Article

Three Representation Types for Systems of Forms and Linear Maps

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Abstract: We consider systems of bilinear forms and linear maps as representations of a graph with undirected and directed edges. Its vertices represent vector spaces; its undirected and directed edges represent bilinear forms and linear maps, respectively. We prove that if the problem of classifying representations of a graph has not been solved, then it is equivalent to the problem of classifying representations of pairs of linear maps or pairs consisting of a bilinear form and a linear map. Thus, there are only two essentially different unsolved classification problems for systems of forms and linear maps.

Keywords: representations of quivers; representations of mixed graphs; wild problems

MSC: 15A04; 15A21; 16G20; 16G60

1. Introduction

We show that the problem of classifying pairs consisting of a bilinear form and a linear map plays the same role in the theory of systems of bilinear forms and linear maps as the problem of classifying pairs of linear maps plays in the theory of representations of finite dimensional algebras.

About fifty years ago, it was noticed that most unsolved classification problems in the theory of representations of groups and algebras "contain" the matrix pair problem, which is the problem of classifying pairs of square matrices of the same size up to similarity transformations

$$(A, B) \mapsto (S^{-1}AS, S^{-1}BS), \qquad S \text{ is nonsingular.} \tag{1}$$

(A classification problem $\mathcal{M}$ contains a classification problem $\mathcal{N}$ if solving $\mathcal{M}$ would solve $\mathcal{N}$.)

For example, Bašev [1] classified representations of the abelian group $(2,2)$ over a field of characteristic 2. The problems of classifying representations of the abelian groups $(2,4)$ and $(2,2,2)$ over a field of characteristic 2 and the abelian group (p,p) over a field of characteristic $p > 2$ are considered as hopeless since these problems contain the matrix pair problem (Brenner [2] and Krugljak [3]).

Donovan and Freislich [4] call a matrix problem "wild" if it contains the matrix pair problem and "tame" otherwise, in analogy with the partition of animals into wild and tame ones. A certain characterization of tame and wild problems is given by Drozd [5] and Crawley-Boevey [6]; a geometric form of the tame–wild theorem is proved by Gabriel, Nazarova, Roiter, Sergeichuk, and Vossieck [7], and by Sergeichuk [8].

The reason for complexity of the matrix pair problem was found by Gelfand and Ponomarev [9], who showed that the problem of classifying pairs of commuting nilpotent matrices over any field up to similarity transformations (1) contains the problem of classifying matrix t-tuples up to similarity transformations

$$(M_1, \ldots, M_t) \mapsto (C^{-1}M_1C, \ldots, C^{-1}M_tC), \qquad C \text{ is nonsingular.}$$

The matrix pair problem also contains the problem of classifying representations of every quiver and every poset (Barot [10] [Section 2.4], Belitskii and Sergeichuk [11], Krause [12] [Section 10]). Moreover, it contains the problem of classifying representations of an arbitrary finite-dimensional algebra (Barot [13] [Proposition 9.14]).

Note that each concrete pair (A, B) of $n \times n$ matrices over an algebraically closed field is reduced to its canonical form $(A_{\text{can}}, B_{\text{can}})$ with respect to similarity transformations (1) by Belitskii's algorithm [14] in such a way that (A, B) is similar to (C, D) if and only if $(A_{\text{can}}, B_{\text{can}}) = (C_{\text{can}}, D_{\text{can}})$. However, there is no nonalgorithmic description of the set of Belitskii's canonical pairs under similarity (i.e., the pairs that are not changed by Belitskii's algorithm). Belitskii's algorithm was extended by Sergeichuk [8] to a wide class of matrix problems that includes the problems of classifying representations of quivers and finite dimensional algebras.

The problem of classifying arrays up to equivalence plays the same role in the theory of tensors as the matrix pair problem in the theory of representations of algebras: Futorny, Grochow, and Sergeichuk [15] proved that the problem of classifying three-dimensional arrays up to equivalence transformations contains the problem of classifying every system of tensors of order at most three.

We show that the problem of classifying pairs consisting of a bilinear form and a linear map contains the problem of classifying arbitrary systems of bilinear forms and linear maps.

2. Main Results

Many classification problems of linear algebra can be formulated and studied in terms of quiver representations introduced by Gabriel [16]. A quiver is a directed graph. Its representation is given by assigning a vector space to each vertex and a linear map of the corresponding vector spaces to each arrow. This notion plays a central role in the representation theory of finite dimensional algebras since each algebra can be given by a quiver with relations and there is a natural correspondence between their representations; see [10,13,17–19].

Following [20,21], we consider systems of forms and linear maps over a field $\mathbb{F}$ as representations of a mixed graph (i.e., of a graph with undirected and directed edges; multiply edges and loops are allowed): Its vertices represent vector spaces, and its undirected and directed edges represent bilinear forms and linear maps between these spaces. Two representations are isomorphic if these are a set of linear bijections of the corresponding vector spaces that transform one representation into the other; see Definition 1.

