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Metastable Periodic Patterns in Singularly Perturbed Delayed Equations

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Abstract We consider the scalar delayed differential equation $\epsilon\dot{x}(t) = -x(t) + f(x(t-1))$, where $\epsilon > 0$ and f verifies either $df/dx > 0$ or $df/dx < 0$ and some other conditions. We present theorems indicating that a generic initial condition with sign changes generates a solution with a transient time of order $\exp(c/\epsilon)$, for some $c > 0$. We call it a metastable solution. During this transient a finite time span of the solution looks like that of a periodic function. It is remarkable that if $df/dx > 0$ then f must be odd or present some other very special symmetry in order to support metastable solutions, while this condition is absent in the case $df/dx < 0$. Explicit ϵ-asymptotics for the motion of zeroes of a solution and for the transient time regime are presented.

Keywords Metastability · Delayed differential equation · Singular perturbation · Transition layer

This paper is dedicated to Professor Jack Hale on the occasion of his 80th birthday.

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1 Introduction

Our goal in this paper is to study the transient solutions of equation

$$\epsilon \dot{x}(t) = -x(t) + f(x(t-1)), \tag{1}$$

where f is either a monotonic increasing (positive feedback) or a monotonic decreasing (negative feedback) function. In either case f must satisfy some additional hypotheses given precisely in Sects. 2 and 3. These hypotheses are necessary to guarantee that the map $f : \mathbb{R} \to \mathbb{R}$ has the following properties: (i) in the positive feedback case f has three fixed points p_-, p_0, p_+, such that $p_0 = 0$ is hyperbolic unstable, $p_- < 0$ ($p_+ > 0$) is hyperbolic stable and attracts all negative (positive) initial conditions; (ii) in the negative feedback case f has the origin as a hyperbolic unstable fixed point, and there is a unique hyperbolic stable period two orbit (p_-, p_+) that attracts almost all initial conditions. So, in both cases the dynamics of f is known and it is simple. Under these hypotheses, a lot is known about the dynamics of the semi-flow generated by Eq. 1 in the phase space $C^0([-1, 0])$, of continuous functions in $[-1, 0]$ with the sup norm.

In the positive feedback case the system has a global attractor that is made of: the three equilibria p_-, p_0, p_+; a finite set of unstable periodic orbits; and a set of global orbits connecting either two equilibria, or one equilibrium and one periodic orbit, or two periodic orbits. Moreover, the system admits a discrete Liapunov function that "more or less" establishes that the number of zeroes of a solution x, in an interval $[t-1, t]$, cannot increase with time. This sets constraints on the directions of the connecting orbits. All this information can be obtained in the following nonexhaustive list of references [11,14–16,21,22] (p. 90). In the negative feedback case a similar picture holds, after we replace the two stable equilibria p_- and p_+ by a stable periodic orbit. The main references here are [12,13].

The motivation for this work came from a numerical study of Eq. 1 with positive feedback that we did in the mid nineties. We obtained, numerically, an abundance of oscillatory solutions, apparently periodic, that after a long time ended up in one of the stable equilibria of the equation. By that time it was well known that very long transients may be displayed by the following scalar parabolic equation

$$\partial_t u = \epsilon^2 \, \partial_x^2 u - \frac{dF}{du}(u), \tag{2}$$

where u is a function of position $x \in [0, 1]$ and time t which satisfies Neumann boundary conditions; F is some smooth function with two local minima and one local maximum (a "double well" function); and $0 < \epsilon \ll 1$ is a small parameter. More precisely, it was shown in [3,7] that if F has the same value at its two minima, then an initial condition that has changes of sign generates a solution $x(t)$ that keeps changing sign for times of order $e^{c/\epsilon}$, for some $c > 0$, until it eventually settles down in one of the two stable equilibria of Eq. 2, corresponding to the two minima of F. These long transient solutions of Eq. 2 have been called "metastable" states by Carr and Pego [3]. Fusco and Hale in [7] described this metastability behavior of Eq. 2 in a way that has suggested to us that Eq. 1 could exhibit a similar phenomenon.

At this point it is convenient to make clear what we mean by metastability in the context of Eq. 1. Under the hypotheses in Sects. 2 and 3, an open and dense set of initial conditions generates solutions of Eq. 1 that are asymptotic to an equilibrium (positive feedback case) or to a periodic orbit (negative feedback case). The transient time is the time it takes for a solution to enter a given small neighborhood U of the asymptotic attracting set. Equation

1 is said to exhibit metastability if the transient time of most solutions grows as $e^{c/\epsilon}$ when $\epsilon \to 0$, where c does not depend on ϵ.

In the positive feedback case, a heuristic explanation for the existence of metastable states is the following. Setting $\epsilon = 0$ in Eq. 1, we obtain the mapping $x(t) = f(x(t-1))$ of continuous functions defined in $[-1, 0]$. Let $x_0 : [-1, 0] \to \mathbb{R}$ be a smooth initial condition that, for simplicity, satisfies $x_0(-1) = x_0(-1/2) = x_0(0) = 0$ and $x_0(t) < 0$, for $-1 < t < -1/2$, and $x_0(t) > 0$, for $-1/2 < t < 0$. The hypothesis on f imply that the iterates of $x(t) = f(x(t-1))$ give a continuous function x that for large t is close to a periodic square-wave function S given by $S(t) = p_-$, for $k < t < k + 1/2$, $S(t) = p_+$, for $k + 1/2 < t < k + 1$, $k \in \mathbb{Z}$. It is not hard to show that the real solution of Eq. 1 with ϵ small, and initial condition x_0, is C^0-close to x in a time interval $[-1, t_1]$, provided the one sided derivative of x times ϵ is small. As time evolves, this condition will eventually cease to hold since x approaches a square-wave function. The right hand side of Eq. 1 must be taken into account near the points where $|\epsilon \dot{x}(t)|$ is large. For ϵ sufficiently small, x will acquire a "rounded square-wave" shape during the time interval $[0, t_1]$. For $t > t_1$ the true solution of Eq. 1 will exhibit sharp transitions, between p_- and p_+, giving rise to the metastable transient of the solution. In the parabolic Eq. 2, the transitions during the transient of the solution have been described using the so called transition layer functions [7] (in this case they are easily obtained by solving a relatively simple ordinary differential equation).

Following very closely what was done in [7], we shall show that Eq. 1 admits metastable states for both positive and negative feedback f. Solutions displaying this metastable behavior are related to particular solutions (transition layer solutions) of the so called transition-layer equations. The concept of transition layer equations for delay differential equations has been established in [5,13]. A transition layer equation is also a functional equation, similar to (1), with one or more parameters to be determined. The existence of transition layer solutions was proved for the negative feedback case in [6,13]. The proof for the positive feedback case is given in the Sect. 5 below.

It is interesting to note that, for negative-feedback function f, Eq. 1 always displays metastable states. In contrast, in the positive-feedback case, Eq. 1 will display metastability only if f is odd or has some other very special symmetry (in analogy with the requirement that F must be degenerate at its two minima for Eq. 2 to display metastability). If the positive-feedback function f does not have this symmetry, the transient time is typically of order $1/\epsilon$.

For Eq. 2 Fusco and Hale gave a geometric description of metastability, in terms of the theory of invariant manifolds, that can essentially be repeated for the delay Eq. 1. For Eq. 1, a typical initial condition with some number of sign changes in $[-1, 0]$ is attracted in a time of order 1 to a small neighborhood of the unstable manifold W_γ of a periodic orbit γ. The number of sign changes of γ is related to the number of sign changes of the initial condition. For the parabolic Eq. 2, the analogue of γ is an equilibrium, and the vector field on most of its unstable manifold is almost null. For this reason, in the parabolic case, part of this unstable manifold is called a "slow manifold". Here, W_γ is by no means a slow manifold. Indeed, the flow in W_γ is such that almost all orbits on it are almost closed, in the sense that after a certain time interval, that is approximately equal to the delay (positive feedback) or twice the delay (negative feedback), the orbit returns very close to its initial point. So, W_γ is almost foliated by periodic orbits. Then the solution, that is close to W_γ, slowly drifts along the "almost periodic orbits" of W_γ until it gets ϵ-close to another periodic orbit. At this point the merging of two zeroes of the solution starts, and in a time interval of order one the solution has two zeroes less in the time interval $[-1, 0]$. Then the slowly drift along "almost closed orbits" takes the solution ϵ-close to a third periodic orbit, and so on. At last, in the positive feedback case the solution approaches an equilibrium and in the negative feedback case the stable-periodic orbit.

We will not give a proof that the picture described in the previous paragraph holds. Nevertheless, similarly to what has been done in [7], we will show that there are several finite dimensional manifolds W_n in $C^0([-1, 0])$, that are foliated by circles, and such that if the semi-flow of (1) acts on a point of W_n, then the resulting trajectory stays $\exp(-c/\epsilon)$-close to W_n, for some $c > 0$, for a finite amount of time. Moreover, we compute the drift velocity of the solution along the circles foliating W_n, and show that up to small errors this velocity coincides with that of the true solution. These computations allow us to obtain ϵ-asymptotics on the transient duration of a typical solution. In Sect. 2, in the positive feedback case, this time is compared to that observed in numerical simulations.

In 1999 [8] we have shown the existence of metastability in (1) for the particular case where f is a piecewise constant function, with two steps. Sharkovsky et al. [20] have also discussed the existence of exponentially long transients for piecewise constant functions, and estimated the time it takes for the solution generated by an initial function with sign changes to get close to a square-wave like function. Recently, Nizette has published an interesting work [17] (see also [18]), related to our 1999 metastability results [8]. The main focus of [17] is to find a criterion for the stability of periodic orbits of Eq. 1, for small ϵ, and with negative feedback but for f's that are not necessarily monotone. Nizette computes a map for the motion of the zeroes, similar to that presented in our Theorem 4, and argues that the periodic orbits associated to them are stable if the corresponding fixed points of the map are stable. In the case of a monotonic f the only stable orbit is the "slowly oscillating" one, which has the minimum possible number of zeroes. If f is nonmonotonic then other stable periodic orbits appear. In [17] the map for the motion of the zeroes is neither derived nor presented explicitly, although numerical results obtained using them are presented. Moreover, besides using the same transition layer solutions his construction of this map is very different from ours. His results for monotonic f are in agreement with ours.

The paper is organized as follows. In the Sects. 2 and 3 we present our main results in the case of positive and negative feedback f, respectively. There are two theorems in each section, one related to the existence and properties of solutions of transition layer equations and a second related to the metastable solutions. In Sect. 4 we make remarks on the global monotonicity hypotheses of the theorems in Sects. 2 and 3. In Sect. 5 we prove the theorem for the transition layer solutions in the positive feedback case. In Sect. 6 we just comment and prove some auxiliary results on the theorem due to Mallet-Pared and Nussbaum [13] and Chow et al. [6] for the transition layer solutions in the negative feedback case. Finally, in Sects. 7 and 8 we prove the theorems on the metastable solutions in the positive and the negative feedback cases, respectively. It is worth mentioning, that the bounds given in our Theorems 2 and 4 are still not enough to prove in a simple way, for instance, using Gronwall's inequality, that, in the positive feedback case, the solution oscillates for a time of order $\exp(c/\epsilon)$, for some $\epsilon > 0$. To prove this it is still necessary to show that the set W_n is close to a true invariant manifold of the system. Indeed, from a mathematical point of view, the results we present in this paper raise more questions than answers them.

2 Metastable Patterns for Positive Feedback Equations

In this section we consider Eq. 1, and assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and verifies the following hypotheses:

(HP1) $f(0) = 0,\ f'(0) > 1,\ f(-\gamma_1) = -\gamma_1,\ f(\gamma_2) = \gamma_2$, where $\gamma_1 > 0,\ \gamma_2 > 0$, and $f(x) \neq x$ for $x \in (-\gamma_1, 0) \cup (0, \gamma_2)$;
(HP2) $f'(x) = \frac{df}{dx}(x) \geq 0, \quad x \in \mathbb{R}$.

A crucial result in this section is the following theorem.

Theorem 1 (Positive feedback transition layer solution) *Consider the ("transition layer") equation*

$$\dot{y}(t) = -y(t) + f(y(t + r)), \tag{3}$$

where r is a real parameter.

There exists a value $r_* > 0$ of r, such that Eq. 3 with $r = r_*$ has a solution $\phi : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\frac{d\phi}{dt}(t) \geq 0, \quad for \quad t \in \mathbb{R}, \quad \phi(0) = 0,$$
$$\lim_{t \to -\infty} \phi(t) \to -\gamma_1, \quad \lim_{t \to \infty} \phi(t) \to \gamma_2.$$

There also exists a value $r_{**} > 0$ of r, such that Eq. 3 with $r = r_{**}$ has a solution $\chi : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\frac{d\chi}{dt}(t) \leq 0, \quad for \quad t \in \mathbb{R}, \quad \chi(0) = 0,$$
$$\lim_{t \to -\infty} \chi(t) \to \gamma_2, \quad \lim_{t \to \infty} \chi(t) \to -\gamma_1.$$

Suppose, in addition that,

(HP3) *f is twice continuously differentiable and*

$$f'(x) > 0 \quad for \quad x \in [-\gamma_1, \gamma_2], \quad f'(-\gamma_1) < 1, \quad and \quad f'(\gamma_2) < 1.$$

Then $r = r_*$ is the unique value of $r \in \mathbb{R}$ such that a solution ϕ as above exists. Moreover, there is only one ϕ with the above properties that, in addition, satisfies $\dot{\phi}(t) > 0$, for $t \in \mathbb{R}$, and has the following asymptotic behavior

$$\phi(t) = -\gamma_1 + b_1 \exp(\nu_1 t)[1 + \mathcal{O}(\exp(kt))] \quad as \quad t \to -\infty,$$
$$\phi(t) = \gamma_2 - b_2 \exp(-\nu_2 t)[1 + \mathcal{O}(\exp(-kt))] \quad as \quad t \to +\infty,$$

where b_1, b_2, ν_1, ν_2, k are all strictly positive constants.

Analogously, $r = r_{**}$ is the unique value of $r \in \mathbb{R}$ such that a solution χ as above exists. Moreover, there is only one χ with the above properties that, in addition, satisfies $\dot{\chi}(t) < 0$, for $t \in \mathbb{R}$, and has the following asymptotic behavior

$$\chi(t) = \gamma_2 - c_2 \exp(\mu_2 t)[1 + \mathcal{O}(\exp(kt))] \quad as \quad t \to -\infty,$$
$$\chi(t) = -\gamma_1 + c_1 \exp(-\mu_1 t)[1 + \mathcal{O}(\exp(-kt))] \quad as \quad t \to +\infty,$$

where c_1, c_2, μ_1, μ_2, k are all strictly positive constants.

The constants $-\mu_1$, μ_2, ν_1, and $-\nu_2$ are solutions of characteristic Eqs. 75 of linearization of Eq. 3 at the equilibria $-\gamma_1$ and γ_2 and the following inequalities are verified $0 < \mu_1 < 1$ and $0 < \nu_2 < 1$.

Finally, if f is an odd function then $r_* = r_{**}$, $\phi(t) = -\chi(t)$, $\gamma_1 = \gamma_2$, $b_2 = c_1$, $b_1 = c_2$, $\mu_1 = \nu_2$, and $\nu_1 = \mu_2$.

The proof of the Theorem 1 will be given in Sect. 5.

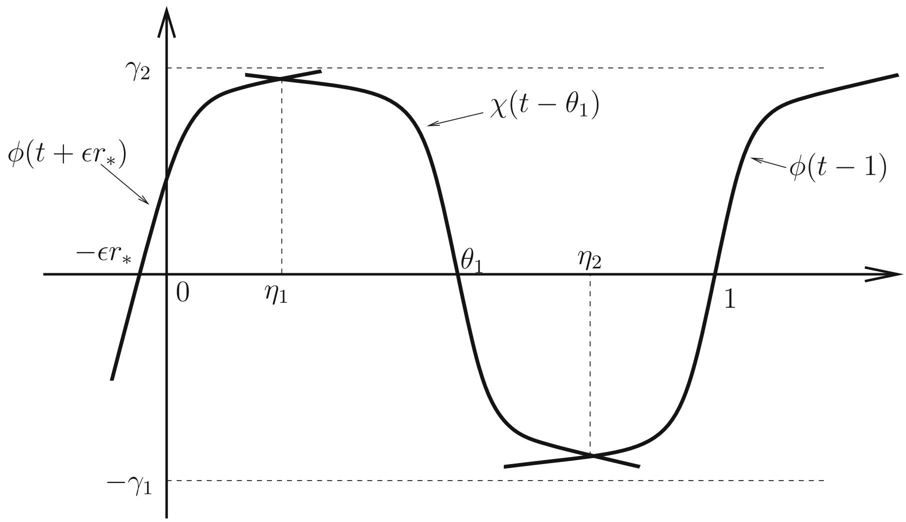

Fig. 1 Outline of the graph of the function given in Eq. 5. This is a function that generates W_2

The importance of the transition layer functions ϕ and χ in the analysis of the singular limit $\epsilon \to 0$ of Eq. 1 lies in the following argument. If ϕ is the function in Theorem 1 then $\phi_\epsilon(t) \stackrel{\text{def}}{=} \phi(t/\epsilon)$ verifies the following equation

$$\epsilon \dot{y}(t) = -y(t) + f(y(t+\epsilon r_*)). \tag{4}$$

Let us define a function $x(t) = \phi_\epsilon(t)$, for $t \in (-\{1+r_*\}/2, \{1+r_*\}/2]$, and let us extend it periodically to $\mathbb{R}$. The period of x is $1+\epsilon r_*$ and it is discontinuous on the set $D_c \stackrel{\text{def}}{=} \{(1+r_*)(2k+1)/2 : \ k \in \mathbb{Z}\}$. The periodicity of x implies that $x(t-1) = x(t+\epsilon r_*)$ and therefore

$$\epsilon \dot{x}(t) = -x(t) + f(x(t-1)) = -x(t) + f(x(t+\epsilon r_*))$$

if $t \in \mathbb{R}\backslash D_\epsilon$ where $D_\epsilon \stackrel{\text{def}}{=} \{t + t_c : \ t_c \in D_c \ \text{ and } \ t \in [-\epsilon r_*, 0]\}$. So, x solves Eq. 1 on the real line except for small intervals of length ϵr_* adjacent to the discontinuity set D_c of x. A similar construction can be made using the function $\chi_\epsilon(t) \stackrel{\text{def}}{=} \chi(t/\epsilon)$ instead of ϕ_ϵ. In this case we obtain a periodic function x with period $1+\epsilon r_{**}$.

