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A note on a coherent system degradation data under a signature point process representation

by

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Abstract A branch of research related to discrete degradation processes is through marked point process and in particular, in this paper we consider a coherent system under its signature point process representation. In this setting the system failure time coincides with a ordered component failure time and the ordered components failure times, before the system failure time are interpreted as degradation times. Degradation models provides inferences about the system failure time distribution.

Keywords: Coherent system, smooth semi-martingale, signature point process, degradation data.

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1 Introduction

High reliability coherent systems generally require its individual components to have extremely high reliability over a long period of time. Short product development time, constructed from such components, puts a severe time constraint on reliability tests and frequently, few or no failures occur during such tests. Thus, it is difficult to assess reliability with traditional life tests which record only systems failure time. When degradation measures can be taken over time, a physical relationship between component failure, and degradation makes possible the use of degradation models to provide inferences about the failure time distribution. Therefore, using degradation measures properly will help engineers to assess coherent systems reliability more quickly and accurately.

Continuous degradation models have been applied to assess device reliability for decades. When the degradation measurements follows a Wiener process, the corresponding failure distribution is well known as an inverse Gaussian distribution (see Park, C. and Padgett, W.J., (2005) and (2006)). However, various types of degradation processes do not all occur in a continuous pattern. Discrete degradation processes have been observed and a branch of research related to then is through marked point process. In particular we can consider a coherent system under its signature point process representation in which the system distribution function lifetime is a linear combination of the ordered component lifetimes distribution functions where the combination coefficients are the signatures. In this setting the system failure time coincides with a ordered component failure time and the ordered components failure times, before the system failure are interpreted as degradation times.

2 The Signature Marked Point Process

2.1 The mathematical details

In our general setup, we consider the vector $(T_1, ..., T_n)$ of n component lifetimes of a coherent system with lifetime S, which are finite and positive random variables defined in a complete probability space $(\Omega, \Im, P)$, with $P(T_i \neq T_j) = 1$, for all $i \neq j, i, j$ in $C = \{1, ..., n\}$, the index set of components. The component lifetimes can be dependent but simultaneous failures are ruled out. As in Barlow and Proschan (1981),the system lifetime and its components can be related by the series parallel decomposition:

$$S = \phi(\mathbf{T}) = \min_{1\leq j\leq k} \max_{i\in K_j} T_i,$$

where $K_j, 1 \leq j \leq k$ are minimal cut sets, that is, a minimal set of components whose joint failure causes the system fail.

However the evolution of components in time define a marked point process given through the failure times and the corresponding marks.

We denote by $T_{(1)} < T_{(2)} < ... < T_{(n)}$ the ordered lifetimes $T_1, T_2, ..., T_n$, as they appear in time and by $X_i = \{j : T_{(i)} = T_j\}$ the corresponding marks. As a convention we set $T_{(n+1)} = T_{(n+2)} = ... = \infty$ and $X_{n+1} = X_{n+2} = ... = e$ where e is a fictitious mark not in C, the index set of the components. Therefore the sequence $(T_n, X_n)_{n\geq 1}$ defines a marked point process.

The mathematical description of our observations, the complete information level, is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t\geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(i)}>s\}}, X_i = j, 1 \leq i \leq n, j \in C, 0 < s \leq t\},$$

satisfies the Dellacherie conditions of right continuity and completeness.

Intuitively, at each time t the observer knows if the event $\{T_{(i)} \leq t, X_i = j\}$ have either occurred or not and if it had, he knows exactly the value $T_{(i)}$ and the mark X_i. Follows that the component and the system lifetimes are $\Im_t$ stopping times.

We consider the lifetimes $T_{(i),j}$ defined by the failure event $\{T_{(i)}, X_i = j\}$ with their subdistribution function $F_{(i),j}(t) = P(T_{(i),j} \leq t) = P(T_{(i)} \leq t, X_i = j)$ suitable standardized.

In what follows we assume that relations between random variables and measurable sets, respectively, always hold with probability one, which means that the term P-a.s., is suppressed.

Remark 2.1.1 An extended and positive random variable τ is an $\Im_t$-stopping time if, and only if, $\{\tau \leq t\} \in \Im_t$, for all $t \geq 0$; an $\Im_t$-stopping time τ is called predictable if an increasing sequence $(\tau_n)_{n\geq 0}$ of $\Im_t$-stopping time, $\tau_n < \tau$, exists such that $\lim_{n\to\infty} \tau_n = \tau$; an $\Im_t$-stopping time τ is totally inaccessible if $P(\tau = \sigma < \infty) = 0$ for all predictable $\Im_t$-stopping time σ. For a mathematical basis of stochastic processes applied to reliability theory see the books of Aven and Jensen(2009) and Bremaud (1981).