Example 1. *Consider a mixed graph Q and its representation $\mathcal{R}$:*

$$(2)$$
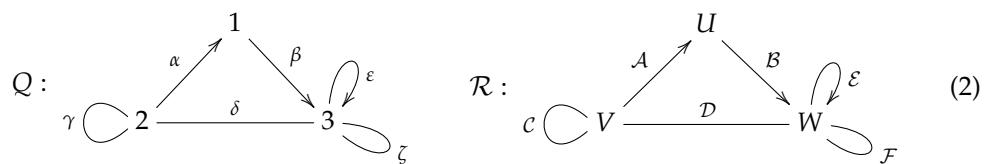
The representation $\mathcal{R}$ consists of vector spaces U, V, W over $\mathbb{F}$, bilinear forms $\mathcal{C} : V \times V \to \mathbb{F}$, $\mathcal{D} : W \times V \to \mathbb{F}$, $\mathcal{F} : W \times W \to \mathbb{F}$, and linear maps $\mathcal{A} : V \to U$, $\mathcal{B} : U \to W$, $\mathcal{E} : W \to W$.

The vector $\underline{n} = (n_1, n_2, n_3) := (\dim U, \dim V, \dim W)$ is called the dimension of $\mathcal{R}$. Changing bases in the spaces U, V, W, we can reduce the matrices of $\mathcal{R}$ as follows:

$$(3)$$
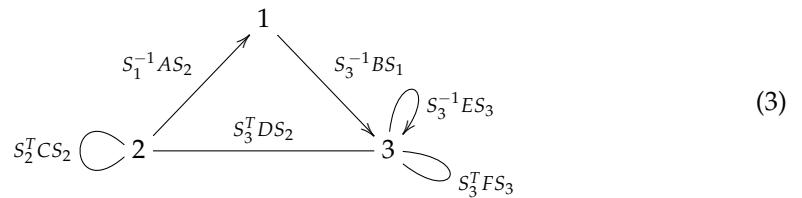
where S_1, S_2, S_3 are nonsingular matrices. Thus, the problem of classifying representations of Q of dimension $\underline{n} = (n_1, n_2, n_3)$ is the problem of classifying all tuples $(A, B, \dots, F)$ of matrices of sizes $n_1 \times n_2, n_3 \times n_1, \dots, n_3 \times n_3$ up to these transformations.

For a mixed graph Q, we denote by $M_{\underline{n}}(Q)$ the set of representations of dimension $\underline{n}$, in which all vector spaces are of the form $\mathbb{F}^k$ with $k = 0, 1, 2, \dots$. The set $M_{\underline{n}}(Q)$ is a vector space over $\mathbb{F}$; its elements are matrix tuples (see Definition 2). We say that the problem of classifying representations of a mixed graph Q is contained in the problem of classifying representations of a mixed graph Q' if for each dimension $\underline{n}$ there exists an affine injection (That is, $F(A) = R + \varphi(A)$ for $R \in M_{\underline{n}'}(Q')$ and all $A \in M_{\underline{n}}(Q)$, in which $\varphi : M_{\underline{n}}(Q) \to M_{\underline{n}'}(Q')$ is a linear injection).

$$F : M_{\underline{n}}(Q) \to M_{\underline{n}'}(Q')$$

of a very special type given in Definition 3 such that $A, B \in M_{\underline{n}}(Q)$ are isomorphic if and only if $F(A), F(B) \in M_{\underline{n}'}(Q')$ are isomorphic.

The problem of classifying representations of Q and Q' are equivalent if each of these problems contains the other.

The main result of this paper is the following theorem, which is proved in Sections 4 and 5.

Theorem 1.

(a) *The problem of classifying representations of $\bigcirc \bullet \circlearrowright$ (i.e., of pairs consisting of a bilinear form and a linear map) contains the problem of classifying representations of each mixed graph.*

(b) *Let a mixed graph Q satisfy the following condition:*

> *Q contains a cycle in which the number of undirected edges is odd, and Q contains an edge outside of this cycle but with a vertex (or both the vertices) in this cycle.* $\qquad (4)$

Then the problem of classifying representations of Q is equivalent to the problem of classifying representations of $\bigcirc \bullet \circlearrowright$.

Let us derive a corollary of Theorem 1.

Let Q be a connected mixed graph that does not satisfy the condition (4). The representations of mixed graphs that are cycles are classified in [20] [§ 3]. Suppose that Q does not contain a cycle in which the number of undirected edges is odd; in particular, it does not contain undirected loops. Then Q can be transformed to a quiver $\overline{Q}$ with the same underlying graph (obtained by deleting the orientation of the edges) using the following procedure described in [20]:

If v is a vertex of Q, then we denote by Q^v the graph that is obtained from Q by deleting (resp., adding) the arrows at the ends of all edges in the vertex v that have it (resp., do not have it); we say that Q^v is obtained from Q by dualization at v. For example:

(we write v^* instead of v). Thus, $v \longrightarrow w$ is replaced by $v^* \longleftrightarrow w$. Then we replace w with w^* and obtain $v^* \longleftarrow w^*$. Since Q does not contain a cycle in which the number of undirected edges is odd, we can reduce Q to some quiver $\overline{Q}$ by these replacements.

