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THE CONCEPT OF SUBTYPE IN BERNSTEIN ALGEBRAS

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Abstract

Given a Bernstein algebra $\mathbf{A} = Fe \oplus U \oplus V$, the ordered pairs of integers $(1 + \dim U, \dim V)$ and $(\dim(UV + V^2), \dim U^2)$ are called, respectively, the type and the subtype of $\mathbf{A}$. It is well known that given integers $r, s \geq 0$ there exists a Bernstein algebra of type $(1 + r, s)$. The similar question for subtypes has no simple answer. In this paper, we generalize the well known concept of exceptional Bernstein algebra $(U^2 = 0)$ introducing n-exceptionality. In this context, we study under which conditions, given a quadruple of non negative integers (r, s, t, z) there exists an n-exceptional algebra of type $(1 + r, s)$ and subtype (t, z). Results are obtained for the cases 0-exceptional and 1-exceptional.

1. Introduction

A Bernstein algebra over a field F is a pair $(\mathbf{A}, \omega)$, where $\mathbf{A}$ is a commutative (not necessarily associative) F-algebra and $\omega : \mathbf{A} \to F$ is a nonzero algebra homomorphism that satisfies

$$(x^2)^2 = \omega(x)^2 x^2 \tag{1}$$

for every $x \in \mathbf{A}$.

From (1) it follows that $N := \ker \omega$ is nil and thus ω, called the weight homomorphism, is uniquely determined. Every Bernstein algebra possesses at least one nonzero idempotent. If F is a field of characteristic not 2, then for every nonzero idempotent e, $\mathbf{A}$ has a Peirce decomposition relative to e, $\mathbf{A} = Fe \oplus U_e \oplus V_e$, where $U_e = \{x \in \mathbf{A} \mid 2ex = x\}$, $V_e = \{x \in \mathbf{A} \mid ex = 0\}$ and $N = U_e \oplus V_e$.

The Peirce subspaces U_e and V_e (relative to the idempotent e) satisfy the relations

$$U_e^2 \subseteq V_e, \qquad U_eV_e \subseteq U_e, \qquad V_e^2 \subseteq U_e \tag{2}$$

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and also the following identities hold for all $u \in U_e$ and $v \in V_e$:

$$u^3 = 0, \quad uv^2 = 0, \quad u(uv) = 0, \quad (uv)^2 = 0, \quad (u^2)^2 = 0. \tag{3}$$

By linearizations of (3), we obtain the following identities for all $u, u_1, u_2, u_3 \in U_e$, $v, v_1, v_2 \in V_e$:

$$u_1^2 u_2 + 2u_1(u_1 u_2) = 0, \tag{4}$$

$$u_1(u_2 u_3) + u_2(u_1 u_3) + u_3(u_1 u_2) = 0, \tag{5}$$

$$u(v_1 v_2) = 0, \tag{6}$$

$$u_1(u_2 v) + u_2(u_1 v) = 0, \tag{7}$$

$$(u_1 v)(u_2 v) = 0, \tag{8}$$

$$(uv_1)(uv_2) = 0, \tag{9}$$

$$(u_1 u_2)(v_1 v_2) = 0. \tag{10}$$

Also for all $x \in N = U_e \oplus V_e$

$$(x^2)^2 = 0 \tag{11}$$

We will use also the following linearized form of this identity:

$$x_1^2(x_1 x_2) = 0, \tag{12}$$

$$x_1^2(x_2 x_3) + 2(x_1 x_2)(x_1 x_3) = 0, \tag{13}$$

$$(x_1 x_2)(x_3 x_4) + (x_1 x_3)(x_2 x_4) + (x_1 x_4)(x_2 x_3) = 0, \tag{14}$$

for all $x_1, x_2, x_3, x_4 \in N$.

In this paper, F is a field with $\mathrm{car}(F) \neq 2,3$ and $\mathbf{A}$ a finite dimensional Bernstein algebra over F. If e is any idempotent of $\mathbf{A}$, $\mathrm{Ip}(\mathbf{A}) = \{e + u + u^2 \mid u \in U_e\}$ is the set of nonzero idempotents of $\mathbf{A}$. For any idempotent $f = e + u_0 + u_0^2$ ($u_0 \in U_e$) the mappings $\sigma : U_e \to U_f$, $\tau : V_e \to V_f$ defined by $\sigma(u) = u + 2u_0 u$ and $\tau(v) = v - 2u_0 v - 2u_0^2 v$ are isomorphisms of vectors spaces. Thus $U_f = \{u + 2u_0 u \mid u \in U_e\}$ and $V_f = \{v - 2u_0 v - 2u_0^2 v \mid v \in V_e\}$. It follows that the dimensions of U_e and V_e do not depend on the idempotent e. The ordered pair $(1 + \dim U_e, \dim V_e)$ is called the *type* of A.

A Bernstein algebra $\mathbf{A}$ is said Jordan-Bernstein if is also Jordan, that is, it satisfies $x^2(yx) = (x^2 y)x$ for all x, y in $\mathbf{A}$. In [5] it is proved that $\mathbf{A} = Fe \oplus U_e \oplus V_e$ is Jordan-Bernstein if and only if $V_e^2 = 0$ and $(uv)v = 0$, for all $u \in U_e$, $v \in V_e$. Let $\mathbf{A} = Fe \oplus U_e \oplus V_e$ be a Peirce decomposition of a Bernstein algebra $\mathbf{A}$. The set $L = \{x \in U_e \mid xu = 0 \text{ for all } u \in U_e\}$ is an ideal of $\mathbf{A}$ contained in U_e, which is independent on the idempotent and the quotient algebra $(\overline{\mathbf{A}}, \overline{\omega})$, where $\overline{\mathbf{A}} = \mathbf{A}/L$ and $\overline{\omega}(a + L) = \omega(a)$, for all $a \in \mathbf{A}$, is Jordan-Bernstein. In the Peirce decomposition $\overline{\mathbf{A}} = F\overline{e} \oplus U_{\overline{e}} \oplus V_{\overline{e}}$ relative to the idempotent $\overline{e} = e + L$, we have $U_{\overline{e}} = \overline{U}_e := U_e/L$ and $V_{\overline{e}} = \overline{V}_e := (V_e + L)/L$. For a subspace X of a Bernstein $\mathbf{A}$, we will denote by $\overline{X}$ the quotient $(X + L)/L$. All these facts are well known and can be found in [6], [8] and [9].

