Standardly stratified lower triangular algebras

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In this article k will denote an algebraicaly closed field, *algebra* always means finite dimensional basic k-algebra and module means finitely generated left module.

If Λ is an algebra, Λ-mod will denote the category of Λ-modules. Let us fix $\bar{e} = \{e_1, \ldots, e_n\}$ an ordered, complete set of primitive, orthogonal idempotents, and we also fix this linear order given by the indices.

Let P_i denote the indecomposable projective corresponding to e_i and S_i the simple module whose projective cover is P_i. For each i, we define the *standard module* $\Delta_\Lambda(i)$ to be the maximal quotient of P_i with composition factors among the S_j with $j \leq i$. Let Δ be the set of all these standard modules $\Delta_\Lambda(i)$. A Λ-module M will be called a *good*, or, more precisely, a Δ-good module, if there is a finite chain of submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$$

such that M_i/M_{i-1} is isomorphic to a module in Δ for all i. The number t does not depend on the filtration, we will call it the Δ-length of M and denote it by $l(M)$. The full subcategory of Λ-mod whose objects are the good modules is denoted by $\mathcal{F}_\Lambda(\Delta)$. The algebra Λ is said to be left standardly stratified if Λ is a good module.

We study here algebras which are given in lower triangular form, with respect to being standardly stratified. Let us begin by quoting two well known results of Dlab and Ringel [13, 8]. that will be useful.

Given Λ, a standardly stratified algebra, with respect to $\bar{e}$, let j be such that $1 \leq j \leq n$ and let us denote by ϵ_j the sum $\epsilon = e_j + \ldots + e_n$.

We now state:

Theorem 1 *1. The algebra $\Lambda/\Lambda\epsilon_j\Lambda$ is standardly stratified and the good-$\Lambda/\Lambda\epsilon_j\Lambda$ modules are the good Λ- modules anihilated by $\Lambda\epsilon_j\Lambda$.*

2. The algebra $\epsilon_j\Lambda\epsilon_j$ is standardly stratified, with respect to $\{e_j, \ldots, e_n\}$.

Now, we fix our notations:

U and V denote finite dimensional k-algebras, M a $V - U$-bimodule and A the finite dimensional k-algebra

$$A = \begin{bmatrix} U & 0 \\ M & V \end{bmatrix}.$$

Also, $\bar{g} = \{e_1, \ldots, e_t, f_{t+1}, \ldots f_{t+r}\}$ is a complete, ordered set of orthogonal, primitive idempotents of A, where $\bar{e} = \{e_1, \ldots, e_t\} \subset U$ and $\bar{f} = \{f_{t+1}, \ldots, f_{t+r}\} \subset V$ are complete sets of idempotents of U, V, respectively.

Remark 1 *We recall that there is an equivalence between the category of A-modules and the category $\mathcal{C}$ whose objects are triples (X, Y, f), with $X \in U$-mod, $Y \in V$-mod and $f : M \otimes_U X \to Y$, a V-module homomorphism. The morphisms between two objects (X, Y, f) and (X', Y', f') are pairs of morphisms (α, β), where $\alpha : X \to X'$ is an U-homomorphism and $\beta : Y \to Y'$ is a V-homomorphism, such that the diagram*

$$\begin{array}{ccc} M \otimes_U X & \stackrel{M \otimes \alpha}{\rightarrow} & M \otimes_U X' \\ \downarrow f & & \downarrow f' \\ Y & \stackrel{\beta}{\rightarrow} & Y' \end{array}$$

comutes.

In what follows, we abuse a little bit of the notations doing as if U and V were contained in A. This we accomplish by identifying elements $(x, 0, 0) \in A$ with $x \in U$ and elements $(0, x, 0) \in A$ with elements $x \in V$.

Remark 2 *It follows easily, from the definitions, that the set of standard A-modules is the union of the set of standard U-modules with the set of standard V-modules.*

Next we recall some well known facts, see [2].

The sequence $(A, B, f) \stackrel{(\alpha,\beta)}{\rightarrow} (A', B', f') \stackrel{(\alpha'\beta')}{\rightarrow} (A'', B'', f'')$ is exact, if and only if, the sequences $A \stackrel{\alpha}{\rightarrow} A' \stackrel{\alpha'}{\rightarrow} A''$ and $B \stackrel{\beta}{\rightarrow} B' \stackrel{\beta'}{\rightarrow} B''$ are exact. Moreover the indecomposable A-projective modules are of the form $(P, M \otimes_V P, Id)$ where $_UP$ is projective, or of the form $(0, Q, 0)$, where $_VQ$ is projective. The indecomposable injective objects in $\mathcal{C}$ are objects of the form $(I, 0, 0)$ where I is an indecomposable injective U-module or of the form $(\text{Hom}_V(M, J), J, \phi)$ where J is an indecomposable injective V-module and $\phi : M \otimes_U \text{Hom}_V(M, J) \to J$ is given by $\phi(m \otimes f) = f(m)$ for $m \in M$ and $f \in \text{Hom}_V(M, J)$.

