An Example of Integrability of Involutive Distributions on Scale of Banach Spaces*

REINALDO SALVITTI

Instituto de Matemática e Estatística, Universidade de São Paulo, Cidade Universitária "Armando Salles de Oliveira" São Paulo, SP

The purpose of this work is to give an example of integrable involutive distribution on scale of Banach spaces, a topological vector space defined in [7]. We will apply the Global Formulation of the Frobenius Theorem [5] to construct Lie subgroups.

In §3 we will see an important example of a distribution given by a Lie subalgebra of the solutions of a Cauchy-Kovalewsky linear system; in this case, the ambient space is the gh(n,ℂ) [2], defined as the germs of analytic transformations around the origin of $\mathbb{C}^n$ that preserve the origin, and the distribution is in the Lie group Gh(n,ℂ) [2], defined as the subset of gh(n,ℂ) whose elements are invertible.

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1. THE SPACE OF GERMS gh(n, ℂ)

We denote by gh(n,ℂ) the topological vector space over ℂ of germs of analytic transformations, around the origin of $\mathbb{C}^n$, that preserve the origin. We will denote in the same way a germ or its representative.

1.1 For each $s > 0$, let X_s be the complex vector space of germs of analytic transformations x of $\mathbb{C}^n$, defined on the ball $D_s(0)$ (ball of center 0 and radius r in $\mathbb{C}^n$), that preserve the origin with

$$\sup_{|t|<s}|x(t)| < \infty$$

where $|t| = |t_1| + \dots + |t_n|$. The space X_s with the norm $\|x\|_s^1 = \sup_{|t|<s}|x(t)|$ is a complex Banach space. We can write that:

(a) $gh(n,\mathbb{C}) = \bigcup_{0<s\leq 1} X_s$.

(b) $X_s \subset X_{s'}$ and $\|\ \|_{s'}^1 \leqq \|\ \|_s^1$ for all s,s′, such that $0 < s' \leqq s \leqq 1$.

(c) The closed ball of center 0 and radius 1 in X_s, $\overline{B_s(0,1)}$, is compact in $X_{s'}$, $0 < s' < s \leqq 1$.

We can see also that the inductive limits are equal:

$$\varinjlim_{0<s\leq 1} X_s = \varinjlim_{n\in\mathbb{N}^*} X_{1/n}.$$

Then gh(n,ℂ) is a Silva space and therefore [1] is Hausdorff and sequentially complete. Hence gh(n,ℂ) is a scale of Banach spaces.

1.2 We can define gh(n,ℂ) in another way. For each $s > 0$, let Y_s be the complex vector space of germs of analytic transformations x of $\mathbb{C}^n$, defined on the ball $D_s(0)$, that preserve the origin with

$$\sup_{|t|<s}|J(x)(t)| < \infty,$$

where $J(x)(t)$ is the matrix whose elements are $\frac{\partial x_i}{\partial t_j}(t)$ $(1 \leqq i,\ j \leqq n)$, and if $A = (a_{ij})_{n\times n}$ is a matrix,

$$|A| = \max_{1\leqq i\leqq n} \sum_{j=1}^{n} |a_{ij}|.$$

The space Y_s with the norm

$$\|x\|_s^2 = \sup_{|t|<s} |J(x)(t)|$$

is a complex Banach space.

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We can write that:

(a) $gh(n,\mathbb{C}) = \bigcup_{0<s\leq 1} Y_s$

(b) $Y_s \subset Y_{s'}$, and $\| \ \|_s \leqq \| \ \|_{s'}$ for all s,s' with $0<s'\leqq s\leqq 1$.

1.3 THEOREM

The locally convex inductive limits $\varinjlim X_s$ and $\varinjlim Y_s$ are the same. We use the lemma below to prove this theorem.

LEMMA:

(a) $\|x\|^2_{s'} \leqq \frac{n}{s-s'}\|x\|_s$ for all s,s' with $0 < s' < s \leqq 1$.

(b) $\|x\|^1_s \leqq s\|x\|^2_s$ for all s, $0 < s \leqq 1$.

Proof: The proof is similar to [2].

