - x_{k-1}^*))_n\|_{\mathfrak{S}_0^*} \\ &= \|(\sum_{k=-\infty}^{n} (x_k^* - x_{k-1}^*))_n\|_{\mathfrak{S}_0^*(e^{-z_0})} \\ &= \|(x_n^*)_n\|_{\mathfrak{S}_0^*(e^{-z_0})} \end{aligned}$$

Similarly, $\|\vec{b}\|_{\mathfrak{S}_1^*(e^{1-z})} = \|(y_n^*)_n\|_{\mathfrak{S}_1^*(e^{1-z_0})}$. Therefore $\|g(z)\|_{\mathcal{X}_z^*} \leq \|x\|_{\mathcal{X}_{z_0}^*}$. □

**Definition 5.3.** A *Daher couple* is a star couple of uniformly convex sequentially structured Banach spaces.

**Theorem 5.4.** *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a Daher couple and $z_0 \in \mathbb{S}$. Let $\Gamma_{z_0} : \mathcal{X}_{z_0} \to \mathcal{H}_\pi^2(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1)$ be the application that sends $x$ to its optimal function (see Theorem 3.6). Then for every $z \in \mathbb{S}$ we have $\|\Gamma_{z_0}(x)(z)\|_z = \|x\|_{z_0}$.*

*Proof.* Let $x \in \mathcal{X}_{z_0}$. Take $x^* \in \mathcal{X}_{z_0}^* = (\mathcal{X}^*)_{z_0}^m$ such that $\|x^*\|_{(\mathcal{X}^*)_{z_0}^m} = \|x\|_{\mathcal{X}_{z_0}} = x^*(x) = 1$ and an optimal representation $\vec{x^*}, \vec{y^*}$ of $x^*$, that is, $x^* = x_k^* + y_k^*$ for every $k$ and

$$1 = \|x^*\|_{\mathcal{X}_{z_0}^*} = (\|\vec{x^*}\|^2_{\mathfrak{S}_0^*(e^{-z_0})} + \|\vec{y^*}\|^2_{\mathfrak{S}_1^*(e^{1-z_0})})^{\frac{1}{2}}$$

Let $Ax^*$ be as in Lemma 5.1 and $g$ be as in Lemma 5.2. Consider the function $F : \mathbb{S} \to \mathbb{C}$ given by

$$F(z) = \langle g(z), \Gamma_{z_0}(x)(z) \rangle$$

Since the series that defines $F(z)$ converges uniformly on compact subsets of $\mathbb{S}$, $F$ is analytic (see the proof of [13, Lemma 4.2]). Also, $F(z_0) = 1$ and for every $z \in \mathbb{S}$

$$\begin{aligned} |F(z)| &= |\langle g(z), \Gamma_{z_0}(x)(z) \rangle| \\ &\leq \|g(z)\|_{\mathcal{X}_{z_0}^*} \|\Gamma_{z_0}(x)(z)\|_{\mathcal{X}_{z_0}} \\ &\leq 1 \end{aligned}$$

because of Lemma 5.2. By the Maximum Modulus Principle, $F \equiv 1$. The result follows. □

**Observation:** For the previous result we only needed reflexivity and strict convexity, not uniform convexity.

Our goal now is to show that the map $\Gamma_z$ from Theorem 5.4 is uniformly continuous, and that the induced map $\Gamma_{z,w} : x \mapsto \Gamma_z(x)(w)$ is a uniform homeomorphism between the spheres of $\mathcal{X}_z$ and $\mathcal{X}_w$ for $z, w \in \mathbb{S}$.