If X and Y are subspaces of a Bernstein algebra $\mathbf{A}$, we define $XY^{(0)} = X$ and $XY^{(k)} = (XY^{(k-1)})Y$, k integer ≥ 1, where $XY = \langle xy \mid x \in X, y \in Y \rangle$. For $a \in \mathbf{A}$, we write simply aX in place of $\langle a \rangle X$.

A Bernstein algebra $\mathbf{A} = Fe \oplus U_e \oplus V_e$ is said to be *normal* if $U_eV_e + V_e^2 = 0$ and is said to be *exceptional* if $U_e^2 = 0$, for some idempotent e. The algebra $\mathbf{A}$ is called *nuclear* if $U_e^2 = V_e$. These definitions do not depend on the choice of the idempotent element. It is known also that $\dim(U_eV_e + V_e^2)$ and $\dim U_e^2$ are invariant under change of the idempotent (see [7]). We will use the dimensions of these subspaces to define the *subtype* of a Bernstein algebra.

Definition 1 *Given a Bernstein algebra $A = Fe \oplus U_e \oplus V_e$, the ordered pair of integers $(\dim(U_eV_e + V_e^2), \dim U_e^2)$ will be called the subtype of A.*

Given non negative integers r and s there exists a Bernstein algebra of type $(1+r, s)$. It is enough to consider a trivial Bernstein algebra $\mathbf{A} = Fe \oplus U_e \oplus V_e$ with $\dim U_e = r$ and $\dim V_e = s$ (see [9]). But this does not hold for the subtype. In this paper we try to determine the conditions satisfied by a quadruple of non negative integers (r, s, t, z) such that there exists a Bernstein algebra of type $(1+r, s)$ and subtype (t, z). The study is made using as a tool the degree of exceptionality of the algebra.

If $\mathbf{A} = Fe \oplus U_e \oplus V_e$ is Bernstein, from (2), it follows that $U_eV_e + V_e^2 \subseteq U_e$ and $U_e^2 \subseteq V_e$, furthermore as $\mathbf{A}$ is commutative, $\dim(U_eV_e + V_e^2) \leq r$ and $\dim U_e^2 \leq \min\{\frac{1}{2}r(r+1), s\}$. Moreover if $r = 0$ or $s = 0$ then $U_eV_e = V_e^2 = U_e^2 = 0$, thus the only possible subtype for $\mathbf{A}$ is $(0,0)$. Therefore we consider only quadruples of integers (r, s, t, z) with $r \geq 1$, $s \geq 1$, $0 \leq t \leq r$ and $0 \leq z \leq \min\{\frac{1}{2}r(r+1), s\}$. Unless necessary, we omit the subscript e in U_e and V_e.

2. *n*-exceptionality

In this section we generalize the concept of exceptional algebra introducing the n-exceptionality. This will made using the subspaces of the chain

$$U^2 \supseteq U(UV) \supseteq U((UV)V) \supseteq \ldots \supseteq U(UV^{(k)}) \supseteq U(UV^{(k+1)}) \supseteq \ldots.$$

which have invariant dimension under change of the idempotent.

Definition 2 *A Bernstein algebra $\mathbf{A} = Fe \oplus U_e \oplus V_e$ is called exceptional of degree n, or n-exceptional, if n is the least non negative integer such that the subspace $U_e(U_eV_e^{(n)}) = 0$, for some $e \in Ip(A)$. The integer n will called the degree of exceptionality of $\mathbf{A}$.*

It was proved in [4, Cor. 4] that every subspace of a Bernstein algebra $\mathbf{A} = Fe \oplus U \oplus V$ contained in V, has invariant dimension under change of the idempotent. Thus this definition does not depend of the idempotent, since for every integer $n \geq 0$, $U(UV^{(n)}) \subseteq V$. Note that the 0-exceptional algebras, such that $U^2 = 0$, are just exceptional.

For every element x of an arbitrary algebra $\mathbf{A}$, R_x denotes the right multiplication by x, that is, $R_x(a) = ax$, for all $a \in \mathbf{A}$.

Lemma 1 *In a Bernstein algebra* $\mathbf{A} = Fe \oplus U \oplus V$ *for every* $u \in U$, $v, v_i \in V$ $(i = 1, 2, \ldots)$ *and for every integer* $k \geq 2$ *we have:*
(i) $R_v^k(u) \in L$*;*
(ii) $\sum_{\sigma \in S_k} R_{v_{\sigma(1)}} R_{v_{\sigma(2)}} \ldots R_{v_{\sigma(k)}}(u) \in L$*, where* S_k *is the symmetric group of degree* k*;*
(iii) $R_{v_1} R_{v_k} R_{v_{k-1}} \ldots R_{v_2} R_{v_1}(u) \in L$*.*

Proof. From (2) it follows that $UV^{(k)} \subseteq U$ for every integer $k \geq 0$. If $u_1 \in U$ then by (7) and (8), $u_1((uv)v) = -(uv)(u_1 v) = 0$. Thus (i) is true for $k = 2$. If $R_v^k(u) = l \in L$, for some $k \geq 2$, then $R_v^{k+1}(u) = R_v(R_v^k(u)) = lv \in L$. By means of consecutive linearizations of $R_v^k(u)$ in v, we obtain (ii). From (ii), for every $u \in U, v_1, v_2 \in V$, there is $l \in L$, such that

$$(uv_2)v_1 = l - (uv_1)v_2 \tag{15}$$

In particular, taking u as uv_1 and using (i), we have $((uv_1)v_2)v_1 = l - ((uv_1)v_1)v_2 \in L$. Therefore (iii) is true for $k = 2$. If $((\ldots((uv_1)v_2)\ldots)v_k)v_1 = l' \in L$, for some $k \geq 2$, using (15) with u as $(\ldots((uv_1)v_2)\ldots)v_k$ and v_2 as v_{k+1}, we have $R_{v_1} R_{v_{k+1}} R_{v_k} \ldots R_{v_2} R_{v_1}(u) = (((\ldots((uv_1)v_2)\ldots)v_k)v_{k+1})v_1 = l - (((\ldots((uv_1)v_2)\ldots)v_k)v_1)v_{k+1} = l - l'v_{k+1} \in L$. □

Theorem 1 *Every Bernstein algebra of type* $(1 + r, s)$ *is n-exceptional for some integer n, with* $0 \leq n \leq s + 1$.

