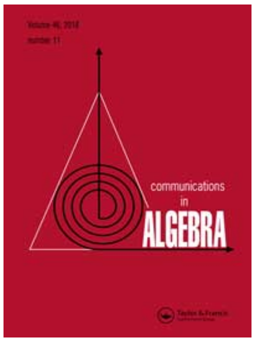

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Jordan derivations of alternative rings

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ABSTRACT
Let $\mathfrak{R}$ be a unital alternative ring with nontrivial idempotent and $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ be a Jordan derivation. Then $\mathfrak{D}$ is of the form $d + \delta$, where d is a derivation of $\mathfrak{R}$ and δ is a singular Jordan derivation of $\mathfrak{R}$. Moreover, d and δ are uniquely determined. This extends the main result of Benkovič and Širovnik's to the case of alternative rings.

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1. Introduction

The structures of derivations and Jordan derivations were studied systematically by many people (cf. [1–4, 8, 10–12, 15, 16]). It is obvious that every derivation is a Jordan derivation. But the converse is in general not true. Herstein [10] showed that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation. Brešar [3] proved that Herstein's result is true for 2-torsion free semiprime rings. Benkovič and Širovnik [2] proved that under certain conditions, every Jordan derivation is the sum of a derivation and a singular Jordan derivation. For the case of alternative rings, we can mention the Ferreira and Ferreira's paper [6] where they proved that under some conditions every Jordan multiplicative derivation on alternative rings are additive. In a more recent paper [7], the same authors make the study of the additivity of the Jordan triple multiplicative derivation on alternative rings, they also prove that under some conditions every Jordan triple multiplicative derivation on alternative rings are additive. These studies for nonassociative rings of Jordan's maps motivated us to ask the same question of Benkovič and Širovnik in the case in which the ring is alternative, that is, under which conditions a Jordan derivation is the sum of a derivation and a singular Jordan derivation? In this article, we give an explicit answer to this question, where the Benkovič and Širovnik's result is a consequence of our case.

2. Jordan derivation and alternative rings

Let $\mathfrak{R}$ be a unital ring not necessarily associative or commutative and consider the following convention for its multiplication operation: $xy \cdot z = (xy)z$ and $x \cdot yz = x(yz)$ for $x, y, z \in \mathfrak{R}$, to

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reduce the number of parentheses. We denote the *associator* of $\mathfrak{R}$ by $(x,y,z) = xy \cdot z - x \cdot yz$ for $x,y,z \in \mathfrak{R}$. And $[x,y] = xy - yx$ is the usual Lie product of x and y, with $x,y \in \mathfrak{R}$.

Let $\mathfrak{R}$ be a ring and $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ an $\mathfrak{R}$ -linear mapping of $\mathfrak{R}$ into itself. We call $\mathfrak{D}$ a *Jordan derivation* of $\mathfrak{R}$ into itself if

$$\mathfrak{D}(x^2) = \mathfrak{D}(x)x + x\mathfrak{D}(x) \tag{2.1}$$

holds true for all $x \in \mathfrak{R}$. Putting $x+y$ for x in (2.1), we obtain

$$\mathfrak{D}(xy+yx) = \mathfrak{D}(x)y + x\mathfrak{D}(y) + \mathfrak{D}(y)x + y\mathfrak{D}(x). \tag{2.2}$$

According to [5], we have the following:

Let $X = \{x_i\}_{i\in\mathbb{N}}$ be an arbitrary set of variables. A *nonassociative monomial of degree 1* is any element of X. Given a natural number $n > 1$, a *nonassociative monomial of degree n* is an expression of the form $(u)(v)$, where u is a nonassociative monomial of some degree i and v a nonassociative monomial of degree $n-i$. A *nonassociative polynomial p* over a ring $\mathfrak{R}$ is any formal linear combination of nonassociative monomials with coefficients in $\mathfrak{R}$. If p includes no variables except $x_1, x_2, \ldots, x_n$ and $a_1, a_2, \ldots, a_n$ is a set of elements of $\mathfrak{R}$, then $p(a_1, a_2, \ldots, a_n)$ is an element of $\mathfrak{R}$ which results by applying the sequence of operations forming p to $a_1, a_2, \ldots, a_n$ in place of $x_1, x_2, \ldots, x_n$.