A better continuous approximation to a solution of Eq. 1 may be obtained by glueing appropriate translations of functions ϕ_ϵ and χ_ϵ. For instance, for $t \in [-\epsilon r_*, 1]$, and for a given $\theta_1 \in [0,1]$, let us define $x(t)$ as (see Fig. 1)

$$\begin{aligned} x(t) &= \phi_\epsilon(t+\epsilon r_*) \quad \text{for} \quad -\epsilon r_* \le t \le \eta_1, \\ x(t) &= \chi_\epsilon(t-\theta_1) \quad \text{for} \quad \eta_1 < t \le \eta_2, \\ x(t) &= \phi_\epsilon(t-1) \quad \text{for} \quad \eta_2 < t \le 1, \end{aligned} \tag{5}$$

where η_1 and η_2 are defined in such a way that x is continuous. If $r_* = r_{**}$ it is possible to extend the function x in Eq. 5 to the real line in a continuous way, so as to obtain a periodic solution of equation (1) on most of $\mathbb{R}$. If $r_* \neq r_{**}$ this is not possible because ϕ_ϵ has periodicity $1+\epsilon r_*$, while χ_ϵ has periodicity $1+\epsilon r_{**}$ so that the extension of x in Eq. 5 cannot approximate a periodic solution of Eq. 1 unless $r_* = r_{**}$. So, from now on, besides (HP1), (HP2), and (HP3), we shall make the following additional hypothesis:

(HP4) The function f is such that its associated numbers r_* and r_{**} given in Theorem (1) are equal, $r_* = r_{**}$.

Under this hypothesis the periodic extension of x has period $1+\epsilon r_*$, is continuous, and satisfies Eq. 1 on the set $\mathbb{R}\backslash D_\epsilon$ where $D_\epsilon \stackrel{\text{def}}{=} \{t+t_c : t_c \in D_c \text{ and } t \in [-\epsilon r_*, 0]\}$, where $D_c \stackrel{\text{def}}{=} \{\eta_1 k : k \in \mathbb{Z}\} \cup \{\eta_2 k : k \in \mathbb{Z}\}$. We shall show that x is a very good approximation to a solution of Eq. 1 on D_ϵ.

To show how the transition layer solutions can provide approximate solutions to Eq. 1 we need several definitions. Let n be a positive **even** integer and A_n be the following open $n-1$-dimensional simplex

$$A_n \stackrel{\text{def}}{=} \{\delta \in \mathbb{R}^n : \delta_i > 0, \quad \text{and} \quad \delta_1+\delta_2+\cdots+\delta_n = 1+\epsilon r_*\}. \tag{6}$$

Given $\delta \in A_n$ let $\theta \in \mathbb{R}^{n+1}$ be defined by $\theta_0 = -\epsilon r_*$ and $\theta_i = \delta_i + \theta_{i-1}$. Notice that a point in A_n determines a unique θ that satisfies $\theta_0 = -\epsilon r_* < \theta_1 < \cdots < \theta_{n-1} < \theta_n = 1$. Now, to each point in A_n we associate a function $z : [-\epsilon r_*, 1] \to \mathbb{R}$ in the following way

$$\begin{aligned} z(t) &= \phi_\epsilon(t-\theta_0) \quad \text{for} \quad \theta_0 = -\epsilon r_* \le t \le \eta_1, \\ z(t) &= \chi_\epsilon(t-\theta_1) \quad \text{for} \quad \eta_1 < t \le \eta_2, \\ z(t) &= \phi_\epsilon(t-\theta_2) \quad \text{for} \quad \eta_2 < t \le \eta_3, \\ &\dots \\ z(t) &= \phi_\epsilon(t-\theta_n) \quad \text{for} \quad \eta_n < t \le \theta_n = 1 \end{aligned} \tag{7}$$

where $\eta_1, \dots, \eta_n$ are uniquely defined (see corollary 1 in Sect. 7) in such a way that z is continuous. In Fig. 1 we sketch the graph of the function z for $n=2$. Now we extend z periodically to $\mathbb{R}$. This extension is continuous and has period $1+\epsilon r_*$. We define a function Φ from the open subset $A_n \times \mathbb{R} \subset \mathbb{R}^n$ to $C^0([-1,0])$ as $\Phi(\delta, t) = z(t+s)$, where $s \in [-1,0]$. The set $W_n \subset C^0([-1,0])$ is defined as the image of this function Φ. Notice that W_n is an n-dimensional submanifold of $C^0([-1,0])$. Moreover, for each fixed δ the image of $\Phi(\delta,\cdot)$ is a circle on W_n. So, W_n is a trivial circle bundle over A_n, which is the image of $A_n \times \mathrm{S}^1 \stackrel{\text{def}}{=} A_n \times \{\mathbb{R} \mod (1+\epsilon r_*)\}$ by Φ. Notice that when we take the quotient $t \mod (1+\epsilon r_*)$ and interpret $z(t)$ as a function on S^1, then $t = \theta_0 = -\epsilon r_*$ is identified with $t = \theta_n = 1$, and $\delta_1, \dots, \delta_n$ represent the angular distance between consecutive zeroes of $z(t)$. The manifold W_n, is homeomorphic to $A_n \times \mathrm{S}^1$, and can be interpreted as a cylinder. The flow $\varphi : \mathbb{R} \times W_n \to W_n$ is naturally defined as

$$\varphi_t(x)(s) = z(t+s), \quad s \in [-1,0], \tag{8}$$

where z is the periodic function on $\mathbb{R}$ that equals x when restricted to $[-1,0]$, and generates the fiber of W_n passing through x. We claim that W_n is an approximation to part of the unstable manifold of a periodic orbit of Eq. 1 and that φ is a good approximation for the flow of Eq. 1 on this invariant manifold. Some support for this claim was given in the introduction and further support is given in our next Theorem 2.

Before we present Theorem 2 it is convenient to introduce the following notation for functions that admit exponential bounds.

Notation

$$R(t,s) = \mathcal{E}(t) \quad \text{if there exists } \beta > 0 \text{, that may depend on } s \text{, but not on } t, \\ \text{such that} \quad \lim_{t\to\infty} R(t,s)\exp(\beta t) = 0. \tag{9}$$

$$R(t,s) = \mathcal{E}_-(t) \quad \text{if there exists } \beta > 0 \text{, that may depend on } s \text{, but not on } t, \text{ such that } \lim_{t\to-\infty} R(t,s)\exp(-\beta t) = 0. \tag{10}$$

The following definitions are also needed. Let X_n be the set of functions x in $C^0([-1,0])$ that are piecewise continuously differentiable, undergo an odd number $n-1$ of sign changes within the interval $(-1,0)$, and satisfy $x(0) = 0$ and $\dot{x}(0) > 0$. We define a projection $P_n : X_n \to W_n$ in the following way. If the zeroes of x are located at $-1+\theta_1 < -1+\theta_2 < \cdots < -1+\theta_{n-1} < \theta_n = 0$, then $P_n(x)(t) = z(t+1)$, $t \in [-1,0]$, where z is given in Eq. 7. Namely, $P_n(x)$ is a function in W_n with the same zeroes as x and with positive derivative at zero. Now, let $\psi_t : C^0([-1,0]) \to C^0([-1,0])$ be the flow of Eq. 1. We define a subset $\overline{X}_n$ of X_n in the following way. A function $x \in X_n$ is in $\overline{X}_n$ if there exists a value of time $T(x) > 1$ such that $\psi_T x \in X_n$ and $\psi_t x \notin X_n$ for any $t \in (1,T)$. Then we define a function $F_n : \overline{X}_n \to X_n$ as $F_n(x) = \psi_T(x)$. Notice that T is the first time larger than one at which the solution $x(t)$ of Eq. 1 has a zero with positive derivative and such that the number of zeroes of $x(t)$ for $t \in (T-1,T]$ is the same as that in $t \in (-1,0]$. Now we can state the main result of this section.

Theorem 2 *Given $\delta \in A_n$, with $n > 0$ even and A_n defined in Eq. 6, let $x \in W_n \cap X_n$ be a function with zeroes at $\theta_i = \delta_i + \theta_{i-1}$, $i = 1,\dots n$, where $\theta_0 \stackrel{\text{def}}{=} -\epsilon r$. Then there exists $\epsilon_0 > 0$, that depends on δ, and positive constants (depending on neither ϵ nor δ) k_3 and k_4 given in Lemma 10, k_5 given in Lemma 11,*

$$a \stackrel{\text{def}}{=} \frac{\mu_1(1+\nu_1)}{\mu_1+\nu_1}, \quad \textit{and} \quad b \stackrel{\text{def}}{=} \frac{\nu_2(1+\mu_2)}{\mu_2+\nu_2},$$

where μ_1, μ_2, ν_1 and ν_2 are given in Theorem 1, such that for $\epsilon < \epsilon_0$ the following holds:

(i) *For $t \in [0, 1+\eta_1]$ ($\eta_1 > 0$ is the number appearing in Eq. 7), the natural flow φ on W_n (Eq. 8, and the flow ψ of Eq. 1, satisfy the following inequality*

$$\sup_{s\in[-1,0]} |\psi_t(x)(s) - \varphi_t(x)(s)| \stackrel{\text{def}}{=} ||\psi_t(x) - \varphi_t(x)||_0 \le \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon)),$$

where

$$\beta \stackrel{\text{def}}{=} \min\left\{\frac{\mu_2\nu_2}{\mu_2+\nu_2}\delta_1, \frac{\mu_1\nu_1}{\mu_1+\nu_1}\delta_2, \frac{\mu_2\nu_2}{\mu_2+\nu_2}\delta_3, \dots, \frac{\mu_2\nu_2}{\mu_2+\nu_2}\delta_n\right\};$$

(ii) *The function x is in $\overline{X}_n$ with*

$$T(x) = 1 + \epsilon r - \epsilon \exp(-a\delta_n/\epsilon)[k_4 + \mathcal{E}(1/\epsilon)], \tag{11}$$

where $r = r_*$ given in Theorem 1, and $F_n(x) = \psi_T(x)$ satisfies

$$||F_n(x) - P_n \circ F_n(x)||_0 \le \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon)); \tag{12}$$

(iii) *If $-1+\hat{\theta}_1 < -1+\hat{\theta}_2 < \cdots < -1+\hat{\theta}_{n-1} < \hat{\theta}_n = 0$ are the zeroes of $F_n(x)$ and $\delta'_i \stackrel{\text{def}}{=} \hat{\theta}_i - \hat{\theta}_{i-1}$, $i = 1,\dots,n$, with $\hat{\theta}_0 \stackrel{\text{def}}{=} -\epsilon r_*$, then $\hat{\delta} \in A_n$ and*

$$\begin{aligned}\delta'_i &= \delta_i + \epsilon[\exp(-a\delta_{i-1}/\epsilon)(k_4 + \mathcal{E}(1/\epsilon)) - \exp(-b\delta_i/\epsilon)(k_3 + \mathcal{E}(1/\epsilon))], \ i \text{ odd},\\ \delta'_i &= \delta_i + \epsilon[\exp(-b\delta_{i-1}/\epsilon)(k_3 + \mathcal{E}(1/\epsilon)) - \exp(-a\delta_i/\epsilon)(k_4 + \mathcal{E}(1/\epsilon))], \ i \text{ even},\end{aligned} \tag{13}$$

where $\delta_0 \overset{\text{def}}{=} \delta_n$. *If* f *is an odd function then* $\mu_1 = \nu_2 = \nu$, $\nu_1 = \mu_2 = \mu$ *so that* $a = b = \dfrac{\nu(1+\mu)}{\mu+\nu}$, *and*

$$\kappa \overset{\text{def}}{=} k_3 = k_4 = \frac{1}{\dot{\phi}(0)}\left\{-b_1 \exp\left[\frac{1+\mu}{\mu+\nu}\ln(c_1/b_1) - (1+\mu)r_*\right] + c_1 \exp\left[\frac{1-\nu}{\mu+\nu}\ln(c_1/b_1) - (1-\nu)r_*\right]\right\},$$

where c_1 *and* b_1 *are given in Theorem 1, and the expressions for* δ' *simplify to*

$$\delta_i' = \delta_i + \epsilon\kappa[\exp(-a\delta_{i-1}/\epsilon) - \exp(-a\delta_i/\epsilon)](1 + \mathcal{E}(1/\epsilon)). \tag{14}$$

Theorem 2 suggests that although W_n is not an invariant set it is close to an invariant set in the C^0 metric. It also says that the vector field associated to Eq. 1 is almost parallel to the circles which are the fibers of W_n. Indeed, from Theorem 2 we conclude that, for ϵ small, a solution of Eq. 1 that starts on W_n rotates around the initial fiber many times, with speed of order one, while it slowly drifts towards other fibers, with a speed of order $\exp(-c\epsilon)$, for some $c > 0$ that can be computed. The functions that generate these circular fibers are what we call the *periodic metastable states*. Notice also that the bounds given in Theorem 2 are not enough to prove in a simple way, for instance using Gronwall's inequality, that the solution oscillates for a time of order $\exp(c/\epsilon)$, for some $\epsilon > 0$. To prove this it is necessary to show that W_n is close to a true invariant manifold of the system.

In the following we assume that f is odd. Let x be an initial condition which satisfies the hypotheses of Theorem 2. Let $\delta_m < \delta_i$, $i = 1, 2, \dots, n$, with $i \neq m$. Now, if all terms in the Eqs. 14 for δ_i' are compared to $\exp(-a\delta_m/\epsilon)$ and all those of order $\exp(-a\delta_m/\epsilon)\mathcal{E}(1/\epsilon)$ are neglected, then we are left with the following set of equations:

$$\begin{aligned} \delta_i' &= \delta_i, \quad \text{for} \quad i = 1, 2, \dots, n, \quad i \neq m-1, \quad i \neq m, \\ \delta_{m-1}' &= \delta_{m-1} + \epsilon\kappa \exp(-a\delta_m/\epsilon) \\ \delta_m' &= \delta_m - \epsilon\kappa \exp(-a\delta_m/\epsilon) \end{aligned} \tag{15}$$

For simplicity let us assume that $m \neq 1$ and $m \neq n$ (these cases can also be analyzed). If we iterate the map (15) we obtain that δ_m decreases very slowly while δ_{m-1} increases at the same rate and all other $\delta' s$ remain the same. Assuming that the estimates for the motion of zeroes given in Theorem 2 remain valid until some fast dynamical process eliminates two sufficiently close zeroes, the above analysis shows that all zeroes θ_i undergo a negligible displacement under iterates of F_n, except θ_m which moves towards θ_{m-1} until both of them get "annihilated". Then the dynamics of a new map F_{n-2} may eventually annihilate another pair of zeroes and so on. A simple computation of the time it takes for the iterates of δ_m under map (15) to be of order ϵ gives the following asymptotic estimate for T_d, the time it takes for the two zeroes θ_{m-1} and θ_m to disappear:

$$\ln T_d = \frac{\delta_m a}{\epsilon} - \ln(a\kappa) + \mathcal{O}(\epsilon) \tag{16}$$

To verify this asymptotic expression we made the following numerical calculations. For $f = 3\tanh(2x)$, varying ϵ, and for a fixed initial condition $(0.15 + \cos(2\pi t))$ with two zeroes, Eq. 1 was integrated until the zeroes disappeared (using a 4th-order Runge–Kutta). The minimum distance between zeroes (δ_m in the table below), and the time instant t_d when the zeroes disappeared were recorded for several values of ϵ. For $t \approx 20$ the solution has a

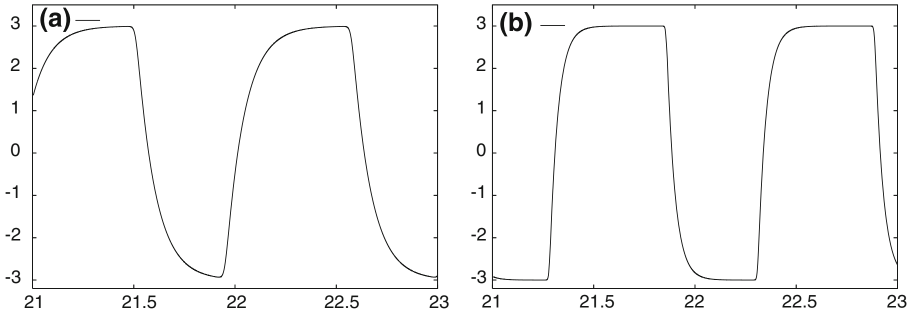

Fig. 2 Graphs illustrating the square-wave like shape of the solutions corresponding to parameter $\epsilon = 0.09$ (plate **a**), and $\epsilon = 0.04$ (plate **b**) for $t \in [21, 23]$. These solutions were used in the table shown. Notice that as the parameter ϵ is decreased, the shape of the solution gets closer to a square wave

square-wave like shape (see Fig. 2 below) so that we have set $T_d = t_d - 20$. The numerical results are displayed in the following table.