Proof. Let $\mathbf{A} = Fe \oplus U \oplus V$ be Bernstein of type $(1 + r, s)$. Initially we show that $UV^{(s+k)} \subseteq L$, for every integer $k \geq 1$. If $r = 0$ or $s = 0$, this is obviously true. Let us assume $r, s \geq 1$. Let $\{u_1, u_2, \ldots, u_r\}$ and let $\{v_1, v_2, \ldots, v_s\}$ be a basis of U and V, respectively. Then $UV^{(s+1)} = \langle(((\ldots((u_i v_{j_1})v_{j_2})\ldots)v_{j_s})v_{j_{s+1}} \mid 1 \leq i \leq r,\ 1 \leq j_1, j_2, \ldots, j_s, j_{s+1} \leq s\rangle$. Let us show that every spanning of $UV^{(s+1)}$ is an element of L. As $1 \leq j_1, j_2, \ldots, j_s, j_{s+1} \leq s$, there exist j_k and j_l, with $1 \leq k < l \leq s + 1$, such that $j_k = j_l$. By Lemma 1, item (iii) we have:
$((\ldots(((\ldots((((\ldots((u_i v_{j_1})v_{j_2})\ldots)v_{j_{k-1}})v_{j_k})v_{j_{k+1}})\ldots)v_{j_l})v_{j_{l+1}})\ldots)v_{j_s})v_{j_{s+1}} =$
$((\ldots(((\ldots((u'v_{j_k})v_{j_{k+1}})\ldots)v_{j_k})v_{j_{l+1}})\ldots)v_{j_s})v_{j_{s+1}} = ((\ldots(l'v_{j_{l+1}})\ldots)v_{j_s})v_{j_{s+1}} \in L$,
where $u' = (\ldots((u_i v_{j_1})v_{j_2})\ldots)v_{j_{k-1}} \in U$ and $l' = (\ldots((u'v_{j_k})v_{j_{k+1}})\ldots)v_{j_k} \in L$. Therefore $UV^{(s+1)} \subseteq L$ and consequently $UV^{(s+k)} \subseteq L$, for every integer $k \geq 1$, since L is an ideal. It follows now that $U(UV^{(n)}) = 0$, for some not negative integer $n \leq s + 1$. □

3. On the Subspace $UV + V^2$

In this section we calculate an upper bound for the dimension of the subspace $UV + V^2$ in algebras with degree of exceptionality ≤ 1.

Throughout this paper, we denote by $R : \mathbb{Z}_+ \to \mathbb{R}$ and $\lceil\ \rceil : \mathbb{R} \to \mathbb{Z}$ the mappings defined by $R(z) = \frac{1}{2}(-1 + \sqrt{1 + 8z})$ and $\lceil x \rceil = n$, where $n - 1 < x \leq n$ and n is an integer.

Proposition 1 *If* $\mathbf{A} = Fe \oplus U_e \oplus V_e$ *is Bernstein of type* $(1 + r, s)$ *with* $r, s \geq 1$*, then* $dimL \leq r - \lceil R(\dim U_e^2) \rceil$*,* R *and* $\lceil\ \rceil$ *as previously defined.*

Proof. Let M_e be a subspace of U_e complementary of L, that is, $U_e = M_e \oplus L$. As the dimensions of U_e and L are invariant under change of the idempotent, the dimension of M_e also is invariant. Moreover, $U_e^2 = M_e^2$. Let $z = \dim U_e^2$ and $k = \dim M_e$. Then

$z = \dim U_e^2 = \dim M_e^2 \leq \frac{1}{2}k(k+1)$. Thus k satisfies the inequality $k^2 + k - 2z \geq 0$. As k is a not negative integer and less than or equal to r, it follows that $[R(z), r] \cap \mathbb{Z}$ contains the possible solutions for $\dim M_e$, that is, $\lceil R(z) \rceil \leq dimM_e \leq r$ and therefore $\dim L \leq r - \lceil R(\dim U_e^2) \rceil$. □

The next step will calculate some upper bound to the dimension of the subspace $UV + V^2$.

Proposition 2 *If* $\mathbf{A} = Fe \oplus U \oplus V$ *is an n-exceptional Bernstein algebra of type* $(1+r,s)$ *with* $n \leq 1$, *then* $(UV + V^2) \subseteq L$.

Proof. As $n \leq 1$, then $UV \subseteq L$ and by identity (6) it follows that $V^2 \subseteq L$. Therefore $UV + V^2 \subseteq L$. □

Corollary 1 *If* $\mathbf{A} = Fe \oplus U \oplus V$ *is an n-exceptional Bernstein algebra of type* $(1+r,s)$ *with* $n \leq 1$, *then* $\dim(UV + V^2) \leq r - \lceil R(\dim U^2) \rceil$.

The case exceptional is immediate. Given integers (r,s,t) with $r \geq 1$, $s = t = 0$ or $s \geq 1$ and $0 \leq t \leq r$, is known that is possible construct an exceptional Bernstein algebra $\mathbf{A} = Fe \oplus U \oplus V$ of type $(1+r,s)$ and subtype $(t,0)$ defining freely the products, with the condition that the UV and V^2 lie in U.

4. Subtypes of 1-exceptional Algebras

In the study of 1-exceptional algebras let us consider firstly the case in which such algebras are non nuclear.

4.1. Non Nuclear 1-exceptional Algebras

The next lemma shows that there exists non nuclear 1-exceptional Bernstein algebra where the dimension of $UV + V^2$ can reach the upper bound given in Corollary 1.

Lemma 2 *Given integers* (r,s,t,z) *with* $r \geq 1$, $s \geq 2$, $1 \leq z \leq \min\{\frac{1}{2}r(r+1), s-1\}$ *and* $0 \leq t \leq r - \lceil R(z) \rceil$ *there exists a non nuclear* 1*-exceptional Bernstein algebra of type* $(1+r,s)$ *and subtype* (t,z).