Consider a mapping $\Xi : \mathfrak{R} \to \mathfrak{R}$ with the following property,

$$\Xi(p(x_1, \ldots, x_n)) = \sum_{i=1}^{n} p(x_1, \ldots, \Xi(x_i), \ldots, x_n) \tag{2.3}$$

where p is a nonassociative polynomial over a ring $\mathfrak{R}$.

Note that a Jordan derivation $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ satisfies this property for example to $p(x,y) = xy + yx, p(x,y,z) = xy \cdot z + zy \cdot x$ and $p(x,y,z) = x \cdot yz + z \cdot yx$.

A ring $\mathfrak{R}$ is said to be *alternative* if $(x,x,y) = 0 = (y,x,x)$ for all $x,y \in \mathfrak{R}$. One easily sees that any associative ring is an alternative ring. However, it is well known the existence of alternative rings that are not associative rings.

We refer the reader to [13, 14, 17] about basic facts of alternative rings.

An alternative ring $\mathfrak{R}$ is called *k-torsion free* if $k\ x = 0$ implies $x = 0$, for any $x \in \mathfrak{R}$, where $k \in \mathbb{Z}$, $k > 0$, and *prime* if $\mathfrak{A}\mathfrak{B} \neq 0$ for any two nonzero ideals $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{R}$. The *nucleus* of an alternative ring $\mathfrak{R}$ is defined by

$$\mathcal{N}(\mathfrak{R}) = \{r \in \mathfrak{R} \mid (x,y,r) = 0 = (x,r,y) = (r,x,y) \text{ for all } x,y \in \mathfrak{R}\}.$$

And the *center* of an alternative ring $\mathfrak{R}$ is defined by

$$\mathcal{Z}(\mathfrak{R}) = \{r \in \mathcal{N} \mid [r,x] = 0 \text{ for all } x \in \mathfrak{R}\}.$$

Theorem 2.1. *Let $\mathfrak{R}$ be a 3-torsion free alternative ring. Then $\mathfrak{R}$ is a prime ring if and only if $a\mathfrak{R} \cdot b = 0$ (or $a \cdot \mathfrak{R}b = 0$) implies $a = 0$ or $b = 0$ for $a,b \in \mathfrak{R}$.*

Proof. See [7, Theorem 1.1]. □

Definition 2.2. A ring $\mathfrak{R}$ is said to be *flexible* if satisfies

$$(x,y,x) = 0 \ \text{ for all } x,y \in \mathfrak{R}.$$

It is well known that alternative rings are flexible.

Proposition 2.3. *Let $\mathfrak{R}$ be an alternative ring. Then $\mathfrak{R}$ satisfies the relation*

$$(x,y,z) + (z,y,x) = 0 \ \text{ for all } x,y,z \in \mathfrak{R}.$$

Proof. It is sufficient to linearize the identity $(x, y, x) = 0$. □

Remark 2.4. Using $2xyx = x(xy + yx) + (xy + yx)x - (x^2y + yx^2)$ together with Proposition 2.3, (2.1), and (2.2), we can get

$$\mathfrak{D}(xyx) = \mathfrak{D}(x)y \cdot x + x\mathfrak{D}(y)x + xy \cdot \mathfrak{D}(x). \tag{2.4}$$

A nonzero element $e_1 \in \mathfrak{R}$ is called an *idempotent* if $e_1e_1 = e_1$ and a *nontrivial idempotent* if it is an idempotent different from the multiplicative identity element of $\mathfrak{R}$. Let us consider $\mathfrak{R}$ an alternative ring and fix a nontrivial idempotent $e_1 \in \mathfrak{R}$. Let $e_2 : \mathfrak{R} \to \mathfrak{R}$ and $e_2' : \mathfrak{R} \to \mathfrak{R}$ be linear operators given by $e_2(a) = a - e_1a$ and $e_2'(a) = a - ae_1$. Clearly $e_2^2 = e_2$, $(e_2')^2 = e_2'$ and we note that if $\mathfrak{R}$ has a unity, then we can consider $e_2 = 1 - e_1 \in \mathfrak{R}$. Let us denote $e_2(a)$ by e_2a and $e_2'(a)$ by ae_2. It is easy to see that $e_ia \cdot e_j = e_i \cdot ae_j$ $(i, j = 1, 2)$ for all $a \in \mathfrak{R}$. Then $\mathfrak{R}$ has a Peirce decomposition $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$, where $\mathfrak{R}_{ij} = e_i\mathfrak{R}e_j$ $(i, j = 1, 2)$ [9], satisfying the following multiplicative relations:

(i) $\mathfrak{R}_{ij}\mathfrak{R}_{jl} \subseteq \mathfrak{R}_{il}(i, j, l = 1, 2)$;
(ii) $\mathfrak{R}_{ij}\mathfrak{R}_{ij} \subseteq \mathfrak{R}_{ji}(i, j = 1, 2)$;
(iii) $\mathfrak{R}_{ij}\mathfrak{R}_{kl} = 0$, if $j \neq k$ and $(i, j) \neq (k, l), (i, j, k, l = 1, 2)$;
(iv) $x_{ij}^2 = 0$, for all $x_{ij} \in \mathfrak{R}_{ij}(i, j = 1, 2;\ i \neq j)$.

The next definition was first proposed in [2].

Definition 2.5. Let $\delta : \mathfrak{R} \to \mathfrak{R}$ be a Jordan derivation. We say that δ is a *singular Jordan derivation* if

$$\delta(\mathfrak{R}_{11}) = 0,\ \ \delta(\mathfrak{R}_{12}) \subseteq \mathfrak{R}_{21},\ \ \delta(\mathfrak{R}_{21}) \subseteq \mathfrak{R}_{12},\ \ \delta(\mathfrak{R}_{22}) = 0. \tag{2.5}$$

Remark 2.6. Nonzero singular Jordan derivations are not derivations.

In this article, we consider that $\mathfrak{R}$ is 2, 3-torsion free unital alternative ring satisfying the following conditions:

(♣) If $[x_{11}, \mathfrak{R}_{12} + \mathfrak{R}_{21}] = 0$, then $x_{11} = 0$;

(♠) If $[x_{22}, \mathfrak{R}_{12} + \mathfrak{R}_{21}] = 0$, then $x_{22} = 0$.

(◇) Let $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ be a Jordan derivation such that the property (2.3) is satisfied to $p(x, y, z) = xy \cdot z + z \cdot yx$.

Remark 2.7. Note that prime alternative rings satisfy the condition (♣), (♠).

Let us first see the condition (♣). Suppose that $x_{11}\mathfrak{R}_{12} = 0 = \mathfrak{R}_{21}x_{11}$. Then $x_{11}(\mathfrak{R}e_2) = 0 = (e_2\mathfrak{R})x_{11}$. Since $\mathfrak{R}$ is a 3-torsion free alternative ring and e_1 is a nontrivial idempotent, by Theorem 2.1, we have $x_{11} = 0$.

Let us now check the condition (♠). Assume that $\mathfrak{R}_{12}x_{22} = 0 = x_{22}\mathfrak{R}_{21}$. Then $(e_1\mathfrak{R})x_{22} = 0 = x_{22}(\mathfrak{R}e_1)$. Thus, $x_{22} = 0$, because e_1 is a nontrivial idempotent.

Observe that (◇) is true for any associative ring, which is due to (2.4).

3. Main result and its proof

We are now in a position to state the main result of this article.

Theorem 3.1. *Let $\mathfrak{R}$ be a 2, 3-torsion free unital alternative ring with nontrivial idempotent satisfying the conditions labeled by ♣, ♠, ◇, and $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ be a Jordan derivation, then $\mathfrak{D}$ is of the form $d + \delta$, where d is a derivation of $\mathfrak{R}$ and δ is a singular Jordan derivation of $\mathfrak{R}$. Moreover, d and δ are uniquely determined.*

First we provide some basic results concerning Jordan derivations of alternative rings. These results are natural generalizations of associative rings for alternative rings, which have appeared in [2]. Let $e_1 \in \mathfrak{R}$ be a nontrivial idempotent and $e_2 = 1 - e_1$.