ϵ	0.09	0.08	0.07	0.06	0.05	0.04
δ_m	0.438	0.445	0.448	0.450	0.450	0.451
T_d	143.3	284.0	653.9	1932.3	8655.3	81886.7

A least square fit of the data in this table to the function in Eq. 16 gives the following values: $a \approx 0.990$ and $\kappa \approx 0.880$. Now, for $f = 3\tanh(2x)$ one can also numerically solve the transition layer Eq. 3 to get $r = 0.717$, so that $a = 0.999$ and $\kappa = 1.005$. The agreement is quite good, specially because as ϵ gets smaller the numerical errors in the integration of Eq. 1 become larger. It is worth noticing that the limit $\lim_{\alpha\to\infty} 3\tanh(\alpha x)$ is a piecewise constant function. In this limit we get [8] $r = \ln 2 = 0.693\ldots$, $a = 1$, and $\kappa = 1$ which are very close to the values obtained numerically for the case $\alpha = 2$.

Finally, let us make some remarks regarding the assumption that there is a smallest gap between the zeroes of the initial condition, namely $\delta_m < \delta_i$, $i = 1, 2, \ldots, n$, with $i \neq m$. If this assumption is not verified then the above analysis that lead to Eq. 15 breaks down. For instance, if the initial condition has two zeroes and $\delta_1 = \delta_2$ then the map (14) degenerates into $\delta_i' = \delta_i + \mathcal{E}(1/\epsilon)$, and we are left with the identity map plus a correction term which we did not compute. Of course, the asymptotics in Eq. 16 does not hold. This is in agreement to the fact that Eq. 1 admits periodic orbits for ϵ arbitrarily small, in particular orbits with only one sign change in a half period. Moreover, it can be shown that this orbit has a square-wave like shape when $\epsilon \to 0$ (in the negative feedback case this is shown in [13]). So, for small ϵ there are initial conditions with $\delta_1 \approx \delta_2$ that are asymptotic to periodic solutions of Eq. 1 for which the distance between zeroes does not decrease at all. Our asymptotic analysis shows that as $\epsilon \to 0$ these initial conditions must satisfy $\delta_1 \to \delta_2$. Moreover, the asymptotics above shows that all these oscillatory periodic orbits are unstable for ϵ-sufficiently small (this fact is known to be true for any ϵ (see [21,22] p. 90). These properties are the basis of the stability analysis of periodic solutions of negative-delayed feedback Eqs. 1, for ϵ small, made by Nizette [17]. We point out that Nizette made an interesting extension of this analysis to f nonmonotonic.

3 Metastable Patterns for Negative Feedback Equations

In this section we consider Eq. 1 and we assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies the following hypotheses (which are the same as in [6]):

(HN1) $f(0) = 0$ and there exists $\gamma_1 > 0$, and $\gamma_2 > 0$, such that $f(-\gamma_1) = \gamma_2$, and $f(\gamma_2) = -\gamma_1$.
(HN2) $f'(x) \le 0$ for all $x \in \mathbb{R}$.
(HN3) $f'(0) < -1$ and $0 \le f'(-\gamma_1)f'(\gamma_2) < 1$.
(HN4) $|f(f(x))| > |x|$ for $x \in (-\gamma_1, 0) \cup (0, \gamma_2)$, and $|f(f(x))| < |x|$ for $x \in (-\infty, -\gamma_1) \cup (\gamma_2, \infty)$.

A crucial result in this section is the theorem regarding the negative feedback transition layer solution due to Mallet-Paret and Nussbaum [13] and Chow et al. [6].

Theorem 3 (Negative feedback transition layer solution; Chow et al.) *Consider the ("transition layer") equation*

$$\begin{aligned}\dot{y}(t) &= -y(t) + f(z(t + \underline{r})),\\ \dot{z}(t) &= -z(t) + f(y(t + \overline{r}))\end{aligned} \tag{17}$$

where $\underline{r}$ and $\overline{r}$ are real parameters.

Among all possible values of $\overline{r}$ and $\underline{r}$ there exists a unique strictly positive pair, denoted by $(\overline{r}, \underline{r})$, such that Eq. 17 has a unique solution $(y, z) = (\phi, \chi) : \mathbb{R} \to \mathbb{R}^2$ with the following properties:

$$\begin{aligned}&\dot{\phi}(t) \ge 0 \quad \text{and} \quad \dot{\chi}(t) \le 0 \quad for \quad t \in \mathbb{R},\\ &\phi(0) = 0 \quad \text{and} \quad \lim_{t\to-\infty} \phi(t) \to -\gamma_1, \quad \lim_{t\to\infty} \phi(t) \to \gamma_2,\\ &\chi(0) = 0 \quad \text{and} \quad \lim_{t\to-\infty} \chi(t) \to \gamma_2, \quad \lim_{t\to\infty} \chi(t) \to -\gamma_1.\end{aligned} \tag{18}$$

Suppose in addition that

(HN5) *f is twice continuously differentiable and $0 < f'(-\gamma_1)f'(\gamma_2)$.*

Then $\dot{\phi}(t) > 0$, $\dot{\chi}(t) < 0$, and the following asymptotic expressions hold:

$$\begin{aligned}\phi(t) &= -\gamma_1 + b_1 \exp(\mu t)[1 + \mathcal{O}(\exp(kt))] \quad as \quad t \to -\infty\\ \chi(t) &= \gamma_2 - c_2 \exp(\mu t)[1 + \mathcal{O}(\exp(kt))] \quad as \quad t \to -\infty\\ \phi(t) &= \gamma_2 - b_2 \exp(-\nu t)[1 + \mathcal{O}(\exp(-kt))] \quad as \quad t \to +\infty\\ \chi(t) &= -\gamma_1 + c_1 \exp(-\nu t)[1 + \mathcal{O}(\exp(-kt))] \quad as \quad t \to +\infty\end{aligned} \tag{19}$$

where b_1, b_2, c_1, c_2, μ, ν, and k are all strictly positive constants which satisfy the following inequalities:

$$\begin{aligned}&\nu < 1,\\ &\ln\left(\frac{c_1c_2}{b_1b_2}\right) + \mu\underline{r} + \nu\overline{r} = \ln\left(\frac{b_1b_2}{c_1c_2}\right) + \mu\overline{r} + \nu\underline{r} = \ln\left(\frac{1+\mu}{1-\nu}\right) > 0.\end{aligned} \tag{20}$$

The statement in the first part of Theorem 3 is trivially different from that of Theorem 2.1 in [6]. The asymptotic expressions (19) are not in [6], and will be commented upon in Sect. 6.

Now, a heuristic analysis similar to that made in the previous section, using Eqs. 4 and 5, can also be made in this case. Then a remarkable difference between the positive and the negative feedback cases is noticed: the condition $r_* = r_{**}$, necessary for the construction of metastable states in the positive feedback case, does not appear in the negative feedback case. So, metastable solutions exist for the negative feedback Eq. 1 even for functions f that are not odd, in contrast with the case in which Eq. 1 has positive feedback. Much of what was said in the previous section about the use of the transition layer solutions in the construction of an "almost invariant" set of metastable states also applies to this section. So, in the following, without further comments, we introduce the main definitions that will allow us to state the Theorem 4, our main result in this section.

Let $n \geq 3$ be a positive **odd** integer and let A_n and $\underline{A}_n$ be the following open $n-1$-dimensional simplexes

$$A_n \overset{\text{def}}{=} \{\delta \in \mathbb{R}^n : \delta_i > 0, \quad \text{and} \quad \delta_1 + \delta_2 + \cdots + \delta_n = 1 + \epsilon\overline{r}\} \tag{21}$$

and

$$B_n \overset{\text{def}}{=} \{\delta \in \mathbb{R}^n : \delta_i > 0, \quad \text{and} \quad \delta_1 + \delta_2 + \cdots + \delta_n = 1 + \epsilon\underline{r}\} \tag{22}$$

Given $\delta \in A_n$ let $\theta \in \mathbb{R}^{2n+1}$ be defined by

$$\begin{aligned} \theta_0 &= -\epsilon\overline{r}, \\ \theta_i &= \delta_i + \theta_{i-1}, \quad \text{for} \quad i = 1, 2, \ldots, n; \\ \theta_{i+n} &= \theta_i + 1 + \underline{r}, \quad \text{for} \quad i = 1, 3, \ldots, n; \\ \theta_i &= \theta_i + 1 + \overline{r}, \quad \text{for} \quad i = 2, 4 \ldots, n-1. \end{aligned}$$

For ϵ sufficiently small a point in A_n determines a unique θ which satisfies $\theta_0 = -\epsilon\overline{r} < \theta_1 < \cdots \theta_n = 1 < \cdots < \theta_{2n} = 2+\epsilon\underline{r}$. Moreover, if $\delta_i = \theta_i - \theta_{i-1}, i = n+1, n+2, \ldots, 2n$, then $(\delta_{n+1}, \ldots, \delta_{2n}) \in \underline{A}_n$. Now, to each point in A_n we associate a function $z : [-\epsilon\overline{r}, 2+\epsilon\underline{r}] \to \mathbb{R}$ in the following way

$$\begin{aligned} &z(t) = \phi_\epsilon(t-\theta_0) \quad \text{for} \quad \theta_0 = -\epsilon\overline{r} \leq t \leq \eta_1 \\ &z(t) = \chi_\epsilon(t-\theta_1) \quad \text{for} \quad \eta_1 < t \leq \eta_2 \\ &z(t) = \phi_\epsilon(t-\theta_2) \quad \text{for} \quad \eta_2 < t \leq \eta_3 \\ &\ldots \\ &z(t) = \phi_\epsilon(t-\theta_{2n}) \quad \text{for} \quad \eta_{2n} < t \leq \theta_{2n} = 2+\epsilon\underline{r} \end{aligned} \tag{23}$$

where $\eta_1, \ldots, \eta_{2n}$ are uniquely defined (see corollary 2 in Sect. 8) in such a way that z is continuous. Now we extend z periodically to $\mathbb{R}$. This extension is continuous and has period $2+\epsilon\underline{r}+\overline{r}$. Then we define a function Φ from the open subset $A_n \times \mathbb{R} \subset \mathbb{R}^n$ to $C^0([-1,0])$ as $\Phi(\delta, t) = z(t+s)$, where $s \in [-1, 0]$. The set $W_n \subset C^0([-1,0])$ is defined as the image of this function Φ. As in the positive feedback case W_n is an n-dimensional submanifold of $C^0([-1,0])$ which is also a circle bundle over A_n. Again a flow $\varphi : \mathbb{R} \times W_n \to W_n$ is naturally defined as $\varphi_t(x)(s) = z(t+s)$, $s \in [-1, 0]$, where z is the periodic function on $\mathbb{R}$ which when restricted to $[-1, 0]$ is equal to x and which generates the fiber of W_n passing through x. Now, let X_n (Y_n) be sets of functions x in $C^0([-1,0])$ that are piecewise continuously differentiable, have an even number $n-1$ of sign changes in $(-1, 0)$, satisfy $x(0) = 0$ and such that $\dot{x}(0) > 0$ ($\dot{x}(0) < 0$) for $x \in X_n$ ($x \in Y_n$). We define projections $P_{Xn} : X_n \to W_n$ and $P_{Yn} : Y_n \to W_n$ in the following way. If the zeroes of x are located at $-1+\theta_1 < -1+\theta_2 < \cdots < -1+\theta_{n-1} < \theta_n = 0$, then $P_{Yn}(x)(t) = z(t+1)$ and $P_{Xn}(x)(t) = z(t+2+\epsilon\underline{r})$, $t \in [-1, 0]$, where z is given in Eq. 23. Namely, $P_{Yn}(x)$ ($P_{Xn}(x)$) is a function in W_n, with negative derivative at zero (positive derivative at zero), having the same zeroes as x. Now, let $\psi_t : C^0([-1,0]) \to C^0([-1,0])$ be the flow of Eq. 1. We define a subset $\overline{X}_n$ of X_n in the following way. A function $x \in X_n$ is in $\overline{X}_n$ if there exists a value of time $T(x) > 1$ such that $\psi_T x \in Y_n$ and $\psi_t x \notin Y_n$ for any $t \in (1, T)$. Then we define a function $F_{Xn} : \overline{X}_n \to Y_n$ as $F_{Xn}(x) = \psi_T(x)$. Analogously, a function $x \in Y_n$ is in a subset

$\overline{Y}_n$ of Y_n if there exists a value of time $T(x) > 1$ such that $\psi_T x \in X_n$ and $\psi_t x \notin X_n$ for any $t \in (1, T)$. Then we define a function $F_{Yn} : \overline{Y}_n \to X_n$ as $F_{Yn}(x) = \psi_T(x)$. Now we can state the Theorem 4 that is the main result of this section.

Theorem 4 *Given $\delta \in A_n$ and $\underline{\delta} \in \underline{A}_n$ with $n \geq 3$ odd let x be a function in either $W_n \cap Y_n$ or $W_n \cap X_n$, such that if $x \in W_n \cap Y_n$ ($x \in W_n \cap X_n$) then its zeroes are at $\theta_i = \delta_i + \theta_{i-1}$, $i = 1, \ldots n$, where $\theta_0 \stackrel{\text{def}}{=} -\epsilon\overline{r}$ ($\theta_i = \underline{\delta}_i + \theta_{i-1}$, $i = 1, \ldots n$, where $\theta_0 \stackrel{\text{def}}{=} -\epsilon\underline{r}$). Then there exists $\epsilon_0 > 0$, that depends on either δ or $\underline{\delta}$, and positive constants (depending on neither ϵ nor δ or $\underline{\delta}$) k_3 and k_4 given in Lemma 14, k_5 given in Lemma 15,*

$$a \stackrel{\text{def}}{=} \frac{\nu(1+\mu)}{\mu+\nu} > 0, \tag{24}$$

where μ, ν are given in Theorem 3, such that for $\epsilon < \epsilon_0$ the following holds:

(i) *The natural flow φ on W_n defined above and the flow ψ of Eq. 1 satisfy the following inequality for $t \in [0, 1 + B]$*

$$\sup_{s\in[-1,0]} |\psi_t(x)(s) - \varphi_t(x)(s)| \stackrel{\text{def}}{=} ||\psi_t(x) - \varphi_t(x)||_0 \leq \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon)),$$

where either $B = \eta_1 > 0$ if $x \in Y_n \cap W_n$ or $B = \eta_{n+1} - 1 - \epsilon\underline{r} > 0$ if $x \in X_n \cap W_n$, and η_1 and η_{n+1} are numbers appearing in Eq. 23, and

$$\beta \stackrel{\text{def}}{=} \frac{\mu\nu}{\mu+\nu} \min\{\delta_1, \delta_2, \delta_3, \ldots, \delta_n\}.$$

(ii) *If $x \in Y_n \cap W_n$ then it is in $\overline{Y}_n$ with $T(x) = 1 + \epsilon\underline{r} - \epsilon\exp(-a\delta_n/\epsilon)[k_3 + \mathcal{E}(1/\epsilon)]$, if $x \in X_n \cap W_n$ then it is in $\overline{X}_n$ with $T(x) = 1 + \epsilon\overline{r} - \epsilon\exp(-a\delta_n/\epsilon)[k_4 + \mathcal{E}(1/\epsilon)]$, where $\underline{r}$ and $\overline{r}$ are given in Theorem 3 and either*

$$||F_{Xn}(x) - P_{Xn} \circ F_{Xn}(x)||_0 \leq \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon))$$

or

$$||F_{Yn}(x) - P_{Yn} \circ F_{Xn}(x)||_0 \leq \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon));$$

depending on $x \in Y_n \cap W_n$ or $x \in X_n \cap W_n$, respectively.

(iii) *For $x \in Y_n \cap W_n$, let the zeroes of F_{Yn} be located at $-1 + \hat{\theta}_1 < -1 + \hat{\theta}_2 < \cdots < -1 + \hat{\theta}_{n-1} < \hat{\theta}_n = 0$. Let us define $\delta'_i \stackrel{\text{def}}{=} \hat{\theta}_i - \hat{\theta}_{i-1}$, $i = 1, \ldots, n$, with $\hat{\theta}_0 \stackrel{\text{def}}{=} -\epsilon\underline{r}$. Then $\delta' \in \underline{A}_n$ and*

$$\begin{aligned}
\delta'_1 &= \delta_1 + \epsilon[\exp(-a\delta_n/\epsilon)(k_3 + \mathcal{E}(1/\epsilon)) - \exp(-a\delta_1/\epsilon)(k_3 + \mathcal{E}(1/\epsilon))] - \epsilon(\overline{r} - \underline{r}) \\
\delta'_i &= \delta_i + \epsilon[\exp(-a\delta_{i-1}/\epsilon)(k_4 + \mathcal{E}(1/\epsilon)) - \exp(-a\delta_i/\epsilon)(k_3 + \mathcal{E}(1/\epsilon))] - \epsilon(\overline{r} - \underline{r}), \\
&\quad i > 1 \text{ odd} \\
\delta'_i &= \delta_i + \epsilon[\exp(-a\delta_{i-1}/\epsilon)(k_3 + \mathcal{E}(1/\epsilon)) - \exp(-a\delta_i/\epsilon)(k_4 + \mathcal{E}(1/\epsilon))] + \epsilon(\overline{r} - \underline{r}), \\
&\quad i \text{ even}.
\end{aligned}$$

For $x \in X_n \cap W_n$, let the zeroes of F_{Xn} be located at $-1 + \hat{\theta}_1 < -1 + \hat{\theta}_2 < \cdots < -1 + \hat{\theta}_{n-1} < \hat{\theta}_n = 0$. Let us define $\delta'_i \stackrel{\text{def}}{=} \hat{\theta}_i - \hat{\theta}_{i-1}$, $i = 1, \ldots, n$, with $\hat{\theta}_0 \stackrel{\text{def}}{=} -\epsilon\overline{r}$.