Proof. The proof is an algorithm to construct such algebra. Let $k = \lceil R(z) \rceil$ and let $\mathbf{A}$ the F-vector space of dimension $1+r+s$ spanned by $\{e, u_1, \ldots, u_r, v_1, \ldots, v_s\}$. We define in $\mathbf{A}$ the commutative product given by:
(1) $e^2 = e$; $2eu_i = u_i$; $ev_j = 0$; $(i = 1, 2, .., r$ and $j = 1, 2, \ldots, s)$;
(2) $u_i u_j = v_{\varepsilon(i,j)}$, if $1 \leq i \leq j \leq k$ and $\varepsilon(i,j) \leq z$, where $\varepsilon(i,j) = \frac{1}{2}(2k-i)(i-1) + j$;
(3) $u_i v_s = u_i$, if $k+1 \leq i \leq k+t$. The other products are zero.
Let $\omega : \mathbf{A} \to F$ defined by $\omega(e) = 1$ and $\omega(u_i) = \omega(v_j) = 0$ on the other elements of the basis. Then $\mathbf{A} = Fe \oplus U \oplus V$, where $U = \langle u_1, u_2, \ldots, u_r \rangle$ and $V = \langle v_1, v_2, \ldots, v_s \rangle$. From (1) to (3) above, it follows that $U^2 = \langle v_1, \ldots, v_z \rangle \subseteq \langle v_1, v_2, \ldots, v_{s-1} \rangle \subsetneq V$; $UV = \langle u_{k+1}, u_{k+2}, \ldots, u_{k+t} \rangle \subseteq U$ and $V^2 = 0$. Moreover, $U(UV) = \langle u_i u_j \mid 1 \leq i \leq r, k+1 \leq$

$j \leq k+t\rangle = 0$, since $u_i u_j = 0$, for all i or $j \geq k+1$. Let $x = \omega(x)e + u + v \in \mathbf{A}$, where $u = \sum_{i=1}^{r} \alpha_i u_i$ and $v = \sum_{j=1}^{s} \beta_j v_j$, with $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s \in F$. By (1) and commutativity of the product we have $x^2 = \omega(x)^2 e + \omega(x)u + u^2 + 2uv + v^2$, with $\omega(x)u + 2uv + v^2 \in U + UV + V^2 = U$ e $u^2 \in U^2 \subseteq V$. Moreover, for any $u \in U$ e $v \in V$, we have: $u^3 \in U^3 \subseteq U\langle v_1, v_2, \ldots, v_z\rangle = 0$; $u(uv) \in U(UV) = 0$; $uv^2 \in UV^2 = 0$; $(u^2)^2 \in (U^2)^2 \subseteq V^2 = 0$; and $(uv)^2 \in (UV)^2 \subseteq U(UV) = 0$. By Theorem 3.4.8 of [6] , it follows that $(\mathbf{A}, \omega)$ is a Bernstein algebra with $U_e = U$ and $V_e = V$. For every idempotent f, $\dim U_f^2 = \dim U_e^2 = z \neq 0$ and $\dim(U_f V_f + V_f^2) = \dim(U_e V_e + V_e^2) = \dim U_e V_e = t$. Therefore $\mathbf{A}$ is 1-exceptional of type $(1+r, s)$ and subtype (t, z). □

Theorem 2 *Given integers (r,s,t,z), there exists a non nuclear 1-exceptional Bernstein algebra of type $(1+r,s)$ and subtype (t,z) if and only if $r \geq 1$, $s \geq 2$, $1 \leq z \leq \min\{\frac{1}{2}r(r+1), s-1\}$ and $0 \leq t \leq r - \lceil R(z) \rceil$.*

Proof. Follows from Corollary 1 and Lemma 2. □

The next example exhibits a non nuclear 1-exceptional Bernstein algebra as shown in Lemma 2 for the quadruple $(8,9,4,8)$.

Example 1 Let $\mathbf{A} = Fe \oplus U \oplus V$ be the Bernstein algebra with $U = \langle u_1, u_2, \ldots, u_8\rangle$, $V = \langle v_1, v_2, \ldots, v_9\rangle$ and the following nonzero products in $N = U \oplus V$:

$$u_1u_1 = v_1;\ u_1u_2 = v_2;\ u_1u_3 = v_3;\ u_1u_4 = v_4;\ u_2u_2 = v_5;\ u_2u_3 = v_6;$$
$$u_2u_4 = v_7;\ u_3u_3 = v_8;\ u_5v_9 = u_5;\ u_6v_9 = u_6;\ u_7v_9 = u_7;\ u_8v_9 = u_8.$$

4.2. Nuclear 1-excepcional Algebras

In the investigation of nuclear 1-exceptional Bernstein algebras the next proposition will be useful.

Given integers $p \geq 1$ and $k_1, k_2, \ldots, k_n \in \{1, 2, \ldots, p\}$, all distinct, let $(k_{j_1}, k_{j_2}, \ldots, k_{j_n})$ be a sequence of these integers, with $k_{j_1} < k_{j_2} < \ldots < k_{j_n}$. Let $o_{k_i}(k_1 k_2 \ldots k_i \ldots k_n)$ denote the position of k_i in this sequence. With this convention, given $M \subseteq U$, a nonzero subspace of the Bernstein algebra $\mathbf{A} = Fe \oplus U \oplus V$, $\{u_1, u_2, \ldots, u_p\}$ an arbitrary basis of M and n_1, n_2, n_3, n_4 distinct integers, we define the following subspaces:

i) For $n_1, n_2, n_3 \in \{1, 2, 3\}$:

$$M_{(n_1n_2)n_3} = \langle (u_i^2 u_j) u_k \mid 1 \leq i,j,k \leq p;\ o_i(ijk) = n_1,\ o_j(ijk) = n_2,\ o_k(ijk) = n_3\rangle\ ;$$
$$M_{n_1(n_2n_3)} = \langle u_i^2 (u_j u_k) \mid 1 \leq i,j,k \leq p;\ o_i(ijk) = n_1,\ o_j(ijk) = n_2,\ o_k(ijk) = n_3\rangle\ ;$$

ii) For $n_1, n_2, n_3, n_4 \in \{1, 2, 3, 4\}$:

$$M_{n_1n_2n_3n_4} = \langle ((u_i u_j) u_k) u_l \mid 1 \leq i,j,k,l \leq p;\ o_i(ijkl) = n_1,\ o_j(ijkl) = n_2,\ o_k(ijkl) = n_3,\ o_l(ijkl) = n_4\rangle;$$
$$M_{(n_1n_2)(n_3n_4)} = \langle (u_i u_j)(u_k u_l) \mid 1 \leq i,j,k,l \leq p;\ o_i(ijkl) = n_1,\ o_j(ijkl) = n_2,\ o_k(ijkl) = n_3,\ o_l(ijkl) = n_4\rangle.$$

Proposition 3 *Let $M \subseteq U$ be an arbitrary subspace of the Bernstein algebra $\mathbf{A} = Fe \oplus U \oplus V$ of type $(1+r,s)$ with $r \geq 1$. If $\dim M = p$, then*
(i) $\dim M^3 \leq \min\{r-1, \frac{1}{3}p(p^2-1)\}$;
(ii) $\dim M^4 \leq \min\{s, \frac{1}{8}(p-2)(p-1)p(p+1)\}$;
(iii) $\dim (M^2)^2 \leq \min\{r-1, \frac{1}{12}p^2(p^2-1)\}$.