Lemma 1 *The A-module $(X, Y, f) \in \mathcal{F}_A(\Delta)$, if and only if, $X \in \mathcal{F}_U(\Delta)$ and $Y \in \mathcal{F}_V(\Delta)$.*

PROOF. Let $L = (X, Y, f)$. Then we have the following filtration

$$L = A\epsilon_1 L \supseteq A\epsilon_2 L \supseteq \ldots \supseteq A\epsilon_{t-1} L \supseteq (0, Y, 0) \supseteq A\epsilon_{t+2} L \supseteq \ldots \supseteq A\epsilon_{t+r+1} L = 0.$$

Assuming that L is A-good we have that $L/(0, Y, 0) \simeq (X, 0, 0)$ is A-good, and it is annihilated by $\begin{bmatrix} 0 & 0 \\ M & V \end{bmatrix}$, it follows that $X \in \mathcal{F}_U(\Delta)$ and $Y \in \mathcal{F}_V(\Delta)$. The converse is analogous.

Proposition 1 *The algebra A is standardly stratified with respect to $\bar{g}$, if and only if the three following conditions are satisfied.*

a) U is standardly stratified with respect to $\bar{e}$.

b) V is standardly stratified with respect to $\bar{f}$.

c) $_VM \in \mathcal{F}_V(\Delta)$.

PROOF. Firstly let us assume that A is standardly stratified and prove that the three conditions hold. Since ${}_AA \in \mathcal{F}_A(\Delta)$ and $A = (U, M, 1) \coprod (0, V, 0)$ then, clearly, ${}_UU \in \mathcal{F}_U(\Delta)$ and $M \coprod V \in \mathcal{F}_V(\Delta)$. The converse follows analogously. **QED**

Corollary 1 *The algebra A is quasi-hereditary, if and only if, U and V are quasi-hereditary and $M \in \mathcal{F}_V(\Delta)$.*

PROOF. The statement follows straightforwardly from the remark and the previous proposition. **QED**

We now want to investigate conditions for lower triangular algebras which imply that the category of good modules is the category of modules of finite projective dimension.

So, let $A = \begin{bmatrix} U & 0 \\ M & V \end{bmatrix}$ and let us keep the notations above. Then there is an exact, full and faithfull functor V-mod$\rightarrow$ A-mod, given by $Y \mapsto (0, Y, 0)$, which takes projectives to projectives, and, also $\mathcal{F}_V(\Delta)$ into $\mathcal{F}_A(\Delta)$.

Theorem 2 [11] *Let $A = \begin{bmatrix} U & 0 \\ M & V \end{bmatrix}$ be such that ${}_VM$ has finite projective dimension. Then $pdL = ((X, Y, f) < \infty$ implies that ${}_VY$ and ${}_UX$ have both finite projective dimension.*

PROOF. It is always true that if L has finite projective dimension then ${}_UX$ also has. (The resolution of L induces a resolution of X).

We show now that ${}_VY$ also has finite projective dimension.

If L is A-projective then ${}_VY$ is in add $(M \coprod V)$ so it has finite projective dimension.

Let L be any A-module with finite projective resolution of the form:

$$0 \to P_n \to P_n - 1 \ldots \to P_0 \to L \to 0$$

This induces an exact sequence

$$0 \to Y_n \to Y_n - 1 \ldots \to Y_0 \to Y \to 0$$

where ${}_VY_i$ are in add $(M \coprod V)$ so, each Y_i has finite projective dimension, it follows that Y has finite projective dimension. **QED**

Theorem 3 *$\mathcal{F}_A(\Delta) = P^{<\infty}(A)$, if and only if, $\mathcal{F}_V(\Delta) = P^{<\infty}(V)$, $\mathcal{F}_U(\Delta) = P^{<\infty}(U)$ and $M \in P^{<\infty}(V)$.*

PROOF. Assume that $\mathcal{F}_A(\Delta) = P^{<\infty}(A)$, then ${}_AA$ is standardly stratified and it follows that $\mathcal{F}_V(\Delta) = P^{<\infty}(V)$ and, by proposition 1, ${}_VM$ has finite projective dimension.

To see that $\mathcal{F}_U(\Delta) = P^{<\infty}(U)$, let us take any U-module X in $P^{\infty}(U)$ and let us show, by induction on the projective dimension of X, that X is A-good and, therefore, U-good. The hypothesis imply that U is standardly stratified and, therefore, that the projective U-modules are good. Hence, let us assume that X has projective dimension equal to n and that all U-modules of projective dimension $n-1$ are U-good. We have an exact sequence:

$$0 \to (\Omega_U(X), M \otimes_U P(X), f) \to (P(X), M \otimes_U P(X), Id) \to (X, 0, 0) \to 0,$$

where $P(X)$ denotes the projective cover of X, and $\Omega_U(X)$ the first syzygy of X, which has projective dimension $n-1$. We also have the following exact sequence:

$$0 \to (0, M \otimes_U P(X), 0) \to (\Omega, M \otimes_U P(X), f) \to (\Omega_U X, 0, 0) \to 0.$$

By induction we have that $(\Omega_U X, 0, 0)$ has finite projective dimension and since $(0, M \otimes_U P(X), 0)$ is projective, it follows that $(\Omega, M \otimes_U P(X), f)$ has finite projective dimension too.

Now, using the first exact sequence, we conclude that $(X, 0, 0)$ has finite projective dimension and therefore it is A-good, which implies that X is U-good.

Let us assume now that $\mathcal{F}_V(\Delta) = P^{<\infty}(V)$, $\mathcal{F}_U(\Delta) = P^{<\infty}(U)$ and $M \in \mathcal{F}_V(\Delta) = P^{<\infty}(V)$. Then by proposition 1 A is standardly stratified. Take any A-module (X, Y, f) of finite projective dimension. Using theorem 2 and the fact that $(0, Y, 0)$ is good, we get that X has finite projective dimension and therefore it is U-good. It follows that (X, Y, f) is A-good. **QED**

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