There is a natural correspondence between the representations of Q and Q^v: If $\mathcal{A}$ is a representation of Q, then the representation $\mathcal{A}^v$ of Q^v is obtained by replacing the vector space V assigned to v with the dual space V^* of all linear forms on V. This correspondence is based on the fact that each bilinear form $B : V \times U \to \mathbb{F}$ defines the linear map $U \to V^*$ via $x \mapsto B(?, x)$, and each linear map $\mathcal{A} : U \to V$ defines the bilinear form $V^* \times U \to \mathbb{F}$ via $(x*, y) \mapsto x^* A y$.

Therefore, the theory of representations of mixed graphs without cycles in which the number of undirected edges is odd is the theory of quiver representations. The representation types of quivers are well known; the representations of tame quivers are classified by Donovan and Freislich [4], and Nazarova [22]. The other quivers are wild; the problem of classifying representations of each wild quiver is equivalent to the problem of classifying representations of $\circlearrowright \bullet \circlearrowleft$ (i.e., of matrix pairs up to similarity transformations (1)); see [10–12] and Lemma 1.

We say that a mixed graph Q is tame if it is reduced by dualizations at vertices to the disjoint union of mixed cycles and a tame quiver (thus, the classification of representations of tame mixed graphs is known). A mixed graph Q is wild if it is reduced by dualizations at vertices to a wild quiver. A mixed graph Q is superwild if the problem of classifying its representations is equivalent to the problem of classifying representations of $\subset \bullet \circlearrowleft$.

Corollary 1. *Each mixed graph is tame, wild, or superwild.*

(i) *A mixed graph is tame if its underlying graph is the disjoint union of some copies of the Dynkin diagrams and the extended Dynkin diagrams*

$$(5)$$
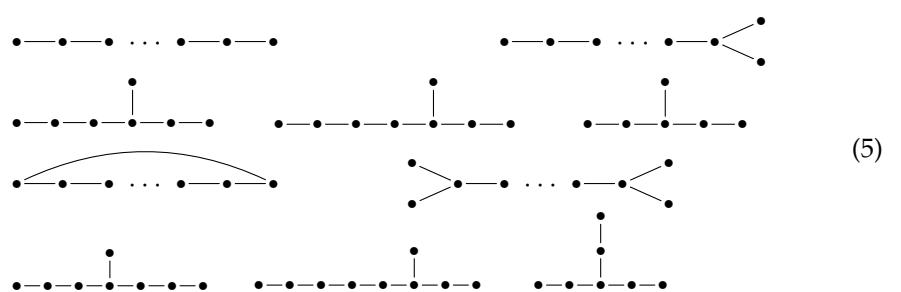
(ii) *A mixed graph is wild if it is not tame and it does not contain a cycle in which the number of undirected edges is odd.*

(iii) *A mixed graph is superwild if it satisfies condition* (4).

Proof. Let Q be a mixed graph. If (4) holds, then Q is superwild by Theorem 1(c). Suppose that (4) does not hold. Then Q is the disjoint union of mixed cycles and a mixed graph Q_0 without a cycle in which the number of undirected edges is odd. The graph Q_0 is reduced to a quiver $\overline{Q}_0$ by dualizations at vertices. By definition, Q is tame or wild if and only if $\overline{Q}_0$ is tame or wild, respectively. By [4,22], $\overline{Q}_0$ is tame if and only if its underlying graph is the disjoint union of some copies of graphs (5). $\square$

3. The Category of Representations

The category of representations of a mixed graph (in general, nonadditive) is defined as follows.

Definition 1 ([21]; see also [18])**.** *Let Q be a mixed graph with vertices $1, \ldots, t$. Its representation $\mathcal{A}$ over a field $\mathbb{F}$ is given by assigning a vector space $\mathcal{A}_v$ over $\mathbb{F}$ to each vertex v, a bilinear form $\mathcal{A}_\alpha : \mathcal{A}_v \times \mathcal{A}_u \to \mathbb{F}$ to each undirected edge $\alpha : u \text{ ——— } v$ with $u \leqslant v$ (this inequality is given for uniqueness), and a linear map $\mathcal{A}_\beta : \mathcal{A}_u \to \mathcal{A}_v$ to each directed edge $\beta : u \longrightarrow v$. A morphism $\varphi : \mathcal{A} \to \mathcal{B}$ of two representations of Q is a family of linear maps $\varphi_1 : \mathcal{A}_1 \to \mathcal{B}_1, \ldots, \varphi_t : \mathcal{A}_t \to \mathcal{B}_t$ such that $\mathcal{A}_\alpha(y, x) = \mathcal{B}_\alpha(\varphi_v y, \varphi_u x)$ for each $\alpha : u \text{ ——— } v$ with $u \leqslant v$, and $\varphi_v \mathcal{A}_\beta = \mathcal{B}_\beta \varphi_u$ for each $\beta : u \longrightarrow v$.*

In the following definition, we consider representations, in which all vector spaces are $\mathbb{F}^k$ with $k = 0, 1, 2, \ldots$. Such representations are given by their matrices, as in (3), and so we call them "matrix representations". The category $M(Q)$ of matrix representations is defined as follows.