**Theorem 5.5.** *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a Daher couple and let $z \in \mathbb{S}$. Let $\Gamma_z : S_{\mathcal{X}_z} \to \mathcal{H}_\pi^2(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1)$ be the application that sends $x$ to its optimal function. Then $\Gamma_z$ is uniformly continuous.*

*Proof.* Let $\delta$ be the modulus of convexity of $\mathcal{H}^2_\pi(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1)$. For $x, y \in S_{\mathcal{X}_z}$ let $f_x = \Gamma_z(x)$ and $f_y = \Gamma_z(y)$. It follows that

$$\left\| \frac{f_x + f_y}{2} \right\|_{\mathcal{H}^2_\pi(\mathbb{S},\mathcal{X}_0,\mathcal{X}_1)} \leq 1 - \delta(\|f_x - f_y\|_{\mathcal{H}^2_\pi(\mathbb{S},\mathcal{X}_0,\mathcal{X}_1)})$$

So

$$\left\| \frac{x + y}{2} \right\|_{\mathcal{X}_{z_0}} \leq 1 - \delta(\|f_x - f_y\|_{\mathcal{H}^2_\pi(\mathbb{S},\mathcal{X}_0,\mathcal{X}_1)})$$

Therefore

$$\begin{aligned} \left\| \frac{x - y}{2} \right\|_{\mathcal{X}_{z_0}} &\geq 1 - \left\| \frac{x + y}{2} \right\|_{\mathcal{X}_{z_0}} \\ &\geq 1 - \Big(1 - \delta(\|f_x - f_y\|_{\mathcal{H}^2_\pi(\mathbb{S},\mathcal{X}_0,\mathcal{X}_1)})\Big) \\ &= \delta(\|f_x - f_y\|_{\mathcal{H}^2_\pi(\mathbb{S},\mathcal{X}_0,\mathcal{X}_1)}) \end{aligned}$$

If we let $\beta(t) = \sup\{u \geq 0 : \delta(u) \leq t\}$ then

$$\|f_x - f_y\|_{\mathcal{H}^2_\pi(\mathbb{S},\mathcal{X}_0,\mathcal{X}_1)} \leq \beta\Big( \|x - y\|_{\mathcal{X}_{z_0}} /2\Big)$$

Since $\lim_{t\to 0} \beta(t) = 0$, it follows that $\Gamma_z$ is uniformly continuous. □

**Theorem 5.6** (Daher's Extended Theorem)**.** *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a Daher couple and let $z, w \in \mathbb{S}$. Let $\Gamma_z : S_{\mathcal{X}_z} \to \mathcal{H}^2_\pi(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1)$ be the application that sends $x$ to its optimal function. Then $\Gamma_{z,w} : S_{\mathcal{X}_z} \to S_{\mathcal{X}_w}$ given by $\Gamma_{z,w}(x) = \Gamma_z(x)(w)$ is a uniform homeomorphism.*

*Proof.* The map $\Gamma_{z,w}$ is surjective because its inverse is $\Gamma_{w,z}$. Since

$$\|\Gamma_{z,w}(x) - \Gamma_{z,w}(y)\|_{\mathcal{X}_w} = \|\Gamma_z(x)(w) - \Gamma_z(y)(w)\|_{\mathcal{X}_w} \leq \|\Gamma_z(x) - \Gamma_z(y)\|_{\mathcal{H}^2_\pi(\mathbb{S},\mathcal{X}_0,\mathcal{X}_1)}$$

it follows from Theorem 5.5 that $\Gamma_{z,w}$ is uniformly continuous. Similarly, $\Gamma_{w,z}$ is uniformly continuous. □

Notice that the interpolation space $\mathcal{X}_z$ is a quotient of $\mathcal{H}^2_\pi(\mathbb{S}, \mathcal{X}_0, \mathcal{X}_1)$ for every $z \in \mathbb{S}$, and therefore we are using the same space of functions to define all interpolation spaces. In turn, in Daher's formulation the complex interpolation space $X_z$ is a quotient of a distinct space $\mathcal{F}_z$ for each $z \in \mathbb{S}$. The use of different spaces of functions is responsible for the following in Daher's work: in principle one may only bound the modulus of continuity of the maps $\Gamma_{z,w}$ for $0 < a < \text{Re}(z), \text{Re}(w) < b < 1$ (see [2, Proposition 9.13]). Since in the discrete formulation all interpolation spaces are defined through the same function space, we get:

**Proposition 5.7.** Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a Daher couple. Then there is a map $\gamma$ satisfying $\lim_{\epsilon\to 0} \gamma(\epsilon) = 0$ such that the modulus of continuity of $\Gamma_{z,w}$ is bounded by $\gamma$ for every $z, w \in \mathbb{S}$.

Of course, the previous result is isometric in nature, and therefore depends on the particular representation of the discrete framework of interpolation.