2. THEOREM [5]

Let X be a scale of Banach spaces and $G \subset X$, a Lie group. Let H be a Lie subalgebra of X with topological supplement. We consider the distribution *D* on G given by L(z)H, where L is the infinitesimal transformation of the group G. We also suppose that for each $z_0 \in G$ there is a projection p onto $L(z_0)H$ such that

$$poL(z): H \longrightarrow L(z_0)H$$

is invertible for all $z \in V(z_0) = \bigcup_{0<s\leq 1} B_s(z_0,R)$ and for each $h \in L(z_0)H$ the map

$$V(z_0) \longrightarrow H$$
$$z \longrightarrow [poL(z)]^{-1}h$$

is LF-analytic [3]. Consider the system

$$(S)\begin{cases} z'(x,a)h = L(z)[poL(z)]^{-1}h, h \in L(z_0)H \\ z(p(a),a) = a \end{cases}$$

where $\|L(z)[poL(z)]^{-1}h\|_{s'} \leqq C\frac{s}{s-s'}\|h\|_s$, $z \in V(z_0)$, $h \in L(z_0)H$. Then for each point $z \in G$ there is a connected maximal integral manifold and *the one which contains the unit of the group, is a Lie subgroup of G.*

3. THE INTEGRABILITY OF AN INVOLUTIVE DISTRIBUTION ON Gh(n,$\mathbb{C}$) GIVEN BY A LIE SUBALGEBRA OF SOLUTIONS OF A CAUCHY-KOVALEWSKY LINEAR SYSTEM

Only for convenience, we will treat this example in the case n = 2.

3.1 *Solutions of a Cauchy-Kovalewsky Linear System*

Consider the system

$$(S)\begin{cases} \frac{\partial z}{\partial t_1} = A\frac{\partial z}{\partial t_2} + Bz \\ z(0,t_2) = \gamma(t_2) \end{cases}$$

where $A(t_1,t_2)$, $B(t_1,t_2)$ are square matrices of analytic functions of order 2, bounded by M for $|t_1| < \eta$ and $|t_2| < 1$, and γ is an analytic map of a neighborhood of 0 in $\mathbb{C}$, taking values in $\mathbb{C}^2$.

We denote as X_s the vector space of analytic maps u, bounded on the ball $D_s(0) = \{x \in \mathbb{C};\ |x| < s\}$, taking values in $\mathbb{C}^2$. We define the norm

$$\|u\|_s = \sup_{|x|<s} |u(x)|.$$

Then X_s is a complex Banach space. Consider the scale $X = \bigcup_{0<s\leq 1} X_s$. For each $u \in X_s$, consider the map

$$F(t_1,u)(t_2) = A(t_1,t_2)\,u'(t_2) + B(t_1,t_2)\,u(t_2).$$

Then, by $|t_1| < \eta$, $|t_2| < s < s'$, and by the Cauchy inequality $|u'(t_2)| \leqslant \frac{\|u\|_s}{s-s'}$, we have

$$|F(t_1,u)(t_2)| \leqslant \frac{M\,\|u\|_s}{s-s'} + M\|u\|_s \leqslant \frac{2M}{s-s'}\|u\|_s$$

and

$$\|F(t_1,u)\|_{s'} \leqslant \frac{2M}{s-s'}\|u\|_s \quad \text{for } |t_1| < \eta. \qquad (A)$$

Consider the system

$$(S')\begin{cases} \frac{du}{dt_1} = F(t_1,u) \\ u(0) = \gamma \end{cases}$$

The map γ belongs to some space X_{s_0}, $0 < s_0 \leqslant 1$. Taking $R > 0$ such that $\|\gamma\|_{s_0} < R$,

$\|u-\gamma\|_s < R$ and taking $|t_1| < \eta$, we have

$$\|u\|_s < R + \|\gamma\|_s < R + \|\gamma\|_{s_0} < 2R, 0 < s \leqslant s_0. \tag{B}$$