Definition 2. *Let Q be a mixed graph with vertices $1, \ldots, t$. A matrix representation A of dimension $\underline{n} = (n_1, \ldots, n_t)$ of Q over a field $\mathbb{F}$ is given by assigning a matrix $A_\alpha \in \mathbb{F}^{n_v \times n_u}$ to each undirected edge $\alpha : u \text{ ——— } v$ with $u \leqslant v$ and to each directed edge $\alpha : u \longrightarrow v$. A morphism $\varphi : A \to B$ of matrix representations of dimensions $\underline{n}$ and $\underline{n}'$ is a family of matrices $S_1, \ldots, S_t$ of sizes $n_1' \times n_1, \ldots, n_t' \times n_t$ such that*

$$A_\alpha = S_v^T B_\alpha S_u, \qquad S_v A_\beta = B_\beta S_u$$

for each undirected edge $\alpha : u \text{ ——— } v$ ($u \leqslant v$) and each directed edge $\beta : u \longrightarrow v$.

The set $M_{\underline{n}}(Q)$ of matrix representations of dimension $\underline{n}$ is a vector space over $\mathbb{F}$.

Definition 3. *Let Q and Q' be mixed graphs with vertices $1, \ldots, t$ and $1, \ldots, t'$, and with edges $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$, respectively. The problem of classifying representations of Q is contained in the problem of classifying representations of Q', we write $Q \lesssim Q'$, if there exists a functor*

$$\mathcal{F} : M(Q) \to M(Q')$$

with the following properties:

- *$\mathcal{F}$ is injective on objects; moreover, for each $\underline{n} = (n_1, \ldots, n_t)$ there exists $\underline{n}' = (n_1', \ldots, n_{t'}')$ (all n_i and n_j' are nonnegative integers) such that $\mathcal{F}$ maps $M_{\underline{n}}(Q)$ to $M_{\underline{n}'}(Q')$ and this map is an affine injection of vector spaces.*
- *$\mathcal{F}$ is injective on morphisms; moreover, $A, B \in M_{\underline{n}}(Q)$ are isomorphic if and only if $\mathcal{F}(A), \mathcal{F}(B) \in M_{\underline{n}'}(Q')$ are isomorphic.*
- *$\mathcal{F}$ is produced by a parameter matrix representation of Q' (compare with [7] [p. 338]), which means that there exists a parameter matrix representation*

$$N(x) := (N_{\beta_1}(x), \ldots, N_{\beta_q}(x)) \tag{6}$$

of Q' with parameters $x_1, x_2, \ldots, x_p$ such that each x_i is an exactly one entry of an exactly one matrix among $N_{\beta_1}(x), \ldots, N_{\beta_q}(x)$, the other of their entries are elements of $\mathbb{F}$, and

$$\mathcal{F}(A) = (N_{\beta_1}(A), \ldots, N_{\beta_q}(A)) \tag{7}$$

for each matrix representation $A = (A_{\alpha_1}, \ldots, A_{\alpha_p})$ of Q. The representation (7) of Q' is constructed on (6) as follows:

- *Rearranging the basis vectors in the vector spaces of the representation $N(x)$, we rearrange the rows and columns of its matrices converting $N(0)$ to a direct sum*

$M_1 \oplus \cdots \oplus M_r$ *of matrix representations of Q' of nonzero dimension with the maximum number r. We say that two rows or columns of $N(x)$ are lined if they (with $x = 0$) are converted to rows or columns from the same summand M_i. Thus, there are r classes $\mathcal{L}_1, \ldots, \mathcal{L}_r$ of linked rows and columns; we require that each class contains a row or column with a parameter.*

– *Let $n_1, \ldots, n_r$ be natural numbers. Denote by $N(K_1, \ldots, K_t)$ a matrix representation of Q' obtained from $N(x)$ by replacing all rows and columns that belong to the same class $\mathcal{L}_j$ by strips of size n_j such that each parameter x_i is replaced by an arbitrary matrix K_i of suitable size, each entry that is a nonzero element $\alpha \in \mathbb{F}$ is replaced by αI, and each zero entry is replaced by the zero block.*

Two mixed graphs Q and Q' are equivalent if $Q \lesssim Q'$ and $Q' \lesssim Q$ (see Definition 3).

Example 2. *The problem of classifying representations of the quiver $\circlearrowright \bullet \circlearrowleft$ is contained in the problem of classifying representations of $\circlearrowright \bullet \longleftarrow \bullet$ (i.e., $\circlearrowright \bullet \circlearrowleft \lesssim \circlearrowright \bullet \longleftarrow \bullet$) via*

$$(N_1(x), N_2(x)) := \left(\begin{bmatrix} 0 & x_1 \\ 1 & x_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

since

- *each morphism $\varphi : A \circlearrowright \bullet \circlearrowleft B \to A' \circlearrowright \bullet \circlearrowleft B'$ (where $A, B \in \mathbb{F}^{n \times n}$ and $A', B' \in \mathbb{F}^{n' \times n'}$) is given by a matrix $S \in \mathbb{F}^{n' \times n}$ such that $SA = A'S$ and $SB = B'S$; the corresponding morphism $\mathcal{F}(\varphi) : \mathcal{F}(A, B) \to \mathcal{F}(A', B')$, where*