Through the rest of this section we let $(X_0, X_1)$ be a regular compatible couple of uniformly convex Banach spaces.

5.1. **Real method.** Let $p_0, p_1 \in (1, \infty)$ and consider the sequence structures $\mathfrak{S}_j = \ell^{p_j}(\mathbb{Z}, X_j)$. The corresponding couple of sequentially structured Banach spaces $(\mathcal{X}_0, \mathcal{X}_1)$ is Daher, and therefore Daher's Extended Theorem applies. We have that $(\mathcal{X}_0, \mathcal{X}_1)_z$ equals the interpolation space $s(p_0, -\mathrm{Re}(z), X_0; p_1, 1-\mathrm{Re}(z), X_1)$ given by the Lions-Peetre mean method of [14] with equivalence of norms. Therefore, $(\mathcal{X}_0, \mathcal{X}_1)_z$ is a renorming of the real interpolation space $(X_0, X_1)_{\mathrm{Re}(z),p}$ given either by the $J$- or the $K$-method, where $\frac{1}{p} = \frac{1-\mathrm{Re}(z)}{p_0} + \frac{\mathrm{Re}(z)}{p_1}$.

5.2. **Complex method.** Since we are dealing with reflexive spaces, the lower and upper complex methods agree, so we only need to describe the lower method. Let $p_0, p_1 \in (1, \infty)$ and consider the sequence structures $\mathfrak{S}_j = \hat{L}^{p_j}(\mathbb{T}, X_j)$ of Fourier coefficients $\hat{f} = (\hat{f}(k))_{k\in\mathbb{Z}}$ of functions in $L^{p_j}(\mathbb{T}, X_j)$, with $\|\hat{f}\|_{\hat{L}^{p_j}(\mathbb{T},X_j)} = (2\pi)^{\frac{1}{p}}\|f\|_{L^{p_j}(\mathbb{T},X_j)}$. Again, $(\mathcal{X}_0, \mathcal{X}_1)$ is a Daher couple. We have that $(\mathcal{X}_0, \mathcal{X}_1)_z$ equals $(X_0, X_1)_{\mathrm{Re}(z)}$, the complex interpolation space of Calderón [4], with equivalence of norms. We therefore recover Daher's theorem.

5.3. **Rademacher and $\gamma$ methods.** We now describe the Rademacher method of [11]. Fix $p \in (1, \infty)$ and let $(\varepsilon_k)_{k\in\mathbb{Z}}$ be a sequence of independent Rademacher random variables on a probability space $(\Omega, P)$. Consider the sequence structures $\mathfrak{S}_j = \epsilon^p(\mathbb{Z}, X_j)$ of all $\vec{x} \in \ell^0(\mathbb{Z}, X_j)$ given by

$$\|\vec{x}\|_{\epsilon^p(\mathbb{Z},X_j)} = \|\sum_{k\in\mathbb{Z}} \varepsilon_k x_k\|_{L^p(\Omega,X_j)} < \infty$$

The couple $(\mathcal{X}_0, \mathcal{X}_1)$ is Daher, and the interpolation space $(\mathcal{X}_0, \mathcal{X}_1)_z$ is the Rademacher interpolation space $(X_0, X_1)_{z,\epsilon}$.

The $\gamma$ interpolation method is defined similarly to the Rademacher method, but instead of Rademacher variables we take Gaussian ones, and Daher's Extended Theorem also applies. It is worth noting that the Rademacher and the $\gamma$ methods agree for spaces of finite cotype ([12], see also the comments after Definition 10.17 and Proposition 10.39 of [18]).

5.4. **$\alpha$-method.** The $\alpha$-method is defined in [12] through the notion of Euclidean structures. Whether the $\alpha$-method gives Daher couples or not depends on the Euclidean structure being considered. For example, the Gaussian Euclidean structure gives the $\gamma$-method, and therefore Daher's Extended Theorem applies. However, if we take the operator norm Euclidean structure on $\ell_2$ uniform convexity is lost.