Proof. If **A** is exceptional or $p=0$, $M^3=M^4=(M^2)^2=0$ and the proposition is trivially true. Let us to assume that **A** is not exceptional and $p\geq 1$. Let $\{u_1,u_2,\ldots,u_p\}$ be a basis of M. If $X=\langle u_i^2 \mid 1\leq i\leq p\rangle$ and $Y=\langle u_iu_j \mid 1\leq i<j\leq p\rangle$, then $M^3=M(X+Y)$. By identities (3) and (4) we have $MX=\langle u_i^2u_j \mid 1\leq i,j\leq p, i\neq j\rangle$ and $MY=\langle u_i^2u_j \mid 1\leq i,j\leq p, i\neq j\rangle+\langle (u_iu_j)u_k \mid 1\leq i<j\leq p, 1\leq k\leq p, k\neq i,j\rangle = MX+\langle (u_iu_j)u_k \mid 1\leq k<i<j\leq p\rangle+\langle (u_iu_j)u_k \mid 1\leq i<k<j\leq p\rangle+\langle (u_iu_j)u_k \mid 1\leq i<j<k\leq p\rangle = M_1+M_2+M_3+M_4$, where $M_1=\langle u_i^2u_j \mid 1\leq i,j\leq p, i\neq j\rangle$; $M_2=\langle (u_iu_j)u_k \mid 1\leq i<j<k\leq p\rangle$; $M_3=\langle (u_iu_j)u_k \mid 1\leq k<i<j\leq p\rangle$ and $M_4=\langle (u_iu_j)u_k \mid 1\leq i<k<j\leq p\rangle$. By identity (5), if i,j,k are such that $1\leq i<k<j\leq p$, then $(u_iu_j)u_k=-(u_iu_k)u_j-(u_ku_j)u_i$. Hence $M_4\subseteq M_2+M_3$ and $M^3=M_1+M_2+M_3$. It follows that $\dim M^3\leq \frac{1}{3}p(p^2-1)$. Now, $M^4=M(M_1+M_2+M_3)$, with M_1, M_2 and M_3 as defined above. Using the identities (3), (4), (7) we have: $MM_1=M_{(12)3}+M_{(31)2}+M_{(21)3}$; $MM_2=M_{(12)3}+M_{(21)3}+M_{2341}+M_{1342}+M_{1234}+M_{1243}$ and $MM_3=M_{(23)1}+M_{(32)1}+M_{2413}+M_{2314}+M_{3412}+M_{3421}$. Therefore $M^4=(M_{(12)3}+M_{(21)3}+M_{(31)2}+M_{(23)1}+M_{(32)1})+(M_{1234}+M_{1243}+M_{1342}+M_{2341}+M_{2314}+M_{2413}+M_{3412}+M_{3421})$. By identities (5) and (7) and commutativity of the product we have:

$$((u_ju_i)u_k)u_l=((u_iu_j)u_k)u_l=-((u_iu_j)u_l)u_k=((u_lu_i)u_j)u_k+((u_lu_j)u_i)u_k, \qquad (16)$$

for all $1\leq i,j,k,l\leq p$. If follows that $M_{n_1n_2n_3n_4}=M_{n_2n_1n_3n_4}=M_{n_2n_1n_4n_3}$ and $M_{n_1n_2n_3n_4}\subseteq M_{n_4n_1n_2n_3}+M_{n_4n_2n_1n_3}$, for all $1\leq n_1,n_2,n_3,n_4\leq 4$. Also from (16), for $i=j$, we obtain $M_{(n_1n_2)n_3}=M_{(n_1n_3)n_2}$, for every $1\leq n_1,n_2,n_3\leq 3$. Therefore $M^4=M_{(12)3}+M_{(21)3}+M_{(31)2}+M_{1234}+M_{2314}+M_{3412}$ and thus $\dim M^4\leq \frac{1}{8}(p-2)(p-1)p(p+1)$. Finally, $(M^2)^2=(X+Y)^2=X^2+XY+Y^2$, X and Y as defined above. Using the identities (11), (12) and (13) we have: $X^2=\langle u_i^2u_j^2 \mid 1\leq i<j\leq p\rangle$, $XY=M_{1(23)}+M_{2(13)}+M_{3(12)}$ and $Y^2=X^2+M_{1(23)}+M_{2(13)}+M_{3(12)}+M_{(12)(34)}+M_{(13)(24)}+M_{(14)(23)}$. Therefore $(M^2)^2=Y^2$. From identity (14), it follows that $(u_iu_j)(u_ku_l)=-(u_iu_k)(u_lu_j)-(u_iu_l)(u_ku_j)$, for every $1\leq i,j,k,l\leq p$. Thus $M_{(n_1n_2)(n_3n_4)}\subseteq M_{(n_1n_3)(n_4n_2)}+M_{(n_1n_4)(n_3n_2)}$, for every $1\leq n_1,n_2,n_3,n_4\leq 4$. In particular, $M_{(14)(23)}\subseteq M_{(12)(34)}+M_{(13)(24)}$. $(M^2)^2=X^2+M_{1(23)}+M_{2(13)}+M_{3(12)}+M_{(12)(34)}+M_{(13)(24)}$. Therefore $\dim(M^2)^2\leq \frac{1}{12}p^2(p^2-1)$. Moreover, as $M\subseteq U$ we have from (2) and Proposition 9 of [3] than $M^3\subsetneq U$, $M^4\subseteq V$ and $(M^2)^2\subsetneq U$. This ends the proof of (i), (ii) and (iii). □

We will see in Example 2 that there exists Bernstein algebras such that the upper bound given in (i) and (iii) is reached.

Let $\mathbf{A}=Fe\oplus U\oplus V$ be a 1-exceptional Bernstein algebra of type $(1+r,s)$ and let M be a subspace of $\mathbf{A}$ such that $U=M\oplus L$. If $\mathbf{A}$ is nuclear, then $U^2=V$, thus $UV+V^2=U^3+(U^2)^2=M^3+(M^2)^2$. Therefore $\dim(UV+V^2)=\dim(M^3+(M^2)^2)\leq \dim M^3+\dim(M^2)^2$. On the other hand, as $\mathbf{A}$ is 1-exceptional by Proposition 2, $\dim(UV+V^2)\leq \dim L=r-\dim M$. By Proposition 1, we might have $\lceil R(s)\rceil\leq \dim M\leq r$. Thus $\dim(UV+V^2)\leq \max\{\min\{r-\dim M, \dim M^3+\dim(M^2)^2\} \mid \lceil R(s)\rceil\leq \dim M\leq r\}$. If $\dim M=p$ and $\dim M^2=\frac{1}{2}p(p+1)$, then we can have a Bernstein algebra with $\dim M^3+\dim(M^2)^2$ as given in Proposition 3 (see Example 2). If $\dim M^2<\frac{1}{2}p(p+1)$ this may be impossible. We will see this in the next proposition.