$$\mathcal{F}(A, B) = \left(\begin{bmatrix} 0_n & A \\ I_n & B \end{bmatrix}, \begin{bmatrix} I_n \\ 0_n \end{bmatrix} \right), \qquad \mathcal{F}(A', B') = \left(\begin{bmatrix} 0_{n'} & A' \\ I_{n'} & B' \end{bmatrix}, \begin{bmatrix} I_{n'} \\ 0_{n'} \end{bmatrix} \right),$$

is given by the matrix pair $(S \oplus S, S)$;
- *(A, B) is reduced to (A', B') by similarity transformations (1) if and only if $\mathcal{F}(A, B)$ is reduced to $\mathcal{F}(A', B')$ by transformations*

$$\left(S^{-1} \begin{bmatrix} 0_n & A \\ I_n & B \end{bmatrix} S, S^{-1} \begin{bmatrix} I_n \\ 0_n \end{bmatrix} R \right), \qquad S \text{ and } R \text{ are nonsingular.}$$

4. Proof of Theorem 1(a)

Let us consider a matrix representation

$$H \circlearrowright \bullet \circlearrowleft J, \qquad J := J_{k_1}(0_{n_1}) \oplus \cdots \oplus J_{k_\tau}(0_{n_\tau}), \tag{8}$$

in which

$$J_{k_i}(0_{n_i}) := \begin{bmatrix} 0_{n_i} & I_{n_i} & & 0 \\ & 0_{n_i} & \ddots & \\ & & \ddots & I_{n_i} \\ 0 & & & 0_{n_i} \end{bmatrix} \quad (k_i \text{ diagonal blocks}).$$

We reduce $H \circlearrowright \bullet \circlearrowleft J$ by those admissible transformations that preserve J:

$$(H, J) \mapsto (S^T H S, S^{-1} J S), \quad S \text{ is nonsingular and } S^{-1} J S = J. \tag{9}$$

Partition $S = [S_{ij}]_{i,j=1}^{\tau}$ conformally to J. Since $JS = SJ$, every S_{ij} has the form

$$
\begin{bmatrix}
 & C_{ij} & C'_{ij} & C''_{ij} & C'''_{ij} & \ddots \\
 & & C_{ij} & C'_{ij} & C''_{ij} & \ddots \\
 & & & C_{ij} & C'_{ij} & \ddots \\
 & & & & C_{ij} & \ddots \\
0 & & & & & \ddots
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
C_{ij} & C'_{ij} & C''_{ij} & C'''_{ij} & \ddots \\
 & C_{ij} & C'_{ij} & C''_{ij} & \ddots \\
 & & C_{ij} & C'_{ij} & \ddots \\
 & & & C_{ij} & \ddots \\
 & & & & \ddots \\
0 & & & &
\end{bmatrix},
$$

where all C_{ij} are arbitrary $n_i \times n_j$ matrices such that S is nonsingular. Replacing all off-diagonal blocks of S by zeros, we obtain the block diagonal matrix

$$
S_{\mathrm{diag}} := \big(\underbrace{S_1 \oplus S_1 \oplus S_1 \oplus \cdots}_{k_1 \text{ summands}} \big) \oplus \cdots \oplus \big(\underbrace{S_\tau \oplus S_\tau \oplus S_\tau \oplus \cdots}_{k_\tau \text{ summands}} \big),
\tag{10}
$$

in which every $S_i := C_{ii}$ is $n_i \times n_i$.

By Belitskii's theorem (see [23] [Section 3.4] or [8] [Theorem 1.2]), the Jordan matrix J is permutation similar to a nilpotent Weyr matrix such that all matrices commuting with it are upper block triangular. Since $JS = SJ$, Belitskii's theorem ensures that the matrix S is permutation similar to a block triangular matrix whose main block diagonal coincides with the sequence of summands in (10), up to permutation of these summands. Hence, S is nonsingular if and only if all blocks $S_1, \dots, S_\tau$ are nonsingular. Moreover, if the matrix H in (8) is partitioned such that the sizes of its diagonal blocks coincide with the sizes of the direct summands in (10), then

> if each horizontal strip and each vertical strip of H contains
> at most one nonzero block, then $S^T H S = S_{\mathrm{diag}}^T H S_{\mathrm{diag}}$. $\tag{11}$