Lemma 3.2. *Let $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ be a Jordan derivation. Then $\mathfrak{D} = \mathfrak{D}_1 + \mathfrak{D}_2$, where $\mathfrak{D}_1 : \mathfrak{R} \to \mathfrak{R}$ is a derivation and $\mathfrak{D}_2 : \mathfrak{R} \to \mathfrak{R}$ is a Jordan derivation such that $\mathfrak{D}_2(e) = 0$.*

Proof. Since $\mathfrak{D}$ is a Jordan derivation and $e_1^2 = e_1$, we know that

$$\mathfrak{D}(e_1^2) = \mathfrak{D}(e_1)e_1 + e_1\mathfrak{D}(e_1).$$

Left and right multiplication of the above relation by e_1 gives

$$e_1\mathfrak{D}(e_1)e_1 = e_1\mathfrak{D}(e_1)e_1 + e_1\mathfrak{D}(e_1)e_1.$$

It follows that $e_1\mathfrak{D}(e_1)e_1 = 0$. Similarly, we obtain $e_2\mathfrak{D}(e_1)e_2 = 0$. And hence $\mathfrak{D}(e_1) = e_1\mathfrak{D}(e_1)e_2 + e_2\mathfrak{D}(e_1)e_1$.

Let us set $x = e_1\mathfrak{D}(e_1)e_2 + e_2\mathfrak{D}(e_1)e_1 \in \mathfrak{R}$ and $z = e_1$. We define a new mapping $\mathfrak{D}_1^{(x,z)} : \mathfrak{R} \to \mathfrak{R}$ by $\mathfrak{D}_1^{(x,z)} = [L_x, L_z] + [L_x, R_z] + [R_x, R_z]$, where L and R denote the left and right multiplication operator, respectively. It is known that $\mathfrak{D}_1^{(x,z)}$ is a derivation with $\mathfrak{D}_1^{(x,z)}(e_1) = e_1\mathfrak{D}(e_1)e_2 + e_2\mathfrak{D}(e_1)e_1$. Clearly, $\mathfrak{D}_2 := \mathfrak{D} - \mathfrak{D}_1^{(x,z)}$ is a Jordan derivation and

$$\mathfrak{D}_2(e_1) = \mathfrak{D}(e_1) - \mathfrak{D}_1^{(x,z)}(e_1) = \mathfrak{D}(e_1) - (e_1\mathfrak{D}(e_1)e_2 + e_2\mathfrak{D}(e_1)e_1) = 0.$$

□

Before we continue to proceed with our discussion, it should be remarked that $[L_y, L_z] + [L_y, R_z] + [R_y, R_z]$ is a derivation. Thus without loss of generality, we may assume that $\mathfrak{D}(e_1) = 0$. Hence, $\mathfrak{D}(e_2) = \mathfrak{D}(1 - e_1) = \mathfrak{D}(1) - \mathfrak{D}(e_1) = 0$.

Proposition 3.3. *Let $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ be a Jordan derivation such that $\mathfrak{D}(e_1) = 0$. Then*

$$\mathfrak{D}(a_{11}) = e_1\mathfrak{D}(a_{11})e_1, \mathfrak{D}(a_{12}) = e_1\mathfrak{D}(a_{12})e_2 + e_2\mathfrak{D}(a_{12})e_1, \tag{3.1}$$

$$\mathfrak{D}(a_{22}) = e_2\mathfrak{D}(a_{22})e_2, \mathfrak{D}(a_{21}) = e_2\mathfrak{D}(a_{21})e_1 + e_1\mathfrak{D}(a_{21})e_2 \tag{3.2}$$

for all $a_{11} \in \mathfrak{R}_{11}, a_{12} \in \mathfrak{R}_{12}, a_{21} \in \mathfrak{R}_{21}$ and $a_{22} \in \mathfrak{R}_{22}$. Moreover, the following relations also hold true:

(a) $\mathfrak{D}(a_{11}a_{12}) = \mathfrak{D}(a_{11})a_{12} + a_{11}\mathfrak{D}(a_{12}) + \mathfrak{D}(a_{12})a_{11}$;
(b) $\mathfrak{D}(a_{21}a_{11}) = \mathfrak{D}(a_{21})a_{11} + a_{21}\mathfrak{D}(a_{11}) + a_{11}\mathfrak{D}(a_{21})$;
(c) $\mathfrak{D}(a_{12}a_{22}) = \mathfrak{D}(a_{12})a_{22} + a_{12}\mathfrak{D}(a_{22}) + a_{22}\mathfrak{D}(a_{12})$;
(d) $\mathfrak{D}(a_{22}a_{21}) = \mathfrak{D}(a_{22})a_{21} + a_{22}\mathfrak{D}(a_{21}) + \mathfrak{D}(a_{21})a_{22}$;
(e) $\mathfrak{D}(a_{12}a_{21}) = \mathfrak{D}(a_{12})a_{21} + a_{12}\mathfrak{D}(a_{21}) + z_1$;
(f) $\mathfrak{D}(a_{21}a_{12}) = \mathfrak{D}(a_{21})a_{12} + a_{21}\mathfrak{D}(a_{12}) + z_2$;
(g) $\mathfrak{D}(a_{12}b_{12}) = e_1\mathfrak{D}(a_{12})e_2b_{12} + a_{12}\mathfrak{D}(b_{12}) + b_{12}e_2\mathfrak{D}(a_{12})e_1$;
(h) $\mathfrak{D}(a_{21}b_{21}) = e_2\mathfrak{D}(a_{21})e_1b_{21} + a_{21}\mathfrak{D}(b_{21}) + b_{21}e_1\mathfrak{D}(a_{21})e_2$

for all $a_{11} \in \mathfrak{R}_{11}, a_{12}, b_{12} \in \mathfrak{R}_{12}, a_{21}, b_{21} \in \mathfrak{R}_{21}, a_{22} \in \mathfrak{R}_{22}$ and $z \in \mathfrak{R}_{12} + \mathfrak{R}_{21}$.

Proof. We will only prove (e)–(h) because (2.4) and (a)–(d) have the same demonstrations with items (3.2) and (i)–(iv) of the Proposition 2.2 in [2].

Let us embark on (e) and (f). Observe that

$$\begin{aligned}\mathfrak{D}(a_{12}a_{21}) + \mathfrak{D}(a_{21}a_{12}) &= \mathfrak{D}(a_{12}a_{21} + a_{21}a_{12}) \\ &= \mathfrak{D}(a_{12})a_{21} + a_{12}\mathfrak{D}(a_{21}) \\ &\quad + \mathfrak{D}(a_{21})a_{12} + a_{21}\mathfrak{D}(a_{12}) \\ &= e_1\mathfrak{D}(a_{12})e_2a_{21} + a_{12}e_2\mathfrak{D}(a_{21})e_1 \\ &\quad + e_2\mathfrak{D}(a_{21})e_1a_{12} + a_{21}e_1\mathfrak{D}(a_{12})e_2.\end{aligned}$$

Therefore, $\mathfrak{D}(a_{12}a_{21}) = \mathfrak{D}(a_{12})a_{21} + a_{12}\mathfrak{D}(a_{21}) + z_1$ and $\mathfrak{D}(a_{21}a_{12}) = \mathfrak{D}(a_{21})a_{12} + a_{21}\mathfrak{D}(a_{12}) + z_2$, where $z_1 = -(e_2\mathfrak{D}(a_{12})e_1a_{21} + a_{12}e_1\mathfrak{D}(a_{21})e_2) \in \mathfrak{R}_{12} + \mathfrak{R}_{21}$ and $z_2 = -(e_1\mathfrak{D}(a_{21})e_2a_{12} + a_{21}e_2\mathfrak{D}(a_{12})e_1) \in \mathfrak{R}_{12} + \mathfrak{R}_{21}$. This gives the statements (e) and (f). In condition ($\diamondsuit$), let us put $x = a_{12}, y = b_{12}, z = e_1$. Then we obtain

$$\begin{aligned}\mathfrak{D}(a_{12}b_{12}) &= \mathfrak{D}(a_{12})b_{12}\cdot e_1 + a_{12}\mathfrak{D}(b_{12})\cdot e_1 + e_1\cdot\mathfrak{D}(b_{12})a_{12} + e_1\cdot b_{12}\mathfrak{D}(a_{12}) \\ &= e_1\mathfrak{D}(a_{12})e_2b_{12} + a_{12}\mathfrak{D}(b_{12}) + b_{12}e_2\mathfrak{D}(a_{12})e_1,\end{aligned}$$

which implies the assertion (g). By an analogous manner, one can prove that property (h). □