Proposition 4 *Let A be a 1-exceptional Bernstein algebra of type $(1+r,s)$ and subtype (t,s), with $s\geq 1$. If $r>\frac{1}{12}p((p^2-1)(p+4)+12)$, where $p=\lceil R(s)\rceil$, then $t<r-\lceil R(s)\rceil$.*

Proof. Let $\mathbf{A} = Fe \oplus U \oplus V$ and let $M \subseteq U$ be such that $U = M \oplus L$. By previous remark, $t \leq \max\{\min\{r - \dim M, \dim M^3 + \dim(M^2)^2\} \mid p \leq \dim M \leq r\}\}$. If $\dim M = p$, by Proposition 3, $\dim M^3 + \dim(M^2)^2 \leq \frac{1}{12}p(p^2-1)(p+4) < r - p$. On the other hand, $\min\{\dim M^3 + \dim(M^2)^2, r - \dim M\} \leq r - \dim M < r - p$, for every M such that $p+1 \leq \dim M \leq r$. Therefore $t < r - p = r - \lceil R(s) \rceil$. □

We show, by construction, that there is an 1-exceptional nuclear Bernstein algebra with $\dim(UV + V^2) = \dim U - \lceil R(\dim U^2) \rceil$.

Theorem 3 *For every triple of integers (r,s,t) with $s = \frac{1}{2}p(p+1)$, for some integer $p \geq 1$, $1 \leq r \leq \frac{1}{3}(s^2 + (p-2)s + 3p)$ and $0 \leq t \leq r - p$, there exists 1-exceptional Bernstein algebras of type $(1+r,s)$ and subtype (t,s).*

Proof. The proof is a construction of a Bernstein algebra satisfies requirements of theorem. Let $\mathbf{A}$ be a vector space over a field F with $\{e, u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\}$ a basis of $\mathbf{A}$. We define in $\mathbf{A}$ the following commutative products:

(1). $e^2 = e$; $\quad eu_i = \frac{1}{2}u_i$; $\quad ev_j = 0 \quad (i = 1,2,\ldots,r;\ j = 1,\ldots,s)$;

(2). $u_iu_j = v_{\varepsilon(i,j)}$, if $1 \leq i \leq j \leq p$, where $\varepsilon(i,j) = \frac{1}{2}(2p-i)(i-1)+j$;

(3). $u_i^2u_j = -2u_i(u_iu_j) = 2u_{\rho_1(i,j)}$;

$u_j^2u_i = -2u_j(u_ju_i) = 2u_{\rho_2(i,j)}$, if $1 \leq i < j \leq p$ and $\rho_w(i,j) \leq p+t$ for $w = 1,2$, where

$\rho_1(i,j) = p + \sum_{x=1}^{i-1}[(p-x)(p-(x-1))] + (j-i)$ and

$\rho_2(i,j) = p + \sum_{x=1}^{i-1}[(p-x)(p-(x-1))] + (p-i) + (j-i)$;

(4). $(u_iu_j)u_k = u_{\delta_1(i,j,k)}$;

$(u_ju_k)u_i = u_{\delta_2(i,j,k)}$, if $1 \leq i < j < k \leq p$ and $\delta_w(i,j,k) \leq p+t$ for $w = 1,2$, where $\delta_w(i,j,k) = p + \sum_{x=1}^{i-1}[(p-x)(p-(x-1)] + 2(p-i) + (2p-(i+j))(j-i-1) + 2(k-j) + (w-2)$;

(5). $(u_iu_k)u_j = -[(u_iu_j)u_k + (u_ju_k)u_i]$, if $1 \leq i < j < k \leq p$;

(6). $u_i^2u_j^2 = -2(u_iu_j)^2 = 2u_{\tau(i,j)}$, if $1 \leq i < j \leq p$ and $\tau(i,j) \leq p+t$, where

$\tau(i,j) = \frac{1}{3}p(p^2+2) + \frac{1}{2}(2p-i)(i-1) + (j-i)$;

(7). $u_i^2(u_ju_k) = -2(u_iu_j)(u_iu_k) = 2u_{\tau_1(i,j,k)}$;

$u_j^2(u_iu_k) = -2(u_iu_j)(u_ju_k) = 2u_{\tau_2(i,j,k)}$;

$u_k^2(u_iu_j) = -2(u_iu_k)(u_ku_j) = 2u_{\tau_3(i,j,k)}$;

if $1 \leq i < j < k \leq p$ and $\tau_w(i,j,k) \leq p+t$ for $w = 1,2,3$, where

$\tau_w(i,j,k) = \frac{1}{6}p(2p+1)(p+1) + \sum_{x=1}^{i-1}\frac{3}{2}[(p-x)(p-(x+1))] + \frac{3}{2}(2p-(i+j))(j-i-1) + 3(k-j) + (w-3)$;

(8). $(u_iu_j)(u_ku_l) = u_{\sigma_1(i,j,k,l)}$;

$(u_iu_k)(u_ju_l) = u_{\sigma_2(i,j,k,l)}$;

if $1 \leq i < j < k < l \leq p$ and $\sigma_w(i,j,k,l) \leq p+t$ for $w = 1,2$, where

$\sigma_w(i,j,k,l) = \frac{1}{6}p(5p^2-6p+7) + \sum_{x=1}^{i-1}\frac{1}{3}[(p-x)(p-(x+1)(p-(x+2)] + \sum_{x=i+1}^{j-1}[(p-x)(p-(x+1)] + (2p-(k+j))(k-j-1) + 2(l-k) + (w-2)$;

(9). $(u_iu_l)(u_ju_k) = -[(u_iu_j)(u_ku_l) + (u_iu_k)(u_ju_l)]$, if $1 \leq i < j < k < l \leq p$.

Other products are zero.