Then $\delta' \in A_n$ *and*

$$\begin{aligned}
\delta_1' &= \delta_1 + \epsilon[\exp(-a\delta_n/\epsilon)(k_4 + \mathcal{E}(1/\epsilon)) - \exp(-a\delta_1/\epsilon)(k_4 + \mathcal{E}(1/\epsilon))] + \epsilon(\overline{r} - \underline{r})\\
\delta_i' &= \delta_i + \epsilon[\exp(-a\delta_{i-1}/\epsilon)(k_3 + \mathcal{E}(1/\epsilon)) - \exp(-a\delta_i/\epsilon)(k_4 + \mathcal{E}(1/\epsilon))]\\
&\quad +\epsilon(\overline{r} - \underline{r}), \quad i > 1 \textit{ odd}\\
\delta_i' &= \delta_i + \epsilon[\exp(-a\delta_{i-1}/\epsilon)(k_4 + \mathcal{E}(1/\epsilon)) - \exp(-a\delta_i/\epsilon)(k_3 + \mathcal{E}(1/\epsilon))]\\
&\quad -\epsilon(\overline{r} - \underline{r}), \quad i \textit{ even}.
\end{aligned}$$

The same comments concerning the "almost invariance" of W_n under the flow ψ made right after the statement of Theorem 2 also apply in this case.

Differently from the positive feedback case neither F_{Yn} nor F_{Xn} are mappings close to the identity. The same is of course true of the mappings $G_{Yn} \stackrel{\text{def}}{=} \delta \to \delta'$ (from A_n to $\underline{A}_n$) and $G_{Xn} \stackrel{\text{def}}{=} \underline{\delta} \to \delta'$ (from $\underline{A}_n$ to A_n) given in Theorem 4. In this case a more natural mapping to be considered is the composition $F_n \stackrel{\text{def}}{=} F_{Xn} \circ F_{Yn} : \overline{Y}_n \to Y_n$, or the corresponding mapping for its zeroes which is $G_n \stackrel{\text{def}}{=} G_{Xn} \circ G_{Yn} : A_n \to A_n$. From the expressions for G_{Yn} and G_{Xn} in Theorem 4 we get that G is given by (for simplicity all terms $\mathcal{E}(1/\epsilon)$ are omitted in the following equations)

$$\begin{aligned}
\delta_1' &= \delta_1 + \epsilon[K_3 \exp(-a\delta_n/\epsilon) - K_3 \exp(-a\delta_1/\epsilon)]\\
\delta_i' &= \delta_i + \epsilon[K_4 \exp(-a\delta_{i-1}/\epsilon) - K_3 \exp(-a\delta_i/\epsilon)], \quad i = 3, 5, \ldots, n\\
\delta_i' &= \delta_i + \epsilon[K_3 \exp(-a\delta_{i-1}/\epsilon) - K_4 \exp(-a\delta_i/\epsilon)], \quad i = 2, 4, \ldots, n-1,
\end{aligned} \tag{25}$$

where

$$K_3 = k_3 + k_4 \exp[a(\overline{r} - \underline{r})] > 0 \quad K_4 = k_3 \exp[-a(\overline{r} - \underline{r})] + k_4 > 0.$$

Now, let x be an initial condition which satisfies the hypotheses of Theorem 4. Let $\delta_m < \delta_i$, $i = 1, 2, \ldots, n$, with $i \neq m$. For simplicity let us suppose that m is even (the case where m is odd can also be analyzed). Now, if all terms in the equations in (25) are compared to $\exp(-a\delta_m/\epsilon)$ and all those of order $\exp(-a\delta_m/\epsilon)\mathcal{E}(1/\epsilon)$ are neglected, then we are left with the following set of equations:

$$\begin{aligned}
\delta_i' &= \delta_i, \quad \text{for} \quad i = 1, 2, \ldots, n, \quad i \neq m-1, \quad i \neq m,\\
\delta_{m-1}' &= \delta_{m-1} + \epsilon K_4 \exp(-a\delta_m/\epsilon),\\
\delta_m' &= \delta_m - \epsilon K_4 \exp(-a\delta_m/\epsilon).
\end{aligned} \tag{26}$$

This Eq. 26 is exactly the same as Eq. 15 obtained for the motion of zeroes in the positive feedback case. So the same analysis made in the Sect. 3 for the annihilation of the closest lying zeroes also applies to this case. In particular, assuming that the estimates for the motion of zeroes given in Theorem 4 remain valid until some fast dynamical process eliminates two zeroes that become sufficiently close, we get the following asymptotic expression for T_d, the time it takes for the pair of zeroes θ_{m-1} and θ_m to disappear:

$$\ln T_d = \frac{\delta_m a}{\epsilon} + \ln\left(\frac{2}{aK_4}\right) + \mathcal{O}(\epsilon^0)$$

The factor 2 appearing inside the logarithm is due to the approximate period $2 + \epsilon(\overline{r} + \underline{r})$ of the metastable solutions of the negative feedback equation.

Finally, the same sort of comparison with numerical results as done at the end of the previous section can also be done here. Moreover, the same comments about the assumption on the initial condition, $\delta_m < \delta_i, i = 1, 2, \ldots, n$, with $i \neq m$, and on the stability of periodic solutions, also apply to the negative feedback case with some simple modifications.

4 A Remark on the Global Monotonicity Hypotheses

The hypotheses (HP2) and (HN2), namely $f'(x) \geq 0$ and $f'(x) \leq 0$ for $x \in \mathbb{R}$, respectively, can be considerably relaxed. This is a consequence of the following propositions.

Proposition 1 *If f satisfies hypotheses* (HP1) *and* (HP3)*, then there exist $\delta > 0$, and a twice continuously differentiable $\tilde{f} : \mathbb{R} \to \mathbb{R}$, such that $\tilde{f}$ satisfies* (HP1), (HP2), (HP3) *and $f(x) = \tilde{f}(x)$ for $x \in [-\gamma_1 - \delta, \gamma_2 + \delta]$.*

Proposition 2 *If f satisfies hypotheses* (HN1), (HN3), (HN5)*, and*

(HN2′)

$$f'(x) = \frac{df}{dx}(x) \leq 0, \quad \text{for} \quad x \in [-\gamma_1, \gamma_2]$$

(HN4′)

$$|f(f(x))| > |x| \quad \text{for} \quad x \in (-\gamma_1, 0) \cup (0, \gamma_2),$$

then there exist $\delta > 0$, and a twice continuously differentiable $\tilde{f} : \mathbb{R} \to \mathbb{R}$, such that $\tilde{f}$ satisfies (HN1), (HN2), (HN3), (HN4), (HN5) *and $f(x) = \tilde{f}(x)$ for $x \in [-\gamma_1 - \delta, \gamma_2 + \delta]$.*

We shall prove proposition 2, the proof of proposition 1 being similar.

Hypotheses (HN2′), (HN3), and (HN5) imply that $f'(-\gamma_1) < 0$, $f'(\gamma_2) < 0$, and $f'(-\gamma_1)f'(\gamma_2) < 1$. Thus, there exists $\delta > 0$ such that if $\tilde{\gamma}_1 \stackrel{\text{def}}{=} \gamma_1 + \delta$ and $\tilde{\gamma}_2 \stackrel{\text{def}}{=} \gamma_2 + \delta$ then $f'(x) < 0$ for $x \in [-\tilde{\gamma}_1, -\gamma_1] \cup [\gamma_2, \tilde{\gamma}_2]$ and

$$k_1 \stackrel{\text{def}}{=} \min\{f'(x) : x \in [-\tilde{\gamma}_1, \gamma_1]\}, \quad k_2 \stackrel{\text{def}}{=} \min\{f'(x) : x \in [\gamma_2, \tilde{\gamma}_2]\}$$

satisfy $0 < k_1 k_2 < 1$. Let $\eta_1 > 0$ and $\eta_2 > 0$ be given by

$$\eta_1 = \min\left\{\frac{-2f'(\tilde{\gamma}_2)}{|f''(\tilde{\gamma}_2)|}, \frac{-2f'(-\tilde{\gamma}_1)}{|f''(-\tilde{\gamma}_1)|}\right\}, \quad \eta_2 = \frac{-A + \sqrt{A^2 + 4CB}}{B},$$

where $A > 0$, $B > 0$, and $C > 0$ are given by

$$A = |f''(-\tilde{\gamma}_1)k_2| + |f''(\tilde{\gamma}_2)k_1|, \quad B = |f''(-\tilde{\gamma}_1)f''(\tilde{\gamma}_2)|, \quad C = 1 - k_1 k_2.$$

Let $\eta > 0$ be such that $\eta < \min\{\eta_1, \eta_2\}$ and $\hat{\gamma}_1 \stackrel{\text{def}}{=} \tilde{\gamma}_1 + \eta$, $\hat{\gamma}_2 \stackrel{\text{def}}{=} \tilde{\gamma}_2 + \eta$. Let $\tilde{f}$ be defined as

$$\tilde{f}(x) = f(x), \quad x \in [-\tilde{\gamma}_1, \tilde{\gamma}_2]$$

$$\tilde{f}(x) = f(-\tilde{\gamma}_1) + f'(-\tilde{\gamma}_1)(x + \tilde{\gamma}_1) + f''(-\tilde{\gamma}_1)\frac{(x + \tilde{\gamma}_1)^2}{2} + f''(-\tilde{\gamma}_1)\frac{(x + \tilde{\gamma}_1)^3}{6\eta},$$
$$x \in [-\hat{\gamma}_1, -\tilde{\gamma}_1]$$

$$\tilde{f}(x) = f(\tilde{\gamma}_2) + f'(\tilde{\gamma}_2)(x - \tilde{\gamma}_2) + f''(\tilde{\gamma}_2)\frac{(x - \tilde{\gamma}_2)^2}{2} - f''(\tilde{\gamma}_2)\frac{(x - \tilde{\gamma}_2)^3}{6\eta}, \quad x \in [\tilde{\gamma}_2, \hat{\gamma}_2]$$

$$\tilde{f}(x) = f(-\tilde{\gamma}_1) - f'(-\tilde{\gamma}_1)\eta + f''(-\tilde{\gamma}_1)\frac{\eta^2}{3} + \left[f'(-\tilde{\gamma}_1) - f''(-\tilde{\gamma}_1)\frac{\eta}{2}\right](x + \hat{\gamma}_1), \; x \leq -\hat{\gamma}_1$$

$$\tilde{f}(x) = f(\tilde{\gamma}_2) + f'(\tilde{\gamma}_2)\eta + f''(\tilde{\gamma}_2)\frac{\eta^2}{3} + \left[f'(\tilde{\gamma}_2) + f''(\tilde{\gamma}_2)\frac{\eta}{2}\right](x - \hat{\gamma}_2), \quad x \geq \hat{\gamma}_2$$

The function $\tilde{f}$ is twice continuously differentiable, and since $\tilde{f}''(x) \neq 0$ for $x \in (-\hat{\gamma}_1, -\tilde{\gamma}_1)$ then $\tilde{f}'(x) \leq \max\{\tilde{f}'(-\tilde{\gamma}_1), \tilde{f}'(-\hat{\gamma}_1)\}$ for $x \in [-\hat{\gamma}_1, -\tilde{\gamma}_1]$. The choice of δ and $\eta < \eta_1$ imply that $\tilde{f}'(-\tilde{\gamma}_1) < 0$ and $\tilde{f}'(-\hat{\gamma}_1) < 0$, respectively. Thus $\tilde{f}'(x) < 0$ for $x \in [-\hat{\gamma}_1, -\tilde{\gamma}_1]$. A similar argument gives $\tilde{f}'(x) < 0$ for $x \in [\tilde{\gamma}_2, \hat{\gamma}_2]$. Since $\tilde{f}'(x) = \tilde{f}'(-\hat{\gamma}_1) < 0$ for $x \leq -\hat{\gamma}_1$ and $\tilde{f}'(x) = \tilde{f}'(\hat{\gamma}_2) < 0$ for $x \geq \hat{\gamma}_2$, we get that $\tilde{f}(x) \leq 0$ for $x \in \mathbb{R}$. Finally, we claim that $|\tilde{f}(\tilde{f}(x))| < |x|$ for $x \in (-\infty, -\gamma_1) \cup (\gamma_2, \infty)$. The proof of our claim is the following. Since $\tilde{f}'(x) < 0$ for $x \in (-\infty, -\gamma_1] \cup [\gamma_2, \infty)$, $\tilde{f}(-\gamma_1) = f(-\gamma_1) = \gamma_2$, and $\tilde{f}(\gamma_2) = f(\gamma_2) = -\gamma_1$ then $x \in (-\infty, -\gamma_1)$ implies $f(x) \in (\gamma_2, \infty)$, and $x \in (\gamma_2, \infty)$ implies $f(x) \in (-\infty, -\gamma_1)$. Let $F(x) = f(f(x))$. Since $F(-\gamma_1) = -\gamma_1$ and $F(\gamma_2) = \gamma_2$, in order to prove the claim it is enough to show that $F'(x) < 1$ for $x \in (-\infty, -\gamma_1] \cup [\gamma_2, \infty)$. If $x_1 \in (-\infty, -\gamma_1)$, then $x_2 = f(x_1) \in (\gamma_2, \infty)$ (if $x_1 \in (\gamma_2, \infty)$, then $x_2 = f(x_1) \in (-\infty, -\gamma_1)$) and $F'(x_1) = f'(x_2)f'(x_1)$. So, to prove that $F'(x) < 1$ for $x \in (-\infty, -\gamma_1] \cup [\gamma_2, \infty)$, it is enough to show that $f'(x_2)f'(x_1) < 1$ for any $x_1 \in (-\infty, -\gamma_1]$ and $x_2 \in [\gamma_2, \infty)$. Since $\tilde{f}''(x) \neq 0$ for $x \in (-\hat{\gamma}_1, -\tilde{\gamma}_1)$ and $\tilde{f}'(x) = \tilde{f}(-\hat{\gamma}_1)$ for $x \leq -\hat{\gamma}_1$ then $\tilde{f}'(x) \geq \min\{f'(-\tilde{\gamma}_1), \tilde{f}'(-\hat{\gamma}_1)\}$ for $x \leq -\tilde{\gamma}_1$. From the definition of k_1 we get $0 > \tilde{f}'(x) \geq \min\{k_1, \tilde{f}'(-\hat{\gamma}_1)\}$ for $x \leq -\gamma_1$ Similarly, $0 > \tilde{f}'(x) \geq \min\{k_2, \tilde{f}'(\hat{\gamma}_2)\}$ for $x \geq \gamma_2$. Therefore

$$\tilde{f}'(x_1)\tilde{f}'(x_2) \leq \max\{k_1k_2, k_1\tilde{f}'(\hat{\gamma}_2), k_2\tilde{f}'(-\hat{\gamma}_1), \tilde{f}'(-\hat{\gamma}_1)\tilde{f}'(\hat{\gamma}_2)\}.$$

Now, the definition of η_2 is such that the right hand side of this inequality is smaller than one.

Propositions 1 and 2 say that if f satifies certain hypotheses on the finite interval $[-\gamma_1, \gamma_2]$, then it is possible to find another function $\tilde{f}$, that coincides with f in the interval $[-\gamma_1 - \delta, \gamma_2 + \delta]$, $\delta > 0$, and such that $\tilde{f}$ satisfies the hypotheses of either Theorem 1 or Theorem 3. Therefore, the transition layer equations with $\tilde{f}$ replacing f admit transition layer solutions, and since these solutions take values in the interval $[-\gamma_1, \gamma_2]$, they are also solutions to the transition layer equations for f. Thus we obtain the existence of transition layer solutions to the original transition layer equations under the weaker hypotheses in propositions 1 and 2. The analysis of long transients made in Theorems 2 and 4 are still valid provided the initial conditions generate solutions that take values inside $[-\gamma_1 - \delta, \gamma_2 + \delta]$. Solutions that take values outside this interval may not be analyzed with the results in this paper. Indeed the reasoning above shows that without a global monotonicity hypothesis, Eq. 1 can have several "invariant intervals" that support metastable patterns. To avoid such problems we decided to state our results under the global monotonicity hypothesis. For those interested in problems where the global monotonicity hypothesis is not valid but the hypotheses in propositions 1 and 2 hold, the following proposition ([13] Proposition 1.1) may be very useful in the characterization of invariant intervals.