Applying (B) on (A),

$$\|F(t_1,u)\|_{s'} \leqslant \frac{4M}{s-s'}.$$

By Ovcyannikov-Treves Theorem [4,7], there is a unique solution u_γ that depends on γ and a number

$$\Delta = \min\left(\frac{\eta}{2}, \frac{R}{32e^2 4MR}\right)$$

where e is the Neper number, such that

$$\|u_\gamma(t_1) - \gamma\|_s < R.$$

Since $\|\gamma\|_{s_0} < R$, we have

$$\|u_\gamma(t_1)\|_s < R + \|\gamma\|_s < 2R.$$

If the system is linear, then the solution depends linearly on the initial conditions. Hence,

$$\|u_\gamma(t_1)\|_s \leqslant 2\|\gamma\|_{s_0}, \text{ for all } \gamma \in X_{s_0}.$$

Taking $|t_1| < \Delta(s_0 - s)$ and $|t_2| < s$, we have $(|t_1|/\Delta) + |t_2| < s_0$. Without loss of generality for $\Delta \leqslant 1$, we have $|t_1| + |t_2| < s_0$ and so $|t| < s_0$, $t = (t_1,t_2)$. Hence, the map $z(t_1,t_2) = u_\gamma(t_1)(t_2)$ is the only solution of the system (S). By §1.1,

$$\|z\|_s^1 = \sup_{|t|<s} |z(t_1,t_2)|,$$

hence

$$|z(t_1,t_2)| = |u_\gamma(t_1)(t_2)| \leqslant \|u_\gamma(t_1)\|_s \leqslant 2\|\gamma\|_{s_0}$$

and

$$\|z\|_{s_0}^1 \leqslant 2\|\gamma\|_{s_0}.$$

Conclusion: The map that associates initial conditions to solutions of the system (S) is a continuous and linear operator.

3.2 *The Integrability of an Involutive Distribution on* Gh(n,ℂ)

Consider the equation

$$\frac{\partial z}{\partial t_1} = A\frac{\partial z}{\partial t_2} + Bz \tag{E}$$

of the system (S), seen in §3.1, and $X = gh(2,\mathbb{C})$, $U = Gh(2,\mathbb{C})$. Let H be the vector subspace

$$H = \{z \in X;\ z \text{ is a solution of (E)}\}.$$

We suppose that H is a Lie subalgebra of X.

The subset H is closed because $\frac{\partial}{\partial t_1} - A\frac{\partial}{\partial t_2} + - BI$ is a continuous operator.

There is a bijective mapping between H and the subspace H_1 of the germs γ, $\gamma \in gh(2,\mathbb{C})$, such that γ depends only on the second variable t_2. In fact, the equation (E) has a unique solution $z(t_1,t_2)$ such that $z(0,t_2) = \gamma$. We denote by T the continuous linear map that associates initial conditions γ to solutions of (E):

$$T: H_1 \to H$$
$$\gamma \to T(\gamma) = z.$$

We define on U the distribution *D* given by J(z)H, where J(z) is the jacobian matrix of z.

We will prove the hypothesis of the Theorem 2.

(a) We fix z_0 and let p be the projection onto $J(z_0)H$ such that

$$p(z) = J(z_0)T[(J(z_0)^{-1}z)(0,t_2)].$$

In fact, $p(p(z)) = p(z)$ and p is linear and continuous because it is a composition of linear and continuous maps. To simplify, we denote by q the projection

$$q: X \to H$$
$$z \to q(z) = T(z(0,t_2))$$

(b) If we take z in a neighborhood of z_0, the map $p \circ J(z)$, restricted to H,

$$p \circ J(z): H \to J(z_0)H$$

is invertible. In fact, $p = J(z_0) \circ q \circ J(z_0)^{-1}$. Formally, we have

$$\begin{aligned}
&[p \circ J(z)]^{-1}h = \\
&= [J(z_0) \circ q \circ J(z_0)^{-1} \circ J(z)]^{-1}h \\
&= [q \circ J(z_0)^{-1} \circ J(z)]^{-1} \circ J(z_0)^{-1}h \\
&= \{q \circ J(z_0)^{-1} \circ [J(z_0) - J(z_0 - z)]\}^{-1} \circ J(z_0)^{-1}h \\
&= [q - q \circ J(z_0)^{-1}J(z_0 - z)]^{-1} \circ J(z_0)^{-1}h
\end{aligned}$$