As mentioned in the introduction, all the methods above satisfy a reiteration theorem with the complex method, as follows: if $(X_0, X_1)$ is a compatible couple of Banach spaces let us denote by $[X_0, X_1]_\theta$ its complex interpolation space at $\theta \in (0, 1)$. According to [13, Example 6.6], if $(\mathcal{X}_0, \mathcal{X}_1)$ is a star couple such that there is a constant $C > 0$ for which $\left\|(e^{iks}x_k)\right\|_{\mathfrak{S}_j} \leq C \left\|\vec{x}\right\|_{\mathfrak{S}_j}$ for every $\vec{x} \in \mathfrak{S}_j$, $j = 0, 1$ and $s \in \mathbb{R}$, then we have

$$[(\mathcal{X}_0, \mathcal{X}_1)_{\theta_0}, (\mathcal{X}_0, \mathcal{X}_1)_{\theta_1}]_\theta = (\mathcal{X}_0, \mathcal{X}_1)_w$$

with equivalence of norms for every $0 < \theta_0 < \theta_1 < 1$ and $\theta \in (0, 1)$ with $w = (1-\theta)\theta_0 + \theta\theta_1$. As such, we could have already obtained uniform homeomorphisms between the unit spheres of interpolation spaces generated by such methods by passing through complex interpolation. Examples that do not satisfy the hypothesis of [13, Example 6.6] may be built using a variation of James' space.

## References


1. A. Andoni, A. Naor, A. Nikolov, I. Razenshteyn, and E. Waingarten, *Hölder homeomorphisms and approximate nearest neighbors*, 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS), 2018, pp. 159–169.
2. Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, American Mathematical Society colloquium publications, no. pt. 1, American Mathematical Soc., 1998.
3. J. Bergh and J. Löfström, *Interpolation spaces: An introduction*, Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, 2012.
4. A. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Mathematica **24** (1964), no. 2, 113–190 (eng).
5. F. Chaatit, *On uniform homeomorphisms of the unit spheres of certain banach lattices*, Pacific J. Math. **168** (1995), 11–31.
6. R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher, and G. Weiss, *Complex interpolation for families of Banach spaces*, Proceedings, Symposia in Pure Mathematics (Providence, R. I.), vol. 35, part 2, Amer. Math. Soc, 1979, pp. 269–282.
7. M. Cwikel, N. Kalton, M. Milman, and R. Rochberg, *A unified theory of commutator estimates for a class of interpolation methods*, Advances in Mathematics **169** (2002), 241–312.
8. M. Daher, *Homéomorphismes uniformes entre les sphères unité des espaces d'interpolation*, Canadian Mathematical Bulletin **38** (1995), no. 3, 286–294.
9. A. Ivtsan, *Stafney's lemma holds for several "classical" interpolation methods*, Proceedings of the American Mathematical Society **140** (2012), no. 3, 881–889.
10. N. Kalton, *Differentials of complex interpolation processes for Köthe function spaces*, Transactions of the American Mathematical Society **333** (1992), 479–529.
11. N. Kalton, P. Kunstmann, and L. Weis, *Perturbation and interpolation theorems for the $H^\infty$-calculus with applications to differential operators*, Mathematische Annalen **336** (2006), 747–801.
12. N. Kalton, E. Lorist, and L. Weis, *Euclidian structures and operator theory in Banach spaces*, To appear in Mem. Amer. Math. Soc. (2022).
13. N. Lindemulder and E. Lorist, *A discrete framework for the interpolation of Banach spaces*, arXiv e-prints (2021), arXiv:2105.08373.
14. J.-L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Publications Mathématiques de l'IHÉS **19** (1964), 5–68 (fr).
15. S. Mazur, *Une remarque sur l'homéomorphie des champs fonctionnels*, Studia Mathematica **1** (1929), no. 1, 83–85 (fre).
16. E. Odell and T. Schlumprecht, *The distortion problem*, Acta Mathematica **173** (1994), no. 2, 259 – 281.
17. G. Pisier, *Some applications of the complex interpolation method to Banach lattices*, Journal d'Analyse Mathématique **35** (1979), 264–281.
18. ———, *Martingales in Banach spaces*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2016.



Instituto de Ciências Matemáticas e de Computação. Universidade de São Paulo. 13566-590. São Carlos, SP, Brazil.

*Email address*: willhans@icmc.usp.br