Proposition 3 (Mallet-Paret and Nussbaum) *Let $x(t; \epsilon, \phi)$ be the solution of Eq. 1 where $\phi \in C^0[-1, 0]$ is the initial condition, $\epsilon > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then the following holds.*

(i) *Let $I \subseteq \mathbb{R}$ be a closed (possibly infinite) interval such that $f(I) \subseteq I$. If*

$$\phi(t) \in I \quad \text{for all} \quad t \in [-1, 0] \tag{27}$$

then $x(t; \epsilon, \phi) \in I$ for all $t \geq 0$. If in addition $\phi(0) \in \text{int}(I)$, where "int" *denotes interior, then $x(t; \epsilon, \phi) \in \text{int}(I)$ for all $t \geq 0$.*

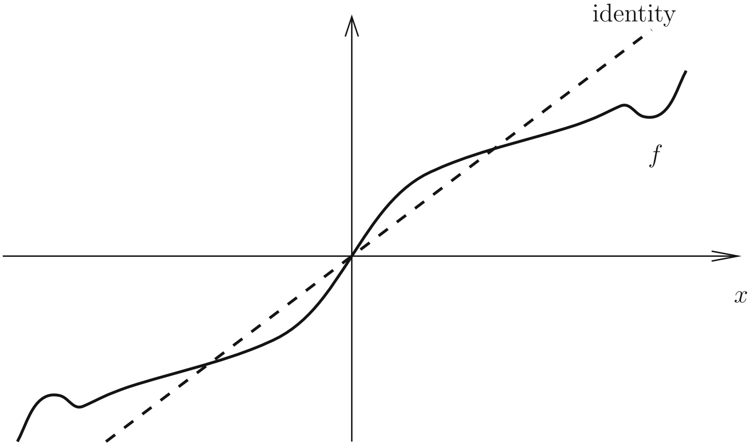

Fig. 3 Example of a function f that is not globally monotone but displays monotonicity in an interval that is attractive

(ii) *Further, define the set*

$$I_\infty = \bigcap_{n=0}^{\infty} \overline{f^n(I)};$$

necessarily I_∞ *is a closed connected subset of* I. *If* $I_\infty \neq \emptyset$, *then the solution* $x(t;\epsilon,\phi)$ *of Eq.* 1 *with* (27) *satisfies*

$$\text{dist}\,(x(t;\epsilon,\phi), I_\infty) \to 0 \quad \text{as} \quad t \to \infty$$

where "dist" *denotes distance from a point to a set.*

Using proposition 3 we easily find examples of function f that are not globally monotone but satisfy the conditions in either propositions 1 or 2 and, moreover, any initial condition generates a solution that after a finite time has values inside the interval $[-\gamma_1-\delta, \gamma_2+\delta]$ (see for instance Fig. 3). So, in this case, the results in the present work apply to any solution of the Eq. 1. This question of attracting monotonicity intervals is further analized in [19] where explicit examples are presented.

5 Proof of Theorem 1

To prove Theorem 1 it is sufficient to show the existence and uniqueness of r_* and ϕ. The existence of r_{**} and χ is a consequence of this result applied to Eq. 3 after the change of variables $y \to -y$. If f is odd then $\phi(t) = -\chi(t)$ is a consequence of the symmetry of Eq. 3 with respect to the change of variables $y \to -y$.

The proof of Theorem 1 will be given in several steps. In Sect. 5.1 we consider a family of auxiliary problems defined in compact sets $[-L, L]$ of the real line. We show that these problems have solutions ϕ_L, r_L for all L. In Sect. 5.2 we show that r_L is uniformly bounded with respect to L, from above and below, and that there is a sequence L_n, $n = 1, 2, \ldots$, of values of L such that $\phi_{L_n}, r_{L_n} \to \phi, r_*$, in compact subsets of $\mathbb{R}$, as $n \to \infty$. In Sect. 5.3 we show that the function ϕ obtained in Sect. 5.2 has the properties in Theorem 1. In Sect. 5.4 we show that for a given r_* there is only one ϕ and we obtain its asymptotic behavior as stated in Theorem 1. Finally, in Sect. 5.5 we prove the uniqueness of r_* among all possible $r \in \mathbb{R}$.

5.1 A Family of Approximating Problems

From Sect. 5.1 through 5.4 we shall assume that $r > 0$. The following definitions are useful:

- For $L > 0$, let C_L be the Banach space defined by

$$C_L \stackrel{\text{def}}{=} \{z : [-L, L] \to \mathbb{R} \mid z \quad \text{continuous}\}, \quad ||z||_L = \sup_{|t|\le L} |z(t)|.$$

- Let Λ_L be the following subset of C_L (endowed with the induced topology)

$$\Lambda_L \stackrel{\text{def}}{=} \{z \in C_L \mid z(0) = 0,\ t \le t' \Rightarrow z(t) \le z(t'), \quad -\gamma_1 \le z(t) \le \gamma_2\}.$$

Proposition 4 *The set Λ_L has the following properties*:

(i) *it is bounded,*
(ii) *it is closed,*
(iii) *it is convex.*

These properties can be easily verified.

Let X be the set of functions given by

$$X \stackrel{\text{def}}{=} \{z : \mathbb{R} \to \mathbb{R} \mid z \quad \text{continuous for } t \in \mathbb{R}, \text{nondecreasing for } \ t < 0,$$
$$\text{strictly increasing for } t > 0, \quad z(0) < 0,$$
$$\lim_{t\to-\infty} z(t) = -\gamma_1, \quad \text{and} \lim_{t\to\infty} z(t) = \gamma_2\}.$$

We endow X with the metric $d(x, z) = \sup_{t\in\mathbb{R}}|x(t) - z(t)|$. Denoting by $z|_L$ the restriction of a function $z : \mathbb{R} \to \mathbb{R}$ to the interval $[-L, L]$, we define the set X_L as

$$X_L \stackrel{\text{def}}{=} \{z : \mathbb{R} \to \mathbb{R} \mid z|_L \in \Lambda_L, \quad z(t) = -\gamma_1,\ t < -L, \quad z(t) = \gamma_2,\ t > L\}.$$

We endow X_L with the the metric $d(x, z) = \sup_{|t|\le L}|x(t) - z(t)|$. Notice that every function in X_L is an extension to $\mathbb{R}$ of a function in Λ_L, originally defined on the interval $[-L, L]$. We denote this extension mapping by $\overline{\Gamma} : \Lambda_L \to X_L$. We define a mapping $\underline{A}_L : X_L \to X$ by

$$\underline{A}_L z(t) \stackrel{\text{def}}{=} \mathrm{e}^{-t} \int_{-\infty}^{t} \mathrm{e}^s f(z(s))ds = \int_{-\infty}^{0} \mathrm{e}^s f(z(s + t))ds.$$

It can be verified that $\underline{A}_L z$ indeed belongs to X. For each $z \in X$ there exists a unique $r(z) \in \mathbb{R}$, $r(z) > 0$, such that $z(r(z)) = 0$. For a fixed L, the composed function $r \circ \underline{A}_L : X_L \to \mathbb{R}_+$ satisfies the following bounds, independent of z.

Proposition 5 *For a given L and any $z \in X_L$ we have*

$$\frac{\mathrm{e}^{-L}\gamma_1}{\gamma_2} + 1 \le \mathrm{e}^{r\circ\underline{A}_L(z)} \le \frac{\gamma_1}{\gamma_2} + \mathrm{e}^L.$$

Proof In the following, to simplify the notation, we will write $r \circ \underline{A}_L(z)$ just as r. The definition of $\underline{A}_L$ implies that

$$\int_{-\infty}^{r} \mathrm{e}^s f(z(s))ds = 0. \tag{28}$$

If $r \leq L$ then the upper bound for r is trivial. So, let us assume that $r > L$. Using that $-f(z(s)) \leq \gamma_1$ for $s \leq 0$ and that $f(z(s)) \geq 0$ for $s \geq 0$, Eq. 28 implies

$$\gamma_1 = \gamma_1 \int_{-\infty}^{0} \mathrm{e}^s ds \geq -\int_{-\infty}^{0} \mathrm{e}^s f(z(s))ds = \int_{0}^{r} \mathrm{e}^s f(z(s))ds \geq \int_{L}^{r} \mathrm{e}^s f(z(s))ds = \gamma_2[\mathrm{e}^r - \mathrm{e}^L].$$

This inequality implies the upper bound for r. Equation (28) implies that

$$\mathrm{e}^{-L}\gamma_1 - \int_{-L}^{0} \mathrm{e}^s f(z(s))ds = \int_{0}^{r} \mathrm{e}^s f(z(s))ds. \tag{29}$$

The lower bound for r comes from the following inequality obtained from Eq. 29

$$\mathrm{e}^{-L}\gamma_1 \leq \int_{0}^{r} \mathrm{e}^s f(z(s))ds \leq \gamma_2(e^r - 1).$$

□

We define the set X_* as

$$X_* \stackrel{\text{def}}{=} \{z : \mathbb{R} \to \mathbb{R} |\, z \quad \text{continuous for } t \in \mathbb{R}, \text{nondecreasing for } \ t < 0,$$
$$\text{strictly increasing for } t > 0,$$
$$\lim_{t\to-\infty} z(t) = -\gamma_1, \text{ and } \lim_{t\to\infty} z(t) = \gamma_2\}.$$

We endow X_* with the metric $d(x, z) = \sup_{t\in\mathbb{R}}|x(t) - z(t)|$. We define the mapping T_r : $X \to X_*$ as $T_r z(t) = z(t + r(z))$ and the restriction mapping $\underline{\Gamma} : X_* \to \Lambda_L$. Finally, we define a mapping $A_L : \Lambda_L \to \Lambda_L$ as

$$A_L = \underline{\Gamma} \circ T_r \circ \underline{A}_L \circ \overline{\Gamma}. \tag{30}$$

$A_L z : [-L, L] \to \mathbb{R}$ is continuous, bounded, nondecreasing, and satisfies $A_L z(0) = 0$. So, $A_L z$ indeed belongs to Λ_L. A more explicit way to write A_L is

$$A_L z(t) \stackrel{\text{def}}{=} \mathrm{e}^{-t-r} \int_{-\infty}^{t+r} \mathrm{e}^s f_L(z(s))ds, \tag{31}$$

where

$$f_L(z(s)) = -\gamma_1 \quad \text{for} \quad s < -L,$$
$$f_L(z(s)) = \gamma_2 \quad \text{for} \quad s > L,$$
$$f_L(z(s)) = f(z(s)) \quad \text{for} \quad |s| \leq L.$$

Proposition 6 *The mapping $A_L : \Lambda_L \to \Lambda_L$ (Eq. 30) is continuous.*

Proof $\overline{\Gamma}$ and $\underline{\Gamma}$ are continuous, and to prove the proposition 6 we show that $\underline{A}_L$ and T_r are continuous. The continuity of $\underline{A}_L$ is proved in the following way. As f is continuously differentiable there exists a constant μ such that

$$|f(x) - f(y)| \leq \mu|x - y| \quad \text{for} \quad -\gamma_1 \leq x \leq \gamma_2, \quad -\gamma_1 \leq y \leq \gamma_2.$$

Thus, if $x \in X_L$, $z \in X_L$, satisfy $d(x, z) < \delta$ then

$$|\underline{A}_L x(t) - \underline{A}_L z(t)| = \left| \int_0^\infty \mathrm{e}^{-s}[f(x(t-s)) - f(z(t-s))]ds \right| \leq \mu\delta \int_0^\infty \mathrm{e}^{-s} ds = \mu\delta,$$

implying the continuity of $\underline{A}_L$. Before proving the continuity of T_r, it should be reminded that r is a function of the point $z \in X_L$ to which T_r is applied. Let us denote by z, z' two points in X and by r and r' their respective zeroes ($z(r) = 0$, $z'(r') = 0$), or, equivalently, the values of the function r at z and z' ($r(z) = r$, $r(z') = r'$). We want to show that for any given $z \in X$ and $\epsilon > 0$ there exists a $\delta > 0$ such that $d(z, z') < \delta$ implies that

$$d(T_{r'}z', T_r z) = \sup_{t \in \mathbb{R}} |T_{r'}z'(t) - T_r z(t)| = \sup_{t \in \mathbb{R}} |z'(t + r')| - |z(t + r)| < \epsilon,$$

where we have used the notation $T_r z(t) = z(t + r)$ and $T_{r'}z'(t) = z'(t + r')$. Let $\delta < \epsilon/2$. Since

$$\begin{aligned} |z'(t + r') - z(t + r)| &= |z'(t + r') - z(t + r') + z(t + r') - z(t + r)| \\ &\leq |z'(t + r') - z(t + r')| + |z(t + r') - z(t + r)|, \end{aligned}$$

and $|z'(t + r') - z(t + r')| < \delta < \epsilon/2$ for any $t \in \mathbb{R}$, we just have to show that it is possible to further decrease $\delta > 0$ such that the following inequality becomes true

$$\sup_{t \in \mathbb{R}} |T_{r'}z(t) - T_r z(t)| = d(z(t + r'), z(t + r)) = d(z(t + r' - r), z(t)) < \epsilon/2.$$

The continuity, monotonicity, and boundedness of z imply that z is uniformly continuous. So, it is possible to find the desired δ if we show that the function $z \to r(z)$ is continuous, namely, that for any given $z \in X$ and $\overline{\epsilon} > 0$ there exists a $\overline{\delta} > 0$ such that $d(z', z) < \overline{\delta}$ implies $|r' - r| < \overline{\epsilon}$. Setting $\epsilon_1 \stackrel{\text{def}}{=} \min\{\overline{\epsilon}, r/2\}$, we have $z(r - \epsilon_1) < 0 < z(r + \epsilon_1)$ as z is strictly increasing on $(0, \infty)$. Taking $\overline{\delta} \stackrel{\text{def}}{=} \min\{|z(r - \epsilon_1)|, z(r + \epsilon_1)\} > 0$, for z' such that $d(z, z') < \overline{\delta}$, we have also $z'(r - \epsilon_1) < 0 < z'(r + \epsilon_1)$ so that z' has a zero r' satisfying $|r - r'| \geq \epsilon_1 \leq \overline{\epsilon}$, which proves that $z \to r(z)$ is continuous and ends the proof of the proposition 6. □

Proposition 7 *The mapping A_L is completely continuous, namely, A_L is continuous and maps bounded sets into compact sets (see* [9] *Sect. 2.2).*

Proof Since $\Lambda_L \subset C_L$ is bounded and $A_L : \Lambda_L \to \Lambda_L$ is continuous by proposition 6, to prove that A_L is completely continuous, it is enough to show that the range of A_L is compact. This is a consequence of the Arzela-Ascoli's theorem if we show that there exists a constant K', independent of $z \in \Lambda_L$, such that

$$|A_L z(t) - A_L z(t')| \leq K'|t - t'| \quad \text{for all} \quad |t| \leq L, \ |t'| \leq L.$$

The definition of A_L, and the fact that $r(z) > 0$, imply that the above inequality is true if there exists a constant K, independent of $z \in X_L$, such that

$$|\underline{A}_L z(t) - \underline{A}_L z(t')| \leq K|t - t'| \quad \text{for all} \quad t > -L,\ t' > -L. \tag{32}$$

For $|t| < L$, $\underline{A}_L z$ is differentiable, and

$$\frac{d}{dt}\underline{A}_L z(t) = -\underline{A}_L z(t) + f(z(t)),$$

which implies

$$\left|\frac{d}{dt}\underline{A}_L z(t)\right| \leq |\underline{A}_L z(t)| + |f(z(t))| \leq 2\max\{\gamma_1, \gamma_2\}. \tag{33}$$

For $t > L$, $\underline{A}_L z$ is explicitly given by

$$\begin{aligned}\underline{A}_L z(t) &= \mathrm{e}^{-t}\left\{-\mathrm{e}^{-L}\gamma_1 + \int_{-L}^{L} \mathrm{e}^s f(z(s))ds + \gamma_2(\mathrm{e}^t - \mathrm{e}^L)\right\}\\ &= \mathrm{e}^{-t}\left\{\underline{A}_L z(L) + \gamma_2(\mathrm{e}^t - \mathrm{e}^L)\right\},\end{aligned}$$

which implies that $\underline{A}_L z$ is differentiable and

$$\left|\frac{d}{dt}\underline{A}_L z(t)\right| \leq \mathrm{e}^{-t}|\underline{A}_L z(L)| + \gamma_2 \leq 2\gamma_2. \tag{34}$$

Inequalities (33) and (34), and the continuity of $\underline{A}_L z$ at $t = L$, imply that inequality (32) is true, thus proving the proposition 7. □

The following proposition is an immediate consequence of the definition of Λ_L.

Proposition 8 *The null function* $\underline{0} \in \Lambda_L$ *is not a fixed point of* A_L.

Finally, propositions 4, 7 and 8, and the Schauder fixed point theorem (see for instance [9], Sect. 2.2), imply the following lemma.

Lemma 1 *The mapping* $A_L : \Lambda_L \to \Lambda_L$ *has a fixed point* ϕ_L *different from* $\underline{0}$.

