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CRITICAL SCHRÖDINGER EQUATION COUPLED WITH BORN-INFELD TYPE EQUATIONS

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Abstract

In this talk we consider a quasilinear Schrödinger-Poisson system with a subcritical nonlinearity f, depending on the two parameters $\lambda, \varepsilon > 0$. We prove existence and behaviour of the solutions with respect to the parameters.

1 Introduction

In the mathematical literature many papers deal with the nonlinear Schrödinger equation coupled with the electrostatic field. These equations are variational in nature, hence the system which describes the phenomenon appear as the Euler-Lagrange equation of some Lagrangian.

The best way to describe the electromagnetic field seems to be by using the Born-Infeld Lagrangian, introduced in the seminal paper [2]. The advantage of working with such a Lagrangian is that it is relativistic invariant which is natural when dealing with electromagnetic phenomena. Explicitly the Lagrangian is

$$\mathcal{L}_{\text{B-I}} = \frac{1}{8\pi\varepsilon^4}\left(1 - \sqrt{1 - 2\varepsilon^4(|\nabla\phi + \partial_t \mathbf{A}|^2 - |\nabla \times \mathbf{A}|^2)}\right)$$

where $\phi, \mathbf{A}$ are the gauge potentials.

Of course dealing with such a Lagrangian implies some mathematical difficulties: in the simplest case, the equation of the electrostatic field generated by a density charge ρ is

$$\nabla \cdot \left(\frac{\nabla\phi}{\sqrt{1 - |\nabla\phi|^2}}\right) = \rho \quad \text{in } \mathbb{R}^3$$

which is not easy to work with.

Note that the first order approximation in ε of $\mathcal{L}_{\text{B-I}}$ is exactly the familiar Maxwell Lagrangian

$$\mathcal{L}_{\text{Max}} = \frac{1}{8\pi}\left(|\nabla\phi + \partial_t \mathbf{A}|^2 - |\nabla \times \mathbf{A}|^2\right)$$

which gives rise to the classical Maxwell equations and, in the electrostatic case, to the well known and more accessible Poisson equation

$$-\Delta\phi = \rho \quad \text{in } \mathbb{R}^3.$$

Here we are interested instead in considering the second order approximation in ε of $\mathcal{L}_{\text{B-I}}$, namely

$$\mathcal{L} = \frac{1}{8\pi}\left(|\nabla\phi + \partial_t \mathbf{A}|^2 - |\nabla \times \mathbf{A}|^2\right) + \frac{\varepsilon^4}{16\pi}\left(|\nabla\phi + \partial_t \mathbf{A}|^2 - |\nabla \times \mathbf{A}|^2\right)^2.$$

Now the equation for the electrostatic field is the quasilinear equation

$$-\Delta\phi - \varepsilon^4\Delta_4\phi = \rho \quad \text{in } \mathbb{R}^3,$$

and the coupling (according to the Abelian Gauge Theories) with the Schrödinger equation led to the system

$$\begin{cases} -\Delta u + u + \phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4 \Delta_4 \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{$P_{\lambda,\varepsilon}$}$$

The main difficulty is to deal with the second equation, which although has a unique solution for every u, an explicit formula and nice properties are not known. To overcome this fact we use a truncation in the energy functional in front of this "bad term" which permits to apply Mountain Pass arguments and prove the existence of solutions.

2 Our result

Let $\lambda > 0$ and $\varepsilon > 0$ parameters, $2^* = 6$ the critical Sobolev exponent, $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ continuous and such that

1. $f(x,t) = 0$ for $t \leq 0$,
2. $\lim_{t\to 0} \dfrac{f(x,t)}{t} = 0$, uniformly on $x \in \mathbb{R}^3$,
3. there exists $q \in (2, 2^*)$ verifying $\lim\limits_{t\to+\infty} \dfrac{f(x,t)}{t^{q-1}} = 0$ uniformly on $x \in \mathbb{R}^3$,
4. there exists $\theta \in (4, 2^*)$ such that $0 < \theta F(x,t) = \theta \int_0^t f(x,s)ds \leq tf(x,t)$ for all $x \in \mathbb{R}^3$ and $t > 0$.

Theorem 2.1. *Under the above assumptions, there exists $\lambda^* > 0$, such that for all $\lambda \geq \lambda^*$ and $\varepsilon > 0$, problem*

$$\begin{cases} -\Delta u + u + \phi u = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4 \Delta_4 \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$P_{\lambda,\varepsilon}$}$$

admits nonnegative solutions $(u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}) \in H^1(\mathbb{R}^3) \times \Big(D^{1,2}(\mathbb{R}^3) \cap D^{1,4}(\mathbb{R}^3)\Big)$.

For every fixed $\overline{\varepsilon} > 0$ we have:

$$\lim_{\lambda\to+\infty} \|u_{\lambda,\overline{\varepsilon}}\|_{H^1} = 0, \quad \lim_{\lambda\to+\infty} \|\phi_{\lambda,\overline{\varepsilon}}\|_{D^{1,2}\cap D^{1,4}} = 0, \quad \lim_{\lambda\to+\infty} |\phi_{\lambda,\overline{\varepsilon}}|_{L^\infty} = 0.$$

For every fixed $\overline{\lambda} \geq \lambda^*$ we have:

$$\lim_{\varepsilon\to 0^+} \|u_{\overline{\lambda},\varepsilon} - u_{\overline{\lambda},0}\|_{H^1} = 0, \quad \lim_{\varepsilon\to 0^+} \|\phi_{\overline{\lambda},\varepsilon} - \phi_{\overline{\lambda},0}\|_{D^{1,2}} = 0,$$

where $(u_{\overline{\lambda},0}, \phi_{\overline{\lambda},0}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a positive solution of the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \phi u = \overline{\lambda} f(x,u) + |u|^{2^*-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

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