Lemma 3.4. *Suppose that $\mathfrak{R}$ is a unital alternative ring with nontrivial idempotent e_1 such that (♣) and (♠) hold true. Assume that $d : \mathfrak{R} \to \mathfrak{R}$ is a linear mapping such that*

$$\begin{aligned}d(a_{11}) = e_1d(a_{11})e_1, d(a_{12}) = e_1d(a_{12})e_2, \\ d(a_{21}) = e_2d(a_{21})e_1, d(a_{22}) = e_2d(a_{22})e_2,\end{aligned}$$

and

(i) $d(a_{11}a_{12}) = d(a_{11})a_{12} + a_{11}d(a_{12})$ and $d(a_{12}a_{22}) = d(a_{12})a_{22} + a_{12}d(a_{22})$,
(ii) $d(a_{22}a_{21}) = d(a_{22})a_{21} + a_{22}d(a_{21})$ and $d(a_{21}a_{11}) = d(a_{21})a_{11} + a_{21}d(a_{11})$,
(iii) $d(a_{12}a_{21}) = d(a_{12})a_{21} + a_{12}d(a_{21})$ and $d(a_{21}a_{12}) = d(a_{21})a_{12} + a_{21}d(a_{12})$,
(iv) $d(a_{12}b_{12}) = d(a_{12})b_{12} + a_{12}d(b_{12})$ and $d(a_{21}b_{21}) = d(a_{21})b_{21} + a_{21}d(b_{21})$

for all $a_{11} \in \mathfrak{R}_{11}, a_{22} \in \mathfrak{R}_{22}, a_{12}, b_{12} \in \mathfrak{R}_{12}$ and $a_{21}, b_{21} \in \mathfrak{R}_{21}$. Then d is a derivation.

Proof. The proof of this lemma is similar to that of [2, Lemma 2.3]. It follows from Proposition 2.3 that

$$\begin{aligned}(x_{11}, y_{11}, z_{12}) + (z_{12}, y_{11}, x_{11}) = 0, \\ (x_{11}, y_{11}, z_{21}) + (z_{21}, y_{11}, x_{11}) = 0, \\ (x_{22}, y_{22}, z_{12}) + (z_{12}, y_{22}, x_{22}) = 0,\end{aligned}$$

and

$$(x_{22}, y_{22}, z_{21}) + (z_{21}, y_{22}, x_{22}) = 0$$

for all $x_{11}, y_{11} \in \mathfrak{R}_{11}, x_{22}, y_{22} \in \mathfrak{R}_{22}, z_{12} \in \mathfrak{R}_{12}$ and $z_{21} \in \mathfrak{R}_{21}$. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: Let $\mathfrak{R}$ be a unital alternative ring with a nontrivial idempotent e_1, satisfying the properties (♣), (♠), ($\diamondsuit$) and let $\mathfrak{D} : \mathfrak{R} \to \mathfrak{R}$ be a Jordan derivation. Without loss of generality, we may assume that $\mathfrak{D}(e_1) = 0 = \mathfrak{D}(e_2)$. Let us define a mapping $d : \mathfrak{R} \to \mathfrak{R}$ in the following way:

$$\begin{aligned} d(a_{11}) &= e_1\mathfrak{D}(a_{11})e_1 = e_1 d(a_{11})e_1,\\ d(a_{22}) &= e_2\mathfrak{D}(a_{22})e_2 = e_2 d(a_{22})e_2,\\ d(a_{12}) &= e_1\mathfrak{D}(a_{12})e_2 = e_1 d(a_{12})e_2,\\ d(a_{21}) &= e_2\mathfrak{D}(a_{21})e_1 = e_2 d(a_{21})e_1 \end{aligned}$$