5.2 Uniform Bounds

Taking the fixed point ϕ_L given by Lemma 1, we set

$$\phi_{L*}(s) = \overline{\Gamma}(\phi_L)(s),$$

i.e.,

$$\begin{aligned}&\phi_{L*}(s) = \phi_L(s) \quad \text{for} \quad |s| \leq L,\\ &\phi_{L*}(s) = -\gamma_1 \quad \text{for} \quad s < -L,\\ &\phi_{L*}(s) = \gamma_2 \quad \text{for} \quad s > L.\end{aligned}$$

Furthermore, we denote by $r_L = r \circ \underline{A}_L \circ \overline{\Gamma}(\phi_L)$ the zero of $\underline{A}_L \circ \overline{\Gamma}(\phi_L)$, i.e.

$$\mathrm{e}^{-r_L}\int_{-\infty}^{r_L} \mathrm{e}^s f(\phi_{L*}(s))ds = 0. \tag{35}$$

Our goal in this section is to find bounds, independent of L, for r_L and for the derivative of ϕ_L. From the definition of A_L (Eq. 31), for $|t| \leq L$, we have

$$\phi_L(t) = e^{-t-r_L} \int_{-\infty}^{t+r_L} e^s f(\phi_{L*}(s))ds. \tag{36}$$

Using (35) we can rewrite Eq. 36 as

$$\phi_L(t) = e^{-t-r_L} \int_{r_L}^{t+r_L} e^s f(\phi_{L*}(s))ds. \tag{37}$$

We shall find an upper bound for r_L in several steps.

Proposition 9 *There exists $M_1 > 0$ such that if $L > M_1$ then $r_L < L$.*

Proof Let us assume that $r_L \geq L$. Then, from (37), we obtain that for $t \in [0, L]$

$$\phi_L(t) = e^{-t-r_L} \gamma_2 \int_{r_L}^{t+r_L} e^s ds = \gamma_2(1 - e^{-t}). \tag{38}$$

Now, using (38), the facts that $|f(z)| \geq |z|$ for $-\gamma_1 \leq z \leq \gamma_2$, and $\phi_L(0) = 0$, we get

$$\begin{aligned}\gamma_1 &\geq - \int_{-\infty}^{0} e^s f(\phi_{L*}(s))ds = \int_{0}^{r_L} e^s f(\phi_{L*}(s))ds \\ &\geq \int_0^{r_L} e^s \phi_{L*}(s)ds \geq \int_0^{L} e^s \gamma_2(1 - e^{-s})ds = \gamma_2[(e^L - 1) - L].\end{aligned}$$

This inequality holds if, and only if, $L \leq M_1$, where M_1 is the positive root of

$$\frac{\gamma_1}{\gamma_2} + 1 = e^{M_1} - M_1.$$

Therefore, if $L > M_1$ then $r_L < L$. □

Proposition 10 *For $L > M_1$ the following two inequalities are true:*

$$\frac{\phi_L(r_L)}{f(\phi_L(r_L))} \geq 1 - e^{-r_L}, \tag{39}$$

$$\frac{\gamma_1}{g(r_L)} \geq \phi_L(r_L), \tag{40}$$

where $g(r_L) = e^{r_L} - 1 - r_L$.

Proof From (37), proposition 9 and $0 \leq t \leq r_L$ we obtain

$$\begin{aligned}\phi_L(t) &= e^{-t-r_L} \int_{r_L}^{t+r_L} e^s f(\phi_{L*}(s))ds \\ &\geq e^{-t-r_L} \int_{r_L}^{t+r_L} e^s f(\phi_{L*}(r_L))ds = f(\phi_L(r_L))[1 - e^{-t}].\end{aligned} \tag{41}$$

For $t = r_L$ this inequality gives (39). From inequality (41), proposition 9, and $\phi_L(0) = 0$, we obtain

$$\gamma_1 \geq -\int_{-\infty}^{0} e^s f(\phi_{L*}(s))ds = \int_{0}^{r_L} e^s f(\phi_L(s))ds$$
$$\geq \int_{0}^{r_L} e^s \phi_L(s)ds \geq \int_{0}^{r_L} e^s f(\phi_L(r_L))(1 - e^{-s})ds$$
$$= f(\phi_L(r_L))[e^{r_L} - 1 - r_L] \geq \phi_L(r_L)g(r_L).$$

□

The fact that f is continuously differentiable, $f(0) = 0$, and $\frac{df}{dz}(0) = \nu > 1$, imply that there exists $b > 0$ such that

$$\frac{f(z)}{z} > \frac{\nu + 1}{2} \quad \text{for} \quad 0 \leq z \leq b. \tag{42}$$

The function g appearing in proposition 10 has the following properties:

$$g(0) = 0, \quad \frac{dg}{dr}(r) > 0 \quad \text{for} \quad r > 0, \quad \lim_{r\to\infty} g(r) = \infty.$$

Therefore, there exists a unique r_* such that $g(r_*) = \gamma_1/b$ and $g(r) > \gamma_1/b$, for $r > r_*$. This and inequality (40) imply that

$$\phi_L(r_L) \leq \frac{\gamma_1}{g(r_L)} < b, \quad \text{if} \quad r_L > r_*. \tag{43}$$

Now, let r_{**} be the only positive root of

$$\frac{2}{\nu + 1} = 1 - e^{-r_{**}}.$$

This implies that

$$\frac{2}{\nu + 1} < 1 - e^{-r} \quad \text{if} \quad r > r_{**}. \tag{44}$$

Lemma 2 *Let* $\overline{r} = \max\{r_*, r_{**}\}$ *and* $L > M_1$. *Then* $r_L \leq \overline{r}$ *independent of* L.

Proof Let us assume that $r_L > \overline{r}$. This and inequality (43) imply that $\phi_L(r_L) < b$. Using (39) and (44) (since $r_L > \overline{r}$) we obtain

$$\frac{\phi_L(r_L)}{f(\phi_L(r_L))} \geq 1 - e^{-r_L} > \frac{2}{\nu + 1}.$$

But this inequality, and the fact that $\phi_L(r_L) < b$, contradict inequality (42). Therefore $r_L \leq \overline{r}$. □

Lemma 3 *Let* $L > M_1$ *and*

$$\underline{r} \overset{\text{def}}{=} \frac{\gamma_1 e^{-\overline{r}}}{\gamma_1 + \gamma_2} > 0.$$

Then $\underline{r} \leq r_L$, *independent of* L.

Proof Since $r_L < L$ (proposition 9) the function ϕ_L is differentiable for $t \in [-L, 0]$. Differentiating expression (36) and using that $f(z) \leq z$ for $z \in [-\gamma_1, 0]$ we obtain that, for $t \in [-L, 0]$,

$$\dot{\phi}_L(t) = -\phi_L(t) + f(\phi_L(t + r_L))$$
$$\leq -f(\phi_L(t)) + f(\phi_L(t + r_L)) = \frac{d}{dt}\int_t^{t+r_L} f(\phi_L(s))ds.$$

Integrating this inequality in the interval $[-L, 0]$, we obtain

$$-\phi_L(-L) \leq \int_0^{r_L} f(\phi_L(s))ds - \int_{-L}^{-L+r_L} f(\phi_L(s))ds \leq (\gamma_1 + \gamma_2)r_L. \qquad (45)$$

Equation 36, Lemma 2, the fact that $r_L < L$, and that $\phi_L(s) < 0$ for $s < 0$, imply that

$$\phi_L(-L) = \mathrm{e}^{L-r_L}\{-\gamma_1\mathrm{e}^{-L} + \int_{-L}^{-L+r_L} \mathrm{e}^s f(\phi_L(s))ds\} \leq -\mathrm{e}^{-r_L}\gamma_1 \leq -\mathrm{e}^{-\bar{r}}\gamma_1. \qquad (46)$$

Adding inequalities (45) and (46) proves the Lemma 3. □

Lemma 4 *There exist infinite sequences L_n, r_n, ϕ_n, $n = 1, 2, \ldots$, with $L_n \to \infty$ as $n \to \infty$, such that the limits*

$$r_n \to r > 0, \quad \textit{and} \quad \phi_n \to \phi \quad s \quad n \to \infty$$

converge. Moreover, ϕ_n converges uniformly, on compact intervals, to a function ϕ having the following properties:

- *it is continuously differentiable and nondecreasing;*
- $\phi(0) = 0$*;*
- $-\gamma_1 \leq \phi(t) \leq \gamma_2$ *for* $t \in \mathbb{R}$*;*
- *it is a solution of the transition layer Eq. 3.*

Also, $\dot{\phi}_n$ converges to $\dot{\phi}$ uniformly on compact intervals.

Proof Let $L = L_1, L_2, L_3, \ldots$ be an infinite sequence of values of L and r_{L_k}, ϕ_{L_k}, $k = 1, 2, \ldots$ be their corresponding sequences of r_L and ϕ_L. Propositions 2 and 3 imply that the sequence r_{L_k} is bounded from above and below by positive numbers. The sequence ϕ_{L_k} is bounded, $-\gamma_1 \leq \phi_{L_k}(t) \leq \gamma_2$, $|t| \leq L_k$, and it is equicontinuous (the equicontinuity is a consequence of estimates (33) and (34) that are independent of L and are also valid for ϕ_L). The remainder of the proof of this lemma involves standard limiting arguments for sub-sequences of ϕ_{L_k} and r_{L_k} using "Helly's 2nd theorem" (see, for instance, [10]), and the fact that ϕ_{L_k}, r_{L_k} satisfy the integral identity (36). □

5.3 The Nontriviality of ϕ

We shall here show that the function ϕ, obtained in the Lemma 4, satisfies

$$\lim_{t\to-\infty} \phi(t) \to -\gamma_1, \quad \lim_{t\to\infty} \phi(t) \to \gamma_2.$$

The following lemma shows that, to prove this, we only need to establish that $\phi(t)$ is nontrivial, i.e., there exists M such that $\phi(M) \neq 0$.

Lemma 5 *If there exists M such that $\phi(M) \neq 0$ then $\phi(-t)\phi(t) < 0$ for all $t \neq 0$ and*

$$\lim_{t\to-\infty} \phi(t) \to -\gamma_1, \quad \lim_{t\to\infty} \phi(t) \to \gamma_2.$$

Proof We remind that ϕ is nondecreasing and $\phi(0) = 0$. Therefore, either $M > 0$ and $\phi(M) > 0$ or, conversely, $M < 0$ and $\phi(M) < 0$. Assume that the former holds, and denote by $t_* = \sup\{t|\phi(t) = 0\}$. As ϕ is a solution of Eq. 3 we have $\dot{\phi}(t_*) = f(\phi(t_* + r)) > 0$. Given that ϕ is nondecreasing, we have $\phi(t) < 0$ for $t < t_*$. This, and the fact that $\phi(0) = 0$ imply that $t_* = 0$ and $\phi(-t)\phi(t) < 0$ for $t \neq 0$. Conversely, assume that $M < 0$ and $\phi(M) < 0$. Then, $\phi(r) > 0$ and we are back to the previous case. Indeed, suppose this is false, i.e. $\phi(r) = 0$. Then $\phi(t) = 0$ for $t \in [0, r]$, because ϕ is nondecreasing. But this contradicts the fact that ϕ is a solution of Eq. 3 (Lemma 4). Indeed, in this case the theorem of uniqueness of backward continuation of solutions of (3) would imply $\phi(t) = 0$ for all $t < 0$, which is false. In summary, we have established so far that if there exists M such that $\phi(M) \neq 0$ then

$$\phi(-t)\phi(t) < 0 \quad \text{for all} \quad t \neq 0. \tag{47}$$

This, the bounds $-\gamma_1 \leq \phi(t) \leq \gamma_2$, $t \in \mathbb{R}$, (47), and the integral equation satisfied by ϕ,

$$\phi(t) = \int_{-\infty}^{0} e^s f(\phi(s + t + r))ds, \tag{48}$$

imply the limits in the statement of Theorem 1, namely

$$\lim_{t\to-\infty} \phi(t) \to -\gamma_1, \quad \lim_{t\to\infty} \phi(t) \to \gamma_2.$$

Indeed, using that $\dot{\phi} \geq 0$ we conclude that the limits $\lim_{t\to\pm\infty} |\phi(t)| \overset{\text{def}}{=} |\phi(\pm\infty)|$ exist and are bounded by $\max\{\gamma_1, \gamma_2\}$. So, we can take the limits on both sides of Eq. 48 to conclude that $\phi(\pm\infty) = f(\phi(\pm\infty))$. This, inequalities $-\gamma \leq \phi(-\infty) < 0$ and $0 < \phi(\infty) \leq \gamma_2$, and the hypothesis (HP1) on f (see Sect. 2) imply the above limits. □

To complete the proof of the first part of Theorem 1, the only thing remaining is to establish that ϕ is nontrivial.

Lemma 6 *ϕ is nontrivial, i.e., there exists M such that $\phi(M) \neq 0$.*

The proof of this is the content of the rest of the section, and will be given in a few steps. We start with the following proposition.

Proposition 11 *We shall assume that Lemma 6 is false. Denoting $\nu = f'(0)$, then, for any $K > r$, where r is the number given in Lemma 4, there exists a continuous function $x : [-K, K] \to \mathbb{R}$, continuously differentiable on $[-K, K - r)$ that satisfies the linear equation*

$$\dot{x}(t) = -x(t) + \nu x(t + r), \quad \text{for} \quad t \in [-K, K - r). \tag{49}$$

and has the following properties: $\dot{x}(t) \geq 0$, $x(0) = 0$, $x(-t)x(t) < 0$ for $t \neq 0$, and $\dot{x}(0) > 0$.

Proof Let us consider the sequences $L_n, r_n, \phi_n, n = 1, 2, \ldots$ of Lemma 4. As we are assuming that Lemma 6 is false, then, for any $K > 0$,

$$||\phi_n||_K = \sup_{-K \le t \le K + r_n} |\phi_n(t)| \to 0 \quad \text{as} \quad n \to \infty.$$

Let N_K be such that $L_n > K + 2r_n$ for $n > N_K$. Since $\dot{\phi}_n(t) \ge 0$, for $t \in (-L_n, L_n - r_n)$, and for each $n > N_K$ there are two possibilities: either (i) $||\phi_n||_K = |\phi_n(-K)|$ or (ii) $||\phi_n||_K > |\phi_n(-K)|$, and $||\phi_n||_K = |\phi_n(K + r_n)|$.

First, assume that there are infinitely many values $n_1, n_2, \ldots$, of $n > N_K$, such that $||\phi_n||_K = |\phi_n(-K)|$. In this case we consider the sub-sequence $\phi_{n_1}, \phi_{n_2}, \ldots$, which, after relabeling, we denote again as $\phi_1, \phi_2, \ldots$ Then we define a sequence of functions $x_n : (-L_n, L_n) \to \mathbb{R}$, as

$$x_n(t) = \frac{\phi_n(t)}{|\phi_n(-K)|}.$$

Clearly $||x_n||_K = 1$. The function ϕ_n is differentiable for $t \in (-L_n, L_n - r_n)$. Differentiating expression (36) we find that in this interval ϕ_n satisfies

$$\dot{\phi}_n(t) = -\phi_n(t) + f(\phi_n(t + r_n)).$$

This implies that $x_n, n > N_K$, is differentiable on $(-L_n, L_n - r_n)$, satisfies $\dot{x}_n(t) \ge 0$, and

$$\dot{x}_n(t) = -x_n(t) + \nu x_n(t + r_n) + R(|\phi_n(-K)|, x_n(t + r_n)), \tag{50}$$

where R is a continuous function such that $R(0, x) = 0$ and, for $\xi \neq 0$,

$$R(\xi, x) \overset{\text{def}}{=} -\nu x + \frac{f(\xi x)}{\xi} \quad \text{with} \quad \nu = f'(0) > 1.$$

Integrating Eq. 50 we obtain that x_n also satisfies the following integral equation:

$$x_n(t) = \mathrm{e}^{t_0 - t} x_n(t_0) - \int_{t + r_n}^{t_0 + r_n} \mathrm{e}^{s - t - r_n} [\nu x_n(s) + R(|\phi_n(-K)|, x_n(s))] ds \tag{51}$$

for $-L_n < t \le t_0 < L_n - r_n$. Given any $\theta > 0, \theta < r/4$, let $N_{K,\theta}$ be such that $r - \theta < r_n$, for $n > N_{K,\theta}$. Each function x_n is nondecreasing and satisfies $|x_n(t)| \le 1$ for $t \in [-K, K + r - \theta]$ and $n > N_{K,\theta}$. Therefore by "Helly's second theorem" (see for instance [10]) there exists a subsequence $x_{n_1}, x_{n_2}, \ldots$, of $x_{N_{K,\theta}+1}, x_{N_{K,\theta}+2}, \ldots$, that converges point-wise to a nondecreasing function x on the interval $[-K, K + r - \theta]$. After relabeling we denote the sequence $x_{n_1}, x_{n_2}, \ldots$ as $x_1, x_2, \ldots$. We claim that this sequence of functions is uniformly equicontinuous on the interval $|t| \le K$. Indeed, for $t \in [-K, K + r_n]$, $|x_n(t)| \le 1$, which implies that $|R(|\phi_n(-K)|, x_n(t + r_n))| \to 0$ as $n \to \infty$, uniformly with respect to t, since by hypothesis $|\phi_n(-K)| \to 0$ as $n \to \infty$. This, Eq. 50, and the uniform boundedness of $|x_n(t)|$, $t \in [-K, K + r_n]$, imply that $|\dot{x}_n(t)|$ is uniformly bounded for $t \in [-K, K]$ and $n > 1$, which implies the uniform equicontinuity of x_n, for $|t| \le K$. Therefore x is continuous for $|t| \le K$ and $x_n \to x$ converges uniformly, for $|t| \le K$, as $n \to \infty$. This, Eq. 50, and the fact that $R(||\phi_n||_K, x(t + r_n)) \to 0$ as $n \to \infty$ uniformly for $t \in [-K, K]$ imply that x is continuously differentiable on the interval $[-K, K - r)$ and satisfies the linear Eq. 49. Finally, taking the limit as $n \to \infty$ in Eq. 51, for $-K < t \le t_0 \le K - \theta$, and using the

Lebesgue dominated convergence theorem we obtain that x satisfies the equation

$$x(t) = \mathrm{e}^{t_0-t}x(t_0) - \int_{t+r}^{t_0+r} \mathrm{e}^{s-t-r}\nu x(s)ds \quad \text{for} \quad -K < t \le t_0 \le K-\theta. \tag{52}$$

Furthermore, $x(-K) = -1$, $\dot{x} \ge 0$, and $x(0) = 0$. These properties, the fact that x is a solution of Eq. 49, and an argument similar to the one that lead us to statement (47), imply that $x(-t)x(t) < 0$, for $t \neq 0$, and $\dot{x}(0) > 0$, as stated in the proposition.