Let $\omega : \mathbf{A} \to F$, defined by $\omega(e) = 1$; $\omega(u_i) = \omega(v_j) = 0$, for $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,s$. As in Lemma 2, in order to show that $(\mathbf{A},\omega)$ is Bernstein we will use Theorem

3.4.8 of [6]. Let $U = \langle u_1, u_2, ..., u_r\rangle$ and $V = \langle v_1, v_2, ..., v_s\rangle$, then:

1) $U^2 = \langle u_iu_j \mid 1 \leq i, j \leq r\rangle = \langle u_iu_j \mid 1 \leq i \leq j \leq p\rangle = \langle v_1, v_2, ..., v_s\rangle = V$, according to (2);

2) $UV = U^3 = \langle (u_iu_j)u_k \mid 1 \leq i, j, k \leq r\rangle = \langle u_{p+1}, u_{p+2}, ..., u_\kappa\rangle \subseteq U$, according to items (3), (4) and (5), where $\kappa = \min\{\frac{1}{3}p(p^2+2), p+t\}$;

3) $V^2 = (U^2)^2 = \langle (u_iu_j)(u_ku_l) \mid 1 \leq i, j, k, l \leq r\rangle = \langle u_{\kappa+1}, u_{\kappa+2}, ..., u_{p+t}\rangle \subseteq U$, by items (6), (7) and (8). We remark that if $\kappa = p+t$, then $V^2 = 0$.

Moreover, as UV and V^2 are included in $\langle u_{p+1}, u_{p+2} ..., u_{p+t}\rangle$, then $U(UV) = (UV)^2 = UV^2 = 0$, because $u_iu_j = 0$ for every i or $j \geq p+1$. Let $x = \alpha e + u + v \in \mathbf{A}$, where $u = \sum_{i=1}^r \alpha_iu_i$ and $v = \sum_{i=1}^s \beta_jv_j$, with $\alpha, \alpha_i, \beta_j \in F$, for every i, j. From (1) and from the commutativity of the product we have $x^2 = \alpha^2 e + (\alpha u + 2uv + v^2) + u^2$, where $\alpha u + 2uv + v^2 \in U + UV + V^2 = U$ and $u^2 \in U^2 = V$. It remains to show that the identities (3) are valid. The identities $u(uv) = (uv)^2 = uv^2 = 0$ are immediate, because $U(UV) = (UV)^2 = UV^2 = 0$. Let $u = \sum_{i=1}^r \alpha_iu_i \in U$, with $\alpha_1, ..., \alpha_r \in F$. By the rules of the product, it is enough to consider $u = \sum_{i=1}^p \alpha_iu_i$. We have $u^2 = \sum_{i=1}^p \alpha_i^2u_i^2 + 2\sum_{1\leq i<j\leq p}\alpha_i\alpha_ju_iu_j$, because the product is commutative. Firstly we show that $u^3 = 0$. Using (3), (4) and (5), we have: $u^3 = \sum_{i=1}^p\sum_{j=1}^p \alpha_i^2\alpha_ju_i^2u_j + 2\sum_{1\leq i<j\leq p}\sum_{k=1}^p \alpha_i\alpha_j\alpha_k(u_iu_j)u_k = \sum_{1\leq i<j\leq p}\alpha_i^2\alpha_ju_i^2u_j + \sum_{i=1}^p \alpha_i^3u_i^3 + \sum_{1\leq j<i\leq p}\alpha_i^2\alpha_ju_i^2u_j$

$$+2\Big(\sum_{1\leq i<j\leq p}(\alpha_i^2\alpha_j(u_iu_j)u_i + \alpha_i\alpha_j^2(u_iu_j)u_j) + \sum_{1\leq i<j\leq p}\underbrace{\sum_{k=1}^{p}}_{k\neq i,j}\alpha_i\alpha_j\alpha_k(u_iu_j)u_k\Big)$$

$$= 2\sum_{1\leq i<j\leq p}\Big(\alpha_i^2\alpha_ju_{\rho_1(i,j)} + \alpha_i\alpha_j^2u_{\rho_2(i,j)} - \alpha_i^2\alpha_ju_{\rho_1(i,j)} - \alpha_i\alpha_j^2u_{\rho_2(i,j)}\Big) + \sum_{1\leq i<j\leq p}\underbrace{\sum_{k=1}^{p}}_{k\neq i,j}\alpha_i\alpha_j\alpha_k(u_iu_j)u_k$$

$$= 2\Big(\sum_{1\leq k<i<j\leq p}\alpha_i\alpha_j\alpha_k(u_iu_j)u_k + \sum_{1\leq i<j<k\leq p}\alpha_i\alpha_j\alpha_k(u_iu_j)u_k + \sum_{1\leq i<k<j\leq p}\alpha_i\alpha_j\alpha_k(u_iu_j)u_k\Big)$$

$$= 2\sum_{1\leq i<j<k\leq p}\Big(\alpha_j\alpha_k\alpha_iu_{\delta_2(i,j,k)} + \alpha_i\alpha_j\alpha_ku_{\delta_1(i,j,k)} - \alpha_i\alpha_j\alpha_k(u_{\delta_1(i,j,k)} + u_{\delta_2(i,j,k)})\Big) = 0.$$

Finally, we have: $(u^2)^2 = \Big(\sum_{i=1}^p \alpha_i^2u_i^2 + 2\sum_{1\leq i<j\leq p}\alpha_i\alpha_ju_iu_j\Big)^2 = \Big(\sum_{i=1}^p \alpha_i^2u_i^2\Big)^2 + 4\sum_{1\leq i<j\leq p}\sum_{k=1}^p \alpha_i\alpha_j\alpha_k^2(u_iu_j)u_k^2 + 4\Big(\sum_{1\leq i<j\leq p}\alpha_i\alpha_ju_iu_j\Big)^2 = 0$, because, according to (6), we have: $\Big(\sum_{i=1}^p \alpha_i^2u_i^2\Big)^2 = \sum_{i=1}^p \alpha_i^4(u_i^2)^2 + 2\sum_{1\leq i<j\leq p}\alpha_i^2\alpha_j^2u_i^2u_j^2 = 4\sum_{1\leq i<j\leq p}\alpha_i^2\alpha_j^2u_{\tau(i,j)}$;

From (7) it follows: $4\sum_{1\leq i<j\leq p}\sum_{k=1}^p \alpha_i\alpha_j\alpha_k^2(u_iu_j)u_k^2$

$$= 4\Big(\sum_{1\leq k<i<j\leq p}\alpha_i\alpha_j\alpha_k^2(u_iu_j)u_k^2 + \sum_{1\leq i<k<j\leq p}\alpha_i\alpha_j\alpha_k^2(u_iu_j)u_k^2 + \sum_{1\leq i<j<k\leq p}\alpha_i\alpha_j\alpha_k^2(u_iu_j)u_k^2\Big)$$

$$= 8\Big(\sum_{1\leq i<j<k\leq p}(\alpha_i^2\alpha_j\alpha_ku_{\tau_1(i,j,k)} + \alpha_i\alpha_j^2\alpha_ku_{\tau_2(i,j,k)} + \alpha_i\alpha_j\alpha_k^2u_{\tau_3(i,j,k)}\Big);$$