for all $a_{11} \in \mathfrak{R}_{11}, a_{22} \in \mathfrak{R}_{22}, a_{12} \in \mathfrak{R}_{12}$ and $a_{21} \in \mathfrak{R}_{21}$. Let $\delta : \mathfrak{R} \to \mathfrak{R}$ be the mapping $\delta = \mathfrak{D} - d$. By invoking (3.1), we know that

$$\begin{aligned} &\delta(a_{11}) = 0, \delta(a_{22}) = 0,\\ &\delta(a_{12}) = e_2\mathfrak{D}(a_{12})e_1 = e_2\delta(a_{12})e_1,\\ &\delta(a_{21}) = e_1\mathfrak{D}(a_{21})e_2 = e_1\delta(a_{21})e_2 \end{aligned}$$

for all $a_{11} \in \mathfrak{R}_{11}, a_{22} \in \mathfrak{R}_{22}, a_{12} \in \mathfrak{R}_{12}$ and $a_{21} \in \mathfrak{R}_{21}$. It is not difficult to see that δ is a singular Jordan derivation. Let us now show that d is a derivation. For this, it suffices to show that d satisfies the conditions (i)–(iv) of Lemma 3.4. Clearly, the conditions (a)–(d) of the Proposition 3.3 imply (i) and (ii) of the Lemma 3.4. By the condition (e) in Proposition 3.3, we have

$$\mathfrak{D}(a_{12}a_{21}) = e_1\mathfrak{D}(a_{12})e_2a_{21} + a_{12}e_2\mathfrak{D}(a_{21})e_1. \tag{3.3}$$

Then (3.3) can be rewritten as

$$\begin{aligned} e_1\mathfrak{D}(a_{12}a_{21})e_1 = {} & e_1\mathfrak{D}(a_{12})e_2a_{21} + e_2\mathfrak{D}(a_{12})e_1a_{21}\\ & + a_{12}e_2\mathfrak{D}(a_{21})e_1 + a_{12}e_1\mathfrak{D}(a_{21})e_2 + z_1. \end{aligned}$$

By the definition of d, we see that $d(a_{12}a_{21}) = d(a_{12})a_{21} + a_{12}d(a_{21})$. Combining the condition (f) of Proposition 3.3 with the definition of d yields $d(a_{21}a_{12}) = d(a_{21})a_{12} + a_{21}d(a_{12})$. It remains to verify condition (iv) of the Lemma 3.4. In view of Proposition 3.3 and condition (g) of Proposition 3.3, we conclude that

$$\mathfrak{D}(a_{12}b_{12}) = e_1\mathfrak{D}(a_{12})e_2b_{12} + a_{12}\mathfrak{D}(b_{12}) + b_{12}e_2\mathfrak{D}(a_{12})e_1.$$

Note that $e_2\mathfrak{D}(a_{12}b_{12})e_1 = e_1\mathfrak{D}(a_{12})e_2b_{12} + a_{12}e_1\mathfrak{D}(b_{12})e_2$. This implies $d(a_{12}b_{12}) = d(a_{12})b_{12} + a_{12}d(b_{12})$, which is due to the definition of d. Similarly, using condition (h) of Proposition 3.3 and the definition of d, we get $d(a_{21}b_{21}) = d(a_{21})b_{21} + a_{21}d(b_{21})$. According to Lemma 3.4, we say that d is a derivation. Note that the uniqueness of d and δ is verified by Remark 2.6. The proof of the Theorem 3.1 is now complete.

As a consequence, we have the following.

Corollary 3.5. [2, Theorem 4.1] *Let A be a 2, 3-torsion free unital associative algebra with a nontrivial idempotent e satisfying the conditions (♣) and (♠). If $\Delta : A \to A$ is a Jordan derivation, then there exist a derivation $d : A \to A$ and a singular Jordan derivation $\delta : A \to A$ such that $\Delta = d + \delta$. Moreover, d and δ are uniquely determined.*

Corollary 3.6. *Let A be a 2, 3-torsion free unital prime associative algebra with a nontrivial idempotent e. Then each Jordan derivation of A is a derivation.*

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