Let us prove Theorem 1(a) for the mixed graph Q in (2); its proof for an arbitrary mixed graph is analogous. Let

$$\tag{12}$$
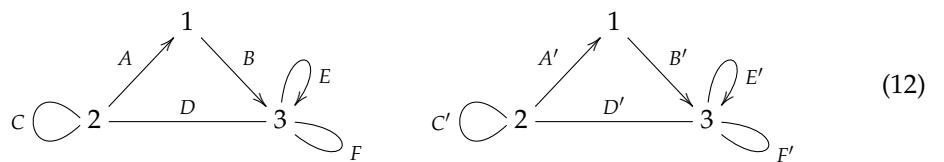
be matrix representations of Q of dimension (n_1, n_2, n_3). We construct the matrix represen-

tations $H \circlearrowleft \bullet \circlearrowright J$ and $H' \circlearrowleft \bullet \circlearrowright J$ as follows:

$$J := J_1(0_{n_1}) \oplus J_3(0_{n_2}) \oplus J_3(0_{n_3}) \oplus J_1(0_{n_4}) \oplus J_2(0_{n_5}), \tag{13}$$

$$H := \quad \begin{array}{c}
\begin{array}{cccccccccc} S_1 & S_2 & S_2 & S_2 & S_3 & S_3 & S_3 & S_4 & S_5 & S_5 \end{array} \\
\begin{array}{c|cccccccccc|}
S_1^T & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot & \cdot \\
S_2^T & \cdot & \cdot & C & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
S_2^T & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
S_2^T & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
S_3^T & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot & \cdot \\
S_3^T & \cdot & \cdot & \cdot & D & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
S_3^T & \cdot & \cdot & \cdot & \cdot & \cdot & F & \cdot & \cdot & \cdot & \cdot \\
S_4^T (= S_1^{-1}) & \cdot & A & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
S_5^T (= S_3^{-1}) & B & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
S_5^T (= S_3^{-1}) & \cdot & \cdot & \cdot & \cdot & E & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{array}, \tag{14}$$

in which the points denote zero blocks; the matrix H' is obtained from H by replacing $A, B, \ldots, F$ with $A', B', \ldots, F'$.

If $JS = SJ$, then

$$S_{\text{diag}} = S_1 \oplus S_2 \oplus S_2 \oplus S_2 \oplus S_3 \oplus S_3 \oplus S_3 \oplus S_4 \oplus S_5 \oplus S_5 \tag{15}$$

(see (10)). The summands of (15) are written in (14) over the vertical strips of H, and their transposes are written to the left of the horizontal strips of H. By (11), each nonzero block of (14) is multiplied by them if (H, J) is reduced by transformations (9).

We must prove that (H, J) is reduced to (H', J) by transformations (9) if and only if the representations (12) are isomorphic; that is, if and only if $A, B, \ldots, F$ are reduced to $A', B', \ldots, F'$ by transformations (3).

Let (H, J) be reduced to (H', J) by transformations (9). By (11),

$$S_1^T I S_4 = I, \quad S_3^T I S_5 = I, \quad S_4^T A S_2 = A', \quad S_5^T B S_1 = B',$$
$$S_2^T C S_2 = C', \quad S_3^T D S_2 = D', \quad S_5^T E S_3 = E', \quad S_3^T F S_3 = F'.$$

The first two equalities ensure that $S_4^T = S_1^{-1}$ and $S_5^T = S_3^{-1}$. Therefore, $A, B, \ldots, F$ are reduced to $A', B', \ldots, F'$ by transformations (3).

Conversely, let $A, B, \ldots, F$ be reduced to $A', B', \ldots, F'$ by transformations (3). Then (H, J) is reduced to (H', J) by transformations (9) with S of the form (15), in which $S_4 := S_1^{-T}$ and $S_5 := S_3^{-T}$.

This completes the proof of Theorem 1(a). The following lemma is proved analogously; it is also proved in [10] [Section 2.4], [11], and [12] [Section 10].

Lemma 1. *The problem of classifying representations of $\circlearrowright \bullet \circlearrowright$ contains the problem of classifying representations of each quiver.*

Proof. Let Q be a quiver, and let A be its matrix representation. We construct a matrix representation $H \circlearrowright \bullet \circlearrowright J$ as follows.

The matrix J is given in (8). The equality $S^{-1} J S = J$ implies that the main block diagonal of S is (10).

We take H in which each horizontal strip and each vertical strip contains at most one nonzero block. By analogy with (11), H is reduced by transformations $S_{\text{diag}}^{-1} H S_{\text{diag}}$. We construct H such that some of its blocks are the matrices of A and they are reduced by the same transformations as in A, and the other blocks are zero. $\quad \square$

5. Proof of Theorem 1(b)

Lemma 2. *Let a connected mixed graph Q contain a cycle in which the number of undirected edges is odd, and let Q not coincide with this cycle. Then the problem of classifying its representations contains the problem of classifying representations of one of the mixed graphs*

$$\subset\bullet\leftrightharpoons , \quad \subset\bullet\supset , \quad \subset\bullet\longleftarrow\bullet , \quad \subset\bullet\longrightarrow\bullet , \quad \subset\bullet\longrightarrow\bullet . \tag{16}$$

Proof. Let a connected mixed graph Q contain a cycle $\mathcal{C}$ in which the number of undirected edges is odd, and let α be an edge outside of $\mathcal{C}$ with vertices $v_1 \in \mathcal{C}$ and u.

Case 1: $u = v_1$. Then Q contains a subgraph

$$G: \quad \alpha \underset{v_1}{\subset} \overset{\beta_1}{\dashrightarrow} v_2 \overset{\beta_2}{\dashrightarrow} v_3 \quad \cdots \quad v_{k-1} \overset{\beta_{k-1}}{\dashrightarrow} v_k \tag{17}$$

in which $k \geqslant 1$ and each dotted line is an undirected or directed edge.