Now, we assume that in the sequences L_n, r_n, ϕ_n, $n = 1, 2, \ldots$ of Lemma 4, there are only finitely many values of $n > N_K$, such that $||\phi_n||_K = |\phi_n(-K)|$. In this case we consider the sub-sequence $\phi_{n_1}, \phi_{n_2}, \ldots$, that does not include these values of n. After relabeling, we denote this sequence as $\phi_1, \phi_2, \ldots$. Then we define a new sequence of functions $x_n : (-L_n, L_n) \to \mathbb{R}$, as

$$x_n(t) = \frac{\phi_n(t)}{|\phi_n(K+r_n)|}$$

and repeat the same steps of the first case to obtain a sequence of functions $x_1, x_2, \ldots$ which converges to a limit function $x : [-K, K+r-\theta]$ which has the following properties: it is nondecreasing, it is continuous for $|t| \le K$, it is continuously differentiable and satisfies Eq. 49 on the interval $[-K, K-r)$, and it satisfies Eq. 52 on the interval $t \in [-K, K-\theta]$. The difference in this case is that $x(-K)$ may be equal to zero. In order to prove that $x(t)$ is not identically zero on the interval $[-K, K-r)$ we need the following argument.

The hypothesis on f imply that $f(x) \ge x$ for $0 \le x \le \gamma_2$. So, for $-r_n \le t < L_n - r_n$ the following inequality holds:

$$\dot{\phi}_n(t) = -\phi_n(t) + f(\phi_n(t+r_n)) \ge -\phi_n(t) + \phi_n(t+r_n) = \frac{d}{dt}\int_t^{t+r_n} \phi_n(s)ds$$

or after integrating

$$\phi_n(t_0) - \int_{t_0}^{t_0+r_n} \phi_n(s)ds \ge \phi_n(t) - \int_t^{t+r_n} \phi_n(s)ds$$

with $-r_n \le t \le t_0 < L_n - r_n$. Let us divide this last expression by $\phi_n(K+r_n)$, take $t = 0$ and $t_0 = K + r/2$ to get

$$x_n(K+r/2) - \int_{K+r/2}^{K+r/2+r_n} x_n(s)ds \ge -\int_0^{r_n} x_n(s)ds.$$

Now, suppose that $x(t) = 0$ for $t \in [-0, K+r-\theta)$. Then $\lim_{n\to\infty} x_n(K+r/2) = 0$, since $\theta < r/4$ implies $K + r/2 < K + r - \theta$, and $\lim_{n\to\infty} x_n(s) \to 0$ uniformly on the interval $[0, r]$. Therefore from the above inequality we get

$$\lim_{n\to\infty} \int_{K+r/2}^{K+r/2+r_n} x_n(s)ds \le 0.$$

But this is false since x_n is a nondecreasing function of t, $x(K + r_n) = 1$, and therefore the integral on the left hand side of the above inequality must be larger then $r/2$ for any n. So, we conclude that $x(t)$ is not identically zero for $t \in [-0, K + r - \theta)$.

Finally, suppose that $x(t) = 0$ for $t \in [-0, K]$ and $x(t) > 0$ on some interval $t \in (K + \alpha, K + r - \theta)$, for some $\alpha > 0$ and $\alpha < r - \theta$. Since x is a solution of Eq. 52 with $t_0 = K - \theta$, it follows that $x(K + \alpha - r) < 0$ which is false. So, $x(K) > 0$ and again an argument similar to the one that lead us to statement (47), imply that $x(-t)x(t) < 0$, for $t \neq 0$, and $\dot{x}(0) > 0$. □

Let us define the function

$$y(t) \stackrel{\text{def}}{=} -x(-t), \quad t \in (-K, K],$$

where x is the function given in proposition 11. This function satisfies the equation

$$\dot{y}(t) = +y(t) - \nu y(t - r), \quad \text{for} \quad t \in (-K + r, K], \tag{53}$$

and has the following properties:

$$\dot{y} \geq 0, \tag{54}$$

$$y(0) = 0, \tag{55}$$

$$y(-t)y(t) < 0 \quad \text{for} \quad t \neq 0. \tag{56}$$

The following lemma contradicts the assumption that we can choose an arbitrarily large $K > 0$, thus proving Lemma 6.

Lemma 7 *There exists $M > 0$ such that if $K > M$ then no function $y : (-K + r, K] \to \mathbb{R}$ which is a solution of Eq. 53 simultaneously satisfies properties* (54)*,* (55)*, and* (56)*.*

To prove this Lemma 7 we need some definitions from the theory of linear delayed differential equations (see [2,9]). The characteristic equation related to Eq. 53 is

$$P(\lambda) \stackrel{\text{def}}{=} \lambda - 1 + \nu e^{-r\lambda} = 0. \tag{57}$$

All the roots of this characteristic equation are on the left hand side of a vertical straight line (c) in the complex plane. The fundamental solution ξ of Eq. 53 is defined as the one that satisfies $\xi(t) = 0$ for $t < 0$, and $\xi(0) = 1$. For $0 \leq t \leq r$ it is explicitly given by $\xi(t) = e^t$. The Laplace transform of ξ can be written in terms of P as

$$\hat{\xi}(u) \stackrel{\text{def}}{=} \int_0^\infty e^{-ut}\xi(t)dt = \frac{1}{P(u)}. \tag{58}$$

The function $\hat{\xi}$ is defined for u complex and is analytic on the left hand side of the line (c). Using the inverse integral for the Laplace transform (see [2,9]) the fundamental solution has the following integral representation in terms of P

$$\xi(t) = \int_{(c)} \frac{1}{P(\lambda)} e^{\lambda t} d\lambda. \tag{59}$$

Let

$$\eta \stackrel{\text{def}}{=} \max\{\text{Re}\lambda | P(\lambda) = 0\}. \tag{60}$$

There is at most one pair of complex conjugate roots $\lambda_1, \overline{\lambda}_1$ of (57) (or a single real root) such that $\mathrm{Re}\lambda_1 = \eta$. In the case that λ_1 is not real, then $\lambda_1, \overline{\lambda}_1$ are simple roots of the characteristic Eq. 57. Let

$$\eta' \stackrel{\text{def}}{=} \max\{\mathrm{Re}\lambda | P(\lambda) = 0, \quad \lambda \neq \lambda_1, \quad \lambda \neq \overline{\lambda}_1 < \eta\}. \tag{61}$$

Using (59) it can be shown [9], [2] that if $\eta < 0$ then there exist $0 < a < -\eta$, and $b > 0$, such that

$$|\xi(t)| < b\mathrm{e}^{-at}, \quad t > 0. \tag{62}$$

If $\eta \geq 0$ and $\lambda_1 = \eta + \omega i$, $\omega \neq 0$, then there exist constants $a \neq 0$, $b > 0$, $c \in [0, 2\pi)$, and $d \in (\eta', \eta)$, such that

$$|\xi(t) - a\mathrm{e}^{\eta t}\cos(\omega t + c)| \leq b\mathrm{e}^{td}, \quad t \geq 0. \tag{63}$$

This estimate is a consequence of (59) and the residue theorem (see [2] p. 116, ex.1). For $-K + r \leq t' < t \leq K$ the following "variation of constants formula" (see [2,9]) is valid

$$y(t) = y(t')\xi(t - t') - \nu \int_{-r}^{0} \xi(t - t' - s - r)y(t' + s)ds. \tag{64}$$

The Eq. 64, and the above properties of ξ, will be used to prove Lemma 7. In order to simplify the explanation we break the proof into the following three propositions (propositions 12, 13, and 14).

Proposition 12 *Assume that η defined in (60) satisfies $\eta < 0$, and that Eq. 53 has a solution y satisfying (55), (56), and such that $\dot{y} \geq 0$ for $t \in [0, r]$. Then, there is $M_1 > 0$ such that $y(K) < y(r)$ for all $K > M_1$. In particular y cannot satisfy (54) if $K > M_1$.*

Proof The variation of constants formula (64) with $t' = r$ and inequality (62) imply

$$\begin{aligned} y(t) &\leq y(r)\left\{|\xi(t-r)| + \nu \int_{-r}^{0} |\xi(t-r-s-r)|ds\right\} \\ &\leq y(r)b\mathrm{e}^{-a(t-r)}\left\{1 + \nu \int_{-r}^{0} \mathrm{e}^{a(s+r)}ds\right\} \\ &= y(r)b\mathrm{e}^{-a(t-r)}\left\{1 + \frac{\nu}{a}(\mathrm{e}^{ar} - 1)\right\}, \end{aligned}$$

where $y(r) > 0$. Now, there is M_1 such that

$$b\mathrm{e}^{-a(K-r)}\left\{1 + \frac{\nu}{a}(\mathrm{e}^{ar} - 1)\right\} < b\mathrm{e}^{-a(M_1-r)}\left\{1 + \frac{\nu}{a}(\mathrm{e}^{ar} - 1)\right\} = 1$$

for all $K > M_1$. This implies that $y(K) < y(r)$, thus proving the proposition. □

Proposition 13 *Assume that η defined in (60) satisfies $\eta \geq 0$, and that $\lambda_1 = \eta + i\omega$, with $\omega > 0$. Moreover, assume that Eq. 53 has a solution y satisfying (55) and such that $y(t) < 0$ for $t \in [-K, 0)$. Then there is $M_2 > 0$ such that for all $K > M_2$ there is $t \in (0, K]$ such that $y(t) < 0$. In particular, y cannot satisfy (56) if $K > M_2$.*

Proof In this case Eq. 63 implies that

$$|e^{-\eta t}\xi(t) - a\cos(\omega t + c)| \leq b e^{-(\eta - d)t}.$$

This equation, the fact that $\eta - d > 0$, and $\xi(t) > 0$ for $t \in [0, r]$, imply that there exists a $t = t_* > r$ such that $\xi(t_*) = 0$ and $\xi(t) > 0$ for $t \in [0, t_*)$. We claim that

$$\xi(t_*) = 0 \quad \Longrightarrow \quad \xi(t) < 0 \quad \text{for} \quad t \in (t_*, t_* + r). \tag{65}$$

Indeed, ξ satisfies Eq. 53 implying that $\dot{\xi}(t_*) = -\nu\xi(t_* - r) < 0$. Therefore, $\xi(t)$ is negative in some interval (t_*, δ). If $\xi(\delta) = 0$ and $\delta < t_* + r$ then $\dot{\xi}(\delta) = -\nu\xi(\delta - r) < 0$, which is absurd. So, $\delta \geq t_* + r$ and $\xi(t) < 0$ for $t \in (t_*, t_* + r)$.

Now, let us take $M_2 = t_* + r$ and $K > M_2$. The variation of constants formula (64) with $t' = 0$ and $t = t_* + r$ implies

$$y(t_* + r) = -\nu \int_{-r}^{0} \xi(t_* - s) y(s) ds.$$

As $y(s) < 0$ for $s < 0$, and $\xi(t) < 0$ for $t \in (t_*, t_* + r)$, it follows that $y(t_* + r) < 0$, thus proving the proposition. □

Proposition 14 *Assume that η defined in (60) satisfies $\eta \geq 0$ and that $\lambda_1 = \eta$. Furthermore, assume that Eq. 53 has a solution y satisfying (54), and (55). Then there is $M_3 > 0$ such that $y(-r) \geq 0$ for all $K > M_3$. In particular, y cannot satisfy (56) if $K > M_3$.*

Proof Let $\zeta : [0, \infty) \to \mathbb{R}$ be the function defined as

$$\zeta(t) \stackrel{\text{def}}{=} \xi(t) - \nu \int_{-r}^{0} \xi(t - s - r) ds. \tag{66}$$

Suppose that there exists $\bar{t} > 0$ such that

$$\zeta(\bar{t}) \leq 0. \tag{67}$$

Let us take $M_3 = \bar{t} + 2r$ and $K > M_3$. The variation of constants formula (64) with $t' = -\bar{t} - r$ and $t = -r$ implies

$$y(-r) = y(-\bar{t} - r)\xi(\bar{t}) - \nu \int_{-r}^{0} \xi(\bar{t} - s - r) y(-\bar{t} - r + s) ds. \tag{68}$$

y is nondecreasing, and $y(0) = 0$, so we obtain that $y(-\bar{t} - r + s) \leq y(-\bar{t} - r) \leq 0$ for $s \in [-r, 0]$. This, together with Eqs. 68 and 67, imply that

$$y(-r) \geq y(-\bar{t} - r) \left\{ \xi(\bar{t}) - \nu \int_{-r}^{0} \xi(\bar{t} - s - r) ds \right\} = y(-\bar{t} - r)\zeta(\bar{t}) \geq 0.$$

To finish the proof of this proposition it remains to be shown that there exists $\bar{t}$ such that (67) is true. To this end, let us define the function $\hat{\zeta} : (\eta, \infty) \to \mathbb{R}$ as

$$\hat{\zeta}(u) \stackrel{\text{def}}{=} \int_{0}^{\infty} e^{-ut} \zeta(t) dt. \tag{69}$$

Substitution of (66) into (69) gives

$$\begin{aligned}\hat{\zeta}(u) &= \int_0^\infty e^{-ut}\zeta(t)dt \\ &= \int_0^\infty e^{-ut}\xi(t)dt - \nu\int_0^\infty\int_{-r}^0 e^{-ut}\xi(t-s-r)dsdt \\ &= \hat{\xi}(u) - \nu\int_{-r}^0\int_0^\infty e^{-ut}\xi(t-s-r)dtds \\ &= \hat{\xi}(u) - \nu\int_{-r}^0 e^{-u(s+r)}\int_{-s-r}^\infty e^{-ut'}\xi(t')dt'ds \\ &= \hat{\xi}(u)\left\{1-\frac{\nu(1-e^{-ur})}{u}\right\} = \frac{\hat{\xi}(u)}{u}[u-\nu+\nu e^{-ur}],\end{aligned} \tag{70}$$

where $\hat{\xi}(u)$, $u \in (\eta, \infty)$, is the Laplace transform of ξ restricted to the infinite interval (η, ∞). Equations 58 and 70 imply that

$$\hat{\zeta}(u) = \frac{\hat{\xi}(u)}{u}[u-\nu+\nu e^{-ur}] = \frac{1}{uP(u)}[u-\nu+\nu e^{-ur}]. \tag{71}$$

Notice that

$$P(u) > 0 \quad \text{for} \quad u > \eta, \tag{72}$$

because η is the largest real root of $P(u) = 0$ and $P(u) \to \infty$ as $u \to \infty$. Using that $P(\eta) = 0$, P is continuous, and $\nu > 1$, we obtain that there is an $\epsilon > 0$ such that

$$u-\nu+\nu e^{-ur} = P(u)-\nu+1 < 0 \quad \text{for} \quad u \in (\eta, \eta+\epsilon]. \tag{73}$$

Combining Eqs. 71–73 we obtain that $\hat{\zeta}(\eta+\epsilon) < 0$. This and the definition (69) of $\hat{\zeta}$ imply that $\zeta(t)$ must be negative on some interval, which implies the existence of $\bar{t}$ as stated in (67). □

Propositions 12, 13, and 14, exhaust all the possibilities for η. Therefore Lemma 7 is proved and so is the existence part of Theorem 1. □

5.4 Asymptotic Behavior of ϕ

The linearization of Eq. 3 at either equilibria, $x(t) = -\gamma_1$ or $x(t) = \gamma_2$, leads to an equation of the following type:

$$\dot{x}(t) = -x(t) + ax(t+r), \tag{74}$$

where $0 < a = a_1 = f'(-\gamma_1) < 1$ at $x(t) = -\gamma_1$, and $0 \leq a = a_2 = f'(\gamma_2) < 1$ at $x(t) = \gamma_2$. The characteristic equation associated to Eq. 74 is

$$\lambda + 1 = ae^{\lambda r}. \tag{75}$$

Since $r > 0$, the characteristic Eq. 75 has only one root λ with real part less than or equal to zero (see [23]). Moreover, this λ is real and has multiplicity one. This implies that the equilibrium $x(t) = \gamma_2$ of Eq. 3 is hyperbolic and it has a one-dimensional stable manifold (see [9]). For a given r_* there is only one family of solutions ϕ connecting $x(t) = -\gamma_1$ to $x(t) = \gamma_2$, parameterized by time translations (one-parameter family). Below we show that $\dot{\phi}(t) > 0$, for $t \in \mathbb{R}$, implying that there is only one solution such that $\phi(0) = 0$. Moreover, as a consequence of a variation of constant formula, ϕ admits an asymptotic expansion for $t \to +\infty$ as in the statement of Theorem 1, where $\lambda = -\nu_2$ is the solution of (74) with negative real part (see [9] chapters 9 and 10, or [4] Theorems 3.2 and 3.4). From Eq. 75 we see that $0 < \nu_2 < 1$.