If $h \in J(z_0)H$, then $J(z_0)^{-1}h \in H$ and q restricted to H is the identity, then

$$[p\circ J(z)]^{-1}h =$$
$$= [I - q\circ J(z_0)^{-1}\circ J(z_0 - z)]^{-1}\circ J(z_0)^{-1}h$$
$$= \sum_{m\geqslant 0} [q\circ J(z_0)^{-1}\circ J(z_0 - z)]^{m}\circ J(z_0)^{-1}h$$
$$= S$$

By §3.1, we have $\|q(z)\|_s^1 \leqslant 2\|z\|_s^1$; then

$$\|q\circ J(z_0)^{-1}\circ J(z_0 - z)k\|_s^1 \leqslant$$
$$\leqslant 2\|J(z_0)^{-1}\circ J(z_0 - z)k\|_s^1 \leqslant$$
$$\leqslant 2 \sup_{|t|<s} |J(z_0)^{-1}(t)| \sup_{|t|<s} |J(z_0 - z)(t)| \sup_{|t|<s} |k(t)|$$
$$\leqslant \underbrace{2 \sup_{|t|<s_0} |J(z_0)^{-1}(t)|}_{M_{z_0}} \|z_0 - z\|_s^2 \|k\|_s^1$$

for $k \in X_s$ and $0 < s \leqslant s_0$. Then

$$\|[q\circ J(z_0)^{-1}J(z_0 - z)]^m k\|_s^1 \leqslant$$
$$\leqslant (2M_{z_0}\|z - z_0\|_s^2)^m \|k\|_s^1$$

for $k \in X_s$ and $0 < s \leqslant s_0$.

The series S converges in $X_s \cap H$ if we take

$$\|z - z_0\|_s^2 < \frac{r}{2M_{z_0}} = R,\ 0 < r < 1,$$

and we get

$$\|S\|_s^1 \leqslant \frac{1}{1 - r}\|k\|_s^1,\ k = J(z_0)^{-1}h$$

Further, if we fix $k \in gh(2,\mathbb{C})$, the map S is LF-analytic in $\bigcup_{0<s\leqslant s_0} B_s(z_0,R)$ with values in X.

(c) Now we examine the majoration for the second member of (S) in §2. By Lemma 1.3 (a),

$$\|J(z)[p\circ J(z)]^{-1}h\|_{s'}^2 \leqslant$$
$$\leqslant \frac{2}{s - s'}\|J(z)[p\circ J(z)]^{-1}h\|_s^1$$
$$\leqslant \frac{2}{s - s'} \sup_{|t|<s} |J(z)(t)|\ \|p\circ J(z)^{-1}h\|_s^1$$
$$\leqslant \frac{2}{s - s'}\|z\|_s^2 \frac{1}{1 - r}\|J(z_0)^{-1}h\|_s^1$$
$$\leqslant \frac{2}{s - s'}(\|z_0\|_{s_0}^2 + R)\frac{1}{1 - r} \sup_{|t|<s} |J(z_0)^{-1}(t)|\|h\|_s^1$$
$$\leqslant \frac{2}{s - s'}(\|z\|_{s_0}^2 + R)\frac{1}{1 - r} M_{z_0}\|h\|_s^1.$$

By Lemma 1.3 (b), $\|h\|_s^1 \leqslant s\|h\|_s^2$. Therefore,

$$\|J(z)[p\circ J(z)]^{-1}h\|_{s'}^2 \leqslant \frac{2}{s - s'}(\|z_0\|_{s_0}^2 + R)\frac{1}{1 - r} M_{s_0} s\|h\|_s^2.$$

We denote $\frac{2}{1 - r}(\|z_0\|_s^2 + R)M_{s_0} = C$; then

$$\|J(z)[p\circ J(z)]^{-1}h\|_{s'}^2 \leqslant \frac{C}{s - s'}\|h\|_s^2.$$

Then, for each z_0 there is a maximal integral manifold passing through z_0, and by §2 the one passing through e is a Lie subgroup of $Gh(2,\mathbb{C})$ with Lie subalgebra H.

In [6], Saraiva proved the same result using §2, when H is a finite Lie subalgebra of $X = gh(n,\mathbb{C})$ and taking on $U = Gh(n,\mathbb{C})$ the distribution J(z)H.

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