Consider the matrix representation

$$R(A,B): \quad A \underset{\mathbb{F}^n}{\subset} \overset{I_n}{\dashrightarrow} \mathbb{F}^n \overset{I_n}{\dashrightarrow} \mathbb{F}^n \quad \cdots \quad \mathbb{F}^n \overset{I_n}{\dashrightarrow} \mathbb{F}^n \tag{18}$$

of Q, in which $A, B \in \mathbb{F}^{n \times n}$ and all vertices outside of $\mathcal{C}$ are assigned by the zero spaces. Let us prove that $\subset\bullet\supset \lesssim Q$ with a suitable direction of the left loop in $\subset\bullet\supset$. We need to show the following equivalence of isomorphisms:

$$A \subset\bullet\supset B \simeq A' \subset\bullet\supset B' \iff R(A,B) \simeq R(A',B') \tag{19}$$

("$\simeq$" means "is isomorphic to").

Let us prove "$\Longleftarrow$". Let $R(A,B) \simeq R(A',B')$ via $(S_1, \ldots, S_k)$. Then $S_1 = S_2^{-T}$ if α_1 is undirected and $S_1 = S_2$ if α_1 is directed. Analogously, $S_2 = S_3^{-T}$ if α_2 is undirected and $S_2 = S_3$ if α_2 is directed, and so on. Since the number of undirected edges of $\mathcal{C}$ is odd, $S_{k-1} = S_k$ if α_k is undirected and $S_{k-1} = S_k^{-T}$ if α_k is directed. Hence, $B' = S_1^T B S_1$, and so $A \subset\bullet\supset B \simeq A' \subset\bullet\supset B'$ via S_1, in which the left loop in $\subset\bullet\supset$ is directed as the left loop in (17). The implication "$\Longrightarrow$" is proved analogously.

Case 2: $u = v_r$ with $r \neq 1$. If $\alpha : v_1 \longleftarrow v_r$, then we take v_r as v_1. We have that $\alpha : v_1 \longrightarrow v_r$ or $\alpha : v_1 \longrightarrow v_r$, and so Q contains the subgraph G_r that is obtained from (17) by replacing $\alpha \subset\bullet$ with $\alpha : v_1 \dashrightarrow v_r$.

Denote by $R_r(A,B)$ the matrix representation of G_r that is obtained from (18) by deleting $A \subset\bullet$ and assigning A to $\alpha : v_1 \dashrightarrow v_r$. Let us prove that $\subset\bullet\supset \lesssim Q$, in which the left loop is directed if and only if

- α in (17) is directed and the number of undirected edges in $\beta_1, \ldots \beta_{r-1}$ is even, or
- α in (17) is undirected and the number of undirected edges in $\beta_1, \ldots \beta_{r-1}$ is odd.

This statement holds since

$$A \subset\bullet\supset B \simeq A' \subset\bullet\supset B' \iff R_r(A,B) \simeq R_r(A',B'),$$

which is proved in (19).

Case 3: $u \notin \mathcal{C}$. Then Q contains the subgraph G' that is obtained from (17) by replacing $\alpha : \stackrel{\frown}{\bullet}$ with $\alpha : u \dashrightarrow v_1$. Denote by $R'(A, B)$ the matrix representation of G' that is obtained from (18) by deleting $A : \stackrel{\frown}{\bullet}$ and assigning A to $\alpha : u \dashrightarrow v_1$. Since

$$\bullet \overset{A}{\dashrightarrow} \bullet \circlearrowright B \simeq \bullet \overset{A'}{\dashrightarrow} \bullet \circlearrowright B' \iff R'(A, B) \simeq R'(A', B'),$$

we have $\bullet \dashrightarrow \bullet \circlearrowright \lesssim Q$. □

Lemma 3. $\subset \bullet \circlearrowleft \lesssim \subset \bullet \circlearrowright$.

Proof. Write

$$A := \begin{bmatrix} X & 0 \\ Y & 0 \end{bmatrix}, \qquad A' := \begin{bmatrix} X' & 0 \\ Y' & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix},$$

in which all blocks are $n \times n$. We need to show the following equivalence of isomorphisms:

$$X \subset \bullet \circlearrowleft Y \simeq X' \subset \bullet \circlearrowleft Y' \iff A \subset \bullet \circlearrowright B \simeq A' \subset \bullet \circlearrowright B \qquad (20)$$

$\Longleftarrow$. Let the right isomorphism hold via S. The equality $S^T B S = B$ takes the form

$$\begin{bmatrix} S_1^T S_3 & S_1^T S_4 \\ S_2^T S_3 & S_2^T S_4 \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad \text{in which } S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}.$$

Therefore, $S_1^T S_4 = I$, $S_2 = S_3 = 0$, and $S = S_1 \oplus S_1^{-T}$. Hence, the left isomorphism in (20) holds via S_1.