And using (6), (7) and (8) we obtain: $4\Big(\sum_{1\leq i<j\leq p}\alpha_i\alpha_ju_iu_j\Big)^2 = 4\sum_{1\leq i<j\leq p}\alpha_i^2\alpha_j^2(u_iu_j)^2 +$

$$8\underbrace{\sum_{1\le i<j\le p}\sum_{1\le k<l\le p}}_{(i,j)<(k,l)}\alpha_i\alpha_j\alpha_k\alpha_l(u_iu_j)(u_ku_l)=-4\sum\nolimits_{1\le i<j\le p}\alpha_i^2\alpha_j^2u_{\tau(i,j)}$$
$$+8\sum\nolimits_{1\le i<j<k\le p}\left(\alpha_i^2\alpha_j\alpha_k(u_iu_j)(u_iu_k)+\alpha_i\alpha_j^2\alpha_k(u_iu_j)(u_ju_k)+\alpha_i\alpha_j\alpha_k^2(u_iu_k)(u_ju_k)\right)$$
$$+8\sum\nolimits_{1\le i<j<k<l\le p}\alpha_i\alpha_j\alpha_k\alpha_l\left((u_iu_j)(u_ku_l)+(u_iu_k)(u_ju_l)+(u_iu_l)(u_ju_k)\right)$$
$$=\ -4\sum\nolimits_{1\le i<j\le p}\alpha_i^2\alpha_j^2u_{\tau(i,j)}\ -\ 8\sum\nolimits_{1\le i<j<k\le p}\left(\alpha_i^2\alpha_j\alpha_ku_{\tau_1(i,j,k)}\ +\ \alpha_i\alpha_j^2\alpha_ku_{\tau_2(i,j,k)}\ +\ \alpha_i\alpha_j\alpha_k^2u_{\tau_3(i,j,k)}\right)$$
$$+8\sum\nolimits_{1\le i<j<k<l\le p}\alpha_i\alpha_j\alpha_k\alpha_l\left(u_{\sigma_1(i,j,k,l)}+u_{\sigma_2(i,j,k,l)}-(u_{\sigma_1(i,j,k,l)}+u_{\sigma_2(i,j,k,l)})\right).$$

It follows that $(\mathbf{A},\omega)$ is Bernstein with $U_e=U$ and $V_e=V$. As $U_eV_e+V_e^2=\langle u_{p+1},u_{p+2},..,u_{p+t}\rangle$, $U_e^2=\langle v_1,v_2,...,v_s\rangle=V_e\neq 0$ and $U_e(U_eV_e)=0$, then $\mathbf{A}$ is 1-exceptional of type $(1+r,s)$ and subtype (t,s). □

The next example exhibits a Bernstein algebra of type $(1+44,10)$ and subtype $(40,10)$, constructed according to Theorem 3 for the triple $(44,10,40)$.

Example 2 Let $\mathbf{A}=Fe\oplus U\oplus V$, with $U=\langle u_1,u_2....,u_{44}\rangle$, $V=\langle v_1,v_2,...,v_{10}\rangle$, and multiplication table in $N=U\oplus V$ given by the following table:

Table of U^2

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	... u_{42}	u_{43}	u_{44}
u_1	v_1	v_2	v_3	v_4									
u_2	v_2	v_5	v_6	v_7									
u_3	v_3	v_6	v_8	v_9									
u_4	v_4	v_7	v_9	v_{10}									
u_5													
...													
u_{43}													
u_{44}													

Table of $UV+V^2$

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
u_1		$-u_5$	$-u_6$	$-u_7$	$2u_8$	u_{12}	u_{14}	$2u_9$	u_{16}	$2u_{10}$
u_2	$2u_5$	$-u_8$	$-u_{11}-u_{12}$	$-u_{13}-u_{14}$		$-u_{17}$	$-u_{18}$	$2u_{19}$	u_{22}	$2u_{20}$
u_3	$2u_6$	u_{11}	$-u_9$	$-u_{15}-u_{16}$	$2u_{17}$	$-u_{19}$	$-u_{21}-u_{22}$		$-u_{23}$	$2u_{24}$
u_4	$2u_7$	u_{13}	u_{15}	$-u_{10}$	$2u_{18}$	u_{21}	$-u_{20}$	$2u_{23}$	$-u_{24}$	
u_5										
...										
u_{44}										
v_1					$2u_{25}$	$2u_{31}$	$2u_{34}$	$2u_{26}$	$2u_{37}$	$2u_{27}$
v_2		$-u_{25}$	$-u_{31}$	$-u_{34}$		$-u_{32}$	$-u_{35}$	$2u_{33}$	u_{43}	$2u_{36}$
v_3		$-u_{31}$	$-u_{26}$	$-u_{37}$	$2u_{32}$	$-u_{33}$	u_{44}		$-u_{38}$	$2u_{39}$
v_4		$-u_{34}$	$-u_{37}$	$-u_{27}$	$2u_{35}$	$-u_{43}-u_{44}$	$-u_{36}$	$2u_{38}$	$-u_{39}$	
v_5	$2u_{25}$		$2u_{32}$	$2u_{35}$				$2u_{28}$	$2u_{40}$	$2u_{29}$
v_6	$2u_{31}$	$-u_{32}$	$-u_{33}$	$-u_{43}-u_{44}$		$-u_{28}$	$-u_{40}$		$-u_{41}$	$2u_{42}$
v_7	$2u_{34}$	$-u_{35}$	u_{44}	$-u_{36}$		$-u_{40}$	$-u_{29}$	$2u_{41}$	$-u_{42}$	
v_8	$2u_{26}$	$2u_{33}$		$2u_{38}$	$2u_{28}$		$2u_{41}$			$2u_{30}$
v_9	$2u_{37}$	u_{43}	$-u_{38}$	$-u_{39}$	$2u_{40}$	$-u_{41}$	$-u_{42}$		$-u_{30}$	
v_{10}	$2u_{27}$	$2u_{36}$	$2u_{39}$		$2u_{29}$	$2u_{42}$		$2u_{30}$		

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