The marked point $N_t((i), j) = 1_{\{T_{(i)}\leq t, X_i=j\}}$ is an $\Im_t$-sub-martingale, that is, $T_{(i),j}$ is $\Im_t$-measurable and $E[N_t((i), j)|\Im_s] \geq N_s((i), j)$ for all $0 \leq s \leq t$.

From the Doob-Meyer decomposition, there exists a unique $\Im_t$-predictable process, $(A_t((i), j)_{t\geq 0}$, called the $\Im_t$-compensator of $N_t((i), j)$, with $A_0((i), j) = 0$ and such that $M_t((i), j) = N_t((i), j) - A_t((i), j)$ is a zero mean uniformly integrable $\Im_t$-martingale. We assume that $T_i, 1 \leq i \leq n$ are totally inaccessible $\Im_t$-stopping time and, under this assumption, $A_t((i), j)$ is continuous. In certain sense, an absolutely continuous lifetime is totally inaccessible. Resuming, we assume a general lifetime model for $T_{(i),j}$, represented by the smooth $\Im_t$-semi-martingale:

$$1_{\{T_{(i),j} \leq t\}} = \int_0^t 1_{\{T_{(i),j} > s\}} \lambda_s((i), j) ds + M_t((i), j).$$

The process $(\lambda_t((i), j))_{t \geq 0}$ is called the intensity process of the semi-martingale representation and generalizes the classical notion of hazard rate. Intuitively indicates the prominence for failure, on the basis of all observations available up to, but not including, the present. As $N_t((i), j)$ can only count on the time interval $(T_{(i-1)}, T_{(i)}]$, the corresponding compensator differential $\lambda_t((i), j)$ must vanish outside this interval.

Note that, to count the $i-th$ failure we let $N_t((i)) = \Sigma_{j \geq 1} N_t((i), j)$ with $\Im_t$-compensator process $A_t((i)) = \Sigma_{j \geq 1} A_t((i), j)$. $N_t(j) = \Sigma_{i \geq 1} N_t((i), j)$, counts the component failure and it has $\Im_t$-compensator process $A_t(j) = \Sigma_{i \geq 1} A_t((i), j)$.

In this fashion we have the survival functions

$$P(T_{(i),j} > t) = E[e^{\int_0^t 1_{\{T_{(i),j} > s\}} \lambda_s((i), j) ds}],$$

and the probability densities

$$f_{T_{(i),j}}(t) = E[1_{\{T_{(i),j} > t\}} \lambda_t((i), j) e^{\int_0^t 1_{\{T_{(i),j} > s\}} \lambda_s((i), j) ds}].$$

2.2 The signature marked point process

The behavior of the stochastic process $P(S > t | \Im_t)$, as the information flows continuously in time is given by Bueno (2013):

Theorem 2.2.1 Let $T_1, T_2, ..., T_n$ be the component lifetimes of a coherent system with lifetime T. Then,

$$P(S \leq t | \Im_t) = \sum_{k,j=1}^{n} 1_{\{S = T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}}.$$

Proof From the total probability rule we have $P(S \leq t | \Im_t) =$

$$\sum_{k,j=1}^{n} P(\{S \leq t\} \cap \{S = T_{(k),j} | \Im_t) = \sum_{k,j=1}^{n} E[1_{\{S = T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}} | \Im_t].$$

As S and $T_{(k),j}$ are $\Im_t$-stopping time and it is well known that the event $\{S = T_{(k),j}\} \in \Im_{T_{(k),j}}$ where

$$\Im_{T_{(k),j}} = \{A \in \Im_\infty : A \cap \{T_{(k),j} \leq t\} \in \Im_t, \forall t \geq 0\},$$

we conclude that $\{S = T_{(k),j}\} \cap \{T_{(k),j} \leq t\}$ is $\Im_t$-measurable.

Therefore $P(S \leq t|\Im_t) =$

$$\sum_{k,j=1}^{n} E[1_{\{S=T_{(k),j}\}}1_{\{T_{(k),j}\leq t\}}|\Im_t] = \sum_{k,j=1}^{n} 1_{\{S=T_{(k),j}\}}1_{\{T_{(k),j}\leq t\}}.$$

The above decomposition allows us to define the signature process at component level.

Definition 2.2.2 The vector $(1_{\{S=T_{(k),j}\}}, 1 \leq k, j \leq n)$ is defined as the marked point signature process of the system ϕ.