$\Longrightarrow$. If the left isomorphism holds via C, then the right isomorphism holds via $C \oplus C^{-T}$. □

Lemma 4. $\subset \bullet \circlearrowleft \lesssim \subset \bullet \longleftarrow \bullet$

Proof. Due to Lemma 3, it is sufficient to prove that $\subset \bullet \circlearrowright \lesssim \subset \bullet \longleftarrow \bullet$. Write

$$A := \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ X & 0 & Y \end{bmatrix}, \qquad A' := \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ X' & 0 & Y' \end{bmatrix}, \qquad B := \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \qquad (21)$$

in which all blocks are $n \times n$. We need to show the following equivalence of isomorphisms:

$$X \subset \bullet \circlearrowright Y \simeq X' \subset \bullet \circlearrowright Y' \iff A \subset \bullet \overset{B}{\longleftarrow} \bullet \simeq A' \subset \bullet \overset{B}{\longleftarrow} \bullet \qquad (22)$$

$\Longleftarrow$. Let the right isomorphism hold. Then

$$(S^T A S, S^{-1} B R) = (A', B) \qquad \text{for nonsingular } S \text{ and } R. \qquad (23)$$

The equality $BR = SB$ implies that S and R have the form

$$S = \begin{bmatrix} S_1 & S_2 & S_3 \\ S_4 & S_5 & S_6 \\ 0 & 0 & S_7 \end{bmatrix}, \qquad R = \begin{bmatrix} S_1 & S_2 \\ S_4 & S_5 \end{bmatrix}. \qquad (24)$$

Substituting them to $S^T A S = A'$, we obtain

$$\begin{bmatrix} S_1^T S_4 & S_1^T S_5 & S_1^T S_6 + S_4^T S_7 \\ S_2^T S_4 & S_2^T S_5 & S_2^T S_6 + S_5^T S_7 \\ S_3^T S_4 + S_7^T X S_1 & S_3^T S_5 + S_7^T X S_2 & S_3^T S_6 + S_6^T S_7 + S_7^T X S_3 + S_7^T Y S_7 \end{bmatrix} = A'.$$

Since the rows of R are linearly independent, $S_2^T[S_4\ S_5] = [0\ 0]$ implies $S_2 = 0$. Since $S_1^T S_5 = I$, $S_1^T S_4 = 0$ implies $S_4 = 0$. Equating the (2,3) blocks gives $S_5^T S_7 = I$. Equating the (1,3) blocks gives $S_6 = 0$. Equating the last horizontal strips gives

$$\begin{bmatrix} S_7^T X S_1 & S_3^T S_5 & S_7^T X S_3 + S_7^T Y S_7 \end{bmatrix} = \begin{bmatrix} X' & 0 & Y' \end{bmatrix}.$$

Since $S_3^T S_5 = 0$ and S_5 is nonsingular, we have $S_3 = 0$. Therefore, $S = S_1 \oplus S_1^{-T} \oplus S_1$, and so $S_1^T X S_1 = X'$ and $S_1^T Y S_1 = Y'$.

$\Longrightarrow$. If the left isomorphism holds via C, then (23) holds for $S = C \oplus C^{-T} \oplus C$ and $R = C \oplus C^{-T}$. $\square$

Lemma 5. $\subset \bullet \supset\ \lesssim\ \subset \bullet \longrightarrow \bullet$ *and* $\subset \bullet \supset\ \lesssim\ \subset \bullet \longrightarrow \bullet$

Proof. It is sufficient to prove that $\subset \bullet \supset\ \lesssim\ \subset \bullet \longrightarrow \bullet$ and $\subset \bullet \supset\ \lesssim\ \subset \bullet \longrightarrow \bullet$ because of Lemma 3. We take A and A' as in (21) and $B = [0\ 0\ I]$. We must prove (22), in which $\longleftarrow$ is replaced by $\longrightarrow$ and by $\longrightarrow$. The proof is the same since the equality $BS = RB$ implies that S as in (24). $\square$

Proof of Theorem 1(b). Let a mixed graph Q satisfy the condition (4). By Theorem 1(a), $Q \lesssim\ \subset \bullet \supset$. Lemma 2 ensures that $H \lesssim Q$, in which H is one of the mixed graphs (16). By Lemmas 3–5, $\subset \bullet \supset\ \lesssim H$. Hence, $\subset \bullet \supset\ \lesssim Q$. $\square$

6. Conclusions

We have proved that the problem of classifying matrix pairs with respect to transformations

$$(A, B) \mapsto (S^T A S, S^{-1} B S), \qquad S \text{ is nonsingular} \tag{25}$$

contains the problem of classifying an arbitrary system of bilinear forms and linear maps. There are only two essentially different unsolved classification problems for systems of forms and linear maps: The classical unsolved problem about matrix pairs under similarity and the problem of classifying matrix pairs under transformations (25). These problems are given by the graphs $\circlearrowright \bullet \supset$ and $\subset \bullet \supset$. There is no sense in studying representations of each of these graphs (and of any graph that is equivalent to one of them) outside the general theory of representations of quivers or mixed graphs, respectively. Likewise, Belitskii's algorithm that was constructed for matrix pairs under similarity can be applied to matrices of an arbitrary system of linear maps.

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