Remark 2.2.3

We can calculate the system reliability as

$$P(S \leq t) = E[P(S \leq t|\Im_t)] =$$

$$E[\sum_{k,j=1}^{n} 1_{\{S=T_{(k),j}\}}1_{\{T_{(k),j}\leq t\}}] = \sum_{k,j=1}^{n} P(\{S = T_{(k),j}\} \cap \{T_{(k),j} \leq t\}).$$

If the component lifetimes are independent and identically distributed we have,

$$P(S \leq t) = \sum_{k,j=1}^{n} P(S = T_{(k),j})P(T_{(k),j} \leq t)$$

recovering the classical result as in Samaniego (1985).

To calculate the $\Im_t$-compensator of $1_{\{S\leq t\}}$, where S is the system lifetime we consider the smooth semi-martingale representation in Section 2.1.

Corollary 2.2.4 Let $T_1, T_2, ..., T_n$, be the components lifetimes of a coherent system with lifetime T. Then, the $\Im_t$-submartingale $P(S \leq t|\Im_t)$, has the $\Im_t$-compensator

$$\sum_{k,j=1}^{n} \int_0^t 1_{\{T_{(k),j}>s\}}1_{\{S=T_{(k),j}\}}\lambda_s((k), j)ds.$$

Proof

We consider the process

$$1_{\{S=T_{(k),j}\}}(w, s) = 1_{\{S=T_{(k),j}\}}(w).$$

It is left continuous and $\Im_t$-predictable. Therefore

$$\int_0^t 1_{\{S=T_{(k),j}\}}(s)dM_s((k), j)$$

is an $\Im_t$-martingale.

As a finite sum of $\Im_t$-martingales is an $\Im_t$-martingale, we have

$$\sum_{k,j=1}^{n}\int_0^t 1_{\{S=T_{(k),j}\}}1_{\{T_{(k),j}\leq t\}} - \sum_{k,j=1}^{n}\int_0^t 1_{\{T_{(k),j}>s\}}1_{\{S=T_{(k),j}\}}\lambda_s((k),j)ds =$$

$$\sum_{k,j=1}^{n} 1_{\{S=T_{(k),j}\}}M_t((k),j)$$

is an $\Im_t$-martingale. As the compensator is unique we finish the proof.

3 Parameter estimation

Using the system distribution representation

$$P(S \leq t|\Im_t) = \sum_{k,j=1}^{n} 1_{\{S=T_{(k),j}\}}1_{\{T_{(k),j}\leq t\}},$$

we note that the system failure is equal to some ordered component failure. Under $\{S = T_{(i),j}\}$, for fixed i and some j, we consider the "components"failure times $t_{(k),j_k},\ \ k < i$, for some $j_k \in C = \{1, ..., n\}$ as degradation times and $t_{(i),j_i}$, for some $j_i, j_i \in C - \{j_1, j_2, ..., j_{i-1}\}$ as the system failure time.

The contribution of these degradation times to the likelihood function is

$$\pi_{k=1}^{i-1}\pi_{j_k\in C}E[\lambda_{t_{(k),j_k}}e^{\int_{t_{(k-1),j_{k-1}}}^{t_{(k),j_k}}\lambda_{s_{(k),j_k}}ds}].$$

The degradation times log-likelihood function is given by

$$\Sigma_{k=1}^{i-1}\Sigma_{j_k\in C}\log\{E[\lambda_{t_{(k),j_k}}e^{\int_{t_{(k-1),j_{k-1}}}^{t_{(k),j_k}}\lambda_{s_{(k),j_k}}ds}]\}.$$

For a random sample of n system, the degradation times log-likelihood function is

$$\ell_D(\theta) = \Sigma_{q=1}^{n}\Sigma_{k=1}^{i_q-1}\Sigma_{j_k\in C}\log\{E[\lambda^q_{t_{(k),j_k}}e^{\int_{t_{(k-1),j_{k-1}}}^{t^q_{(k),j_k}}\lambda^q_{s_{(k),j_k}}ds}]\}.$$

where θ denotes the vector of parameters to be estimated.

The contribution of the system failure data to the likelihood function is

$$\ell_S(\theta) = \Sigma_{q=1}^{n}E[\lambda_t((i),j)e^{\int_{t_{(i-1),j_{i-1}}}^{t}\lambda_s((i),j)ds}].$$

Consequently, the general log-likelihood function for either, failures times and degradation is given by

$$\ell(\theta) = \ell_D(\theta) + \ell_S(\theta).$$

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