We shall now show that $\dot{\phi}(t) > 0$, for $t \in \mathbb{R}$. Suppose there is a value $\bar{t}$ such that $\dot{\phi}(\bar{t}) = 0$. Then differentiating Eq. 3 we get $\ddot{x}(\bar{t}) = f'(\phi(\bar{t} + r_*))\dot{\phi}(\bar{t} + r_*)$. Since $f' > 0$ (hypothesis (HP3)) and $\dot{\phi} \geq 0$ then $\ddot{x}(\bar{t}) = 0$, and $\dot{\phi}(\bar{t} + r_*) = 0$. Repeating this argument inductively, we get that for a sufficiently large integer $k > 0$ there exists a $\tilde{t} = \bar{t} + kr_* > 0$ such that $\dot{\phi}(\tilde{t}) = 0$. Notice that $\phi(\tilde{t} + r_*) < \gamma_2$ because ϕ is a solution on the stable manifold of the equilibrium γ_2, and must approach γ_2 in a exponential way, as described above. But then, from Eq. 3 and hypothesis (HP1), we get $\phi(\tilde{t}) = f(\phi(\tilde{t}+r_*)) > \phi(\tilde{t}+r_*)$, which is absurd.

Now, let us consider the linearization of Eq. 3 at $x(t) = -\gamma_1$, namely Eq. 74 with $0 < a = a_1 = f'(-\gamma_1) < 1$. In this case it is possible to show (see [23]) that among all solutions λ of Eq. 74 the one with smallest positive real part is real and has multiplicity one. It will be denoted as $\lambda = \nu_1$. The proof that the asymptotic expression in Theorem 1 for ϕ holds, as $t \to -\infty$, is more difficult than in the case $t \to +\infty$ because the unstable manifold of $x(t) = -\gamma_1$ is infinite dimensional. Nevertheless, using that $\dot{\phi}(t) > 0$ for all t, we can show that the discrete Liapunov function V given in [4] is one and, as a consequence of Theorems 3.2 and 3.4 of Cao [4], we obtain the asymptotic expression in the statement of Theorem (1). We remark that to apply the results in [4] to Eq. 3 we must change variables $t \to -t$ to obtain a delayed equation. Moreover, at this point it is used the hypothesis $f'(-\gamma_1) > 0$. If $f'(-\gamma_1) = 0$ then it could happen that, as $t \to -\infty$, $\phi(t) \to -\gamma_1$ faster than any exponential (see [4] and [1]). If we exclude the possibility of super-exponential solutions, then it is natural that the dominant term in the expansion of ϕ in terms of eigenfunctions of the linearized problem starts with $\exp(\nu_1 t)$. All other eigenfunctions have real part larger than ν_1 and have nontrivial imaginary part. So, if the term $\exp(\nu_1 t)$ would not be the dominant one in the expansion of ϕ, then $\phi(t)$ would oscillate around $-\gamma_1$, as $t \to -\infty$, and this would violate the property $\dot{\phi}(t) > 0$.

5.5 Uniqueness of r_*

Now, consider Eq. 3 with $r \leq 0$. The real part of any root λ of the characteristic Eq. 75 is strictly negative, for all a such that $|a| < 1$. So, for $r \leq 0$ both equilibria $x(t) = -\gamma_1$ and $x(t) = \gamma_2$ of Eq. 3 are stable and there cannot exist a solution that connects them. Therefore, Eq. 3 cannot admit a solution with the properties of ϕ if $r \leq 0$. So, only the case $r > 0$ needs to be considered.

Let us assume that $\underline{r}$ and $\bar{r}$, $0 < \underline{r} < \bar{r}$, are values of r associated to solutions $\underline{\phi}$ and $\overline{\phi}$, respectively, as in Theorem 1. Let $\underline{\nu}_1 > 0$ and $-\underline{\nu}_2 < 0$ be the two real roots of the characteristic Eq. 75 with $r = \underline{r}$. Let $\overline{\nu}_1 > 0$ and $-\overline{\nu}_2 < 0$ be the corresponding roots for $r = \bar{r}$. It can be shown that $\underline{\nu}_1 > \overline{\nu}_1$ and $\overline{\nu}_2 > \underline{\nu}_2$. These inequalities and the asymptotic expressions in Theorem 1 imply that there exists M sufficiently large such that $\overline{\phi}(t) > \underline{\phi}(t)$ for $|t| > M$. Since both $\dot{\underline{\phi}}$ and $\dot{\overline{\phi}}$ are continuously differentiable and nondecreasing, we conclude that there is a translation $\underline{\phi}_\alpha$ of $\underline{\phi}$, $\underline{\phi}_\alpha(t) = \underline{\phi}(t-\alpha)$, and a value $\tilde{t}$ of t such that $\overline{\phi}(\tilde{t}) = \underline{\phi}_\alpha(\tilde{t})$, $\dot{\overline{\phi}}(\tilde{t}) = \dot{\underline{\phi}}_\alpha(\tilde{t})$, and $\overline{\phi}(t) > \underline{\phi}_\alpha(t)$, for $t > \tilde{t}$. At the point $\tilde{t}$ Eq. 3 implies $f(\underline{\phi}_\alpha(\tilde{t}+\underline{r})) = f(\overline{\phi}(\tilde{t}+\bar{r}))$ and since

$f'(x) > 0$ for $-\gamma_1 < x < \gamma_2$ (hypothesis (HP3)) we conclude that $\underline{\phi}_\alpha(\tilde{t}+\underline{r}) = \overline{\phi}(\tilde{t}+\overline{r})$. But $\underline{r} < \overline{r}$ and $\dot{\underline{\phi}}_\alpha(t) \geq 0$ imply $\underline{\phi}_\alpha(\tilde{t}+\overline{r}) \geq \underline{\phi}_\alpha(\tilde{t}+\underline{r}) = \overline{\phi}(\tilde{t}+\overline{r})$ which is absurd because $\overline{\phi}(t) > \underline{\phi}_\alpha(t)$, for $t > \tilde{t}$. Therefore, there can exist only one value of r_* as stated in Theorem 1.

6 Comments on the Proof of Theorem 3

As already mentioned, the statement of Theorem 3 (given in Sect. 3) differs slightly from the original statement given in Chow, Lin and Mallet-Paret [6]. In this section we analyze the modifications introduced by us. The Eq. (1.1) $_r$ in [6], after the time change $t \to -t$, is given by

$$\begin{aligned} \dot{y}(t) &= -y(t) + f(z(t+r)), \\ \dot{z}(t) &= -z(t) + f(y(t+r)). \end{aligned} \tag{76}$$

For this equation Chow, Lin and Mallet-Paret ([6] Theorem 2.1) proved the existence of a unique $r > 0$ and a unique solution (y, z), up to time translation, such that $y(-\infty) = -\gamma_1$, $y(\infty) = \gamma_2$, $z(-\infty) = \gamma_2$, $z(\infty) = -\gamma_1$, $\dot{y}(t) \geq 0$, and $\dot{z}(t) \leq 0$. Furthermore, they showed that the strict inequalities, $\dot{y}(t) > 0$ and $\dot{z}(t) < 0$, hold as long as $z(t) < \gamma_2$ and $-\gamma_1 < y(t)$. Therefore, the time translation indeterminacy of their solution may be removed by imposing $y(0) = 0$. Under this condition the following proposition holds.

Proposition 15 *Let (y, z), with $y(0) = 0$, be the solution given by Theorem 2.1 in [6]. Let c be the only value of t such that $z(c) = 0$. Then $|c| < r$.*

Proof Equation 76 implies that $\dot{y}(0) = f(z(r)) > 0$, because $\dot{y}(t) > 0$ if $y(t) < \gamma_2$. Then $f'(z) < 0$ implies $z(r) < 0$ and using that $\dot{z}(t) < 0$ if $z(t) > -\gamma_1$ we get $c < r$. In the same way Eq. 76 implies $\dot{z}(c) = f(y(c+r)) < 0$ which implies $y(c+r) > 0$. Since $y(0) = 0$ and $\dot{y}(0) > 0$ it follows that $c + r > 0$. □

The functions $\phi(t)$ and $\chi(t)$ (Eq. 18 in Theorem 3) are then defined as $\phi(t) = y(t)$, $\chi(t) = z(t+c)$, $\underline{r} = r - c > 0$, and $\overline{r} = r + c > 0$.

The asymptotic expressions (19) given in the Theorem 3 are not given explicitly in [6], so we shall obtain them now. Equation 17 either linearized at the equilibrium $(y, z) = (-\gamma_1, \gamma_2)$ or at $(y, z) = (\gamma_2, -\gamma_1)$ can be written as

$$\begin{aligned} \dot{y}(t) &= -y(t) + pz(t+\underline{r}), \\ \dot{z}(t) &= -z(t) + qy(t+\overline{r}), \end{aligned} \tag{77}$$

where either $(p, q) = (f'(\gamma_2), f'(-\gamma_1))$ or $(p, q) = (f'(-\gamma_1), f'(\gamma_2))$, respectively. If λ is the eigenvalue of the system with eigenfunction $(y(t), z(t)) = (\tilde{c}_1, \tilde{c}_2)e^{\lambda t}$, Eq. 77 gives

$$\begin{pmatrix} \lambda+1 & -pe^{\lambda\underline{r}} \\ -qe^{\lambda\overline{r}} & \lambda+1 \end{pmatrix} \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{78}$$

and λ satisfies the characteristic equation

$$\lambda + 1 = \pm h e^{\lambda r} \tag{79}$$

where $h = \sqrt{pq}$, $0 < h < 1$ due to hypotheses (HN3) and (HN5), and $r = (\underline{r} + \overline{r})/2 > 0$. Equation 79 has exactly two real roots $(-\nu_a, -\nu_b)$, non positive, satisfying $-\nu_a < -1 < -\nu_b < 0$; furthermore, the $(\tilde{c}_1, \tilde{c}_2)$ coefficients in the eigenfunctions associated to them satisfy $c_{a1}c_{a2} > 0$ and $c_{b1}c_{b2} < 0$ (see [6], proposition 2.1). Now, consider the asymptotic behavior of the solution $(\phi(t), \chi(t))$ as $t \to \infty$. This solution is in the two dimensional stable manifold of the hyperbolic equilibrium $(y(t), z(t)) = (\gamma_2, -\gamma_1)$. Therefore, $(\phi(t), \chi(t))$ has an asymptotic expansion as (see [9], Chaps. 8 and 9, or use the variation of constants formula, write Eq. 77 as a perturbation of Eq. 17, and use that f is twice differentiable):

$$\phi(t) = \gamma_2 + \alpha c_{a1}\mathrm{e}^{-\nu_a t} + \beta c_{b1}\mathrm{e}^{-\nu_b t} + R_\phi(t),$$
$$\chi(t) = -\gamma_1 + \alpha c_{a2}\mathrm{e}^{-\nu_a t} + \beta c_{b2}\mathrm{e}^{-\nu_b t} + R_\chi(t)$$

where α and β are real parameters, $|\alpha| + |\beta| > 0$, and

$$\frac{|R_\phi(t)| + |R_\chi(t)|}{|\alpha|\mathrm{e}^{-\nu_a t} + |\beta|\mathrm{e}^{-\nu_b t}} \to 0 \quad \text{as} \quad t \to \infty. \tag{80}$$

We claim that $\beta \neq 0$. Indeed, if $\beta = 0$ then

$$\begin{aligned}(\gamma_2 - \phi(t))(\chi(t) + \gamma_1) &= -(\alpha c_{a1}\mathrm{e}^{-\nu_a t} + R_\phi(t))(\alpha c_{a2}\mathrm{e}^{-\nu_a t} + R_\chi(t)) \\ &= -\alpha^2 c_{a1}c_{a2}\mathrm{e}^{-2\nu_a t} + R(t)\end{aligned}$$

where $|R(t)|/\mathrm{e}^{-2\nu_a t} \to 0$, as $t \to \infty$, due to Eq. 80. This equation and $c_{a1}c_{a2} > 0$ imply that $(\gamma_2 - \phi(t))(\chi(t) + \gamma_1) < 0$ for t sufficiently large which is impossible because $\phi(t) \leq \gamma_2$ and $\chi(t) \geq -\gamma_1$ for all t. So, β must be different from zero. Since $0 < \nu_b < 1 < \nu_a$, the expressions (19) in the Theorem 3 for the asymptotic behavior of (ϕ, χ), for $t \to \infty$, hold with $0 < \nu = \nu_b < 1$, $b_2 = -\beta c_{b1} > 0$, and $c_1 = \beta c_{b2} > 0$.

Now, let us turn to the more complicated case of the asymptotic behavior for $t \to -\infty$. Among all solutions of Eq. 79, the solution with smallest positive real part is real. It will be denoted by $\mu > 0$. The $(\tilde{c}_1, \tilde{c}_2)$ coefficients in the eigenfunctions associated to μ satisfy $\tilde{c}_1\tilde{c}_2 < 0$. The following lemma eliminates the possibility of super-exponential convergence for $t \to -\infty$.

Lemma 8 *Given any constant $\sigma > \mu$ there exist constants $\overline{K}(\sigma)$ and $T(\sigma)$ such that $|\phi(t)| + |\chi(t)| \geq \overline{K}\mathrm{e}^{\sigma t}$, for $t < T(\sigma)$.*

This shows that the solution $(\phi(t), \chi(t))$ does not converge super exponentially fast to $(-\gamma_1, \gamma_2)$ as $t \to -\infty$. This and a standard argument using the variation of constants formula for linear advanced equations, imply that as $t \to -\infty$ the solution $(\phi(t), \chi(t))$ has the asymptotic expressions (19) as given in the Theorem 3. To prove Lemma 8 is the only thing that remains in order to complete the proof of the Theorem 3. The proof of Lemma 8 will be made in several steps, and for the solution of Eq. 76 instead of Eq. 77.

Proposition 16 *Given any $\sigma > \mu$ there exists a pair (P, Q), $0 < P < p$, $0 < Q < q$, where $(p, q) = (f'(\gamma_2), f'(-\gamma_1))$ such that σ is a real positive solution of*

$$\lambda + 1 = H\mathrm{e}^{\lambda r}$$

where $H = \sqrt{PQ}$.

Proof Notice that the equation in the proposition is the characteristic Eq. 79 with the + choice of sign and $h = \sqrt{pq}$ replaced by H. σ is the only positive root of the above characteristic equation, and the function $H \to \sigma$, defined for $0 < H < 1$ is strictly decreasing and onto $(0, \infty)$. So, given $\sigma > \mu$ there exists a unique $H < h$ such that σ solves the above characteristic equation, and $P = p\sqrt{H/h}$ and $Q = q\sqrt{H/h}$ have the required properties. □

Now, for a given σ we choose P and Q as in proposition 16, and rewrite Eq. 76 as

$$\begin{aligned}\dot{Y}(t) &= -Y(t) + PZ(t+r) + F(Z(t+r)),\\ \dot{Z}(t) &= -Z(t) + QY(t+r) + G(Y(t+r))\end{aligned} \tag{81}$$

where

$$\begin{aligned} Y(t) &= y(t) - (-\gamma_1) > 0, \quad \lim_{t\to-\infty} Y(t) = 0,\\ Z(t) &= z(t) - \gamma_2 < 0, \quad \lim_{t\to-\infty} Z(t) = 0,\end{aligned}$$

and

$$F(Z) = f(Z+\gamma_2) - PZ, \quad G(Y) = f(Y-\gamma_1) - QY.$$

Notice that $F'(0) = p - P > 0$ and $G'(0) = q - Q > 0$. Therefore, the asymptotic behavior of Y and Z implies that given $\sigma > \mu$ there exists a $T(\sigma) \in \mathbb{R}$ such that

$$\begin{aligned} F(Z(t+r)) &\le 0, \quad \text{for} \quad t < T(\sigma),\\ G(Y(t+r)) &\ge 0, \quad \text{for} \quad t < T(\sigma).\end{aligned} \tag{82}$$

Consider the adjoint equation to Eq. 81 after neglecting F and G, namely

$$\begin{aligned}\dot{\eta}(t) &= \eta(t) - Q\psi(t-r),\\ \dot{\psi}(t) &= \psi(t) - P\eta(t-r).\end{aligned} \tag{83}$$

Notice that $(\eta(t), \psi(t)) = \mathrm{e}^{-\sigma t}(a_1, a_2)$ is a solution of Eq. 83, with $a_1 > 0$ and $a_2 < 0$ satisfying the following equation

$$\begin{pmatrix} -\sigma - 1 & Q\mathrm{e}^{\sigma r} \\ P\mathrm{e}^{\sigma r} & -\sigma - 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{84}$$

Now, multiplying by $\eta(t)$ ($\psi(t)$) the equation in the first (second) line of (81), integrating both equations from t to $T(\sigma) > t$ (integrating by parts the left hand side of the equations), adding the resulting two equations, and using that (η, ψ) is a solution of (83) (this is a standard procedure to handle linear systems of delayed differential equations as described in [9]), we obtain

$$u(t) = u(T(\sigma)) + \int_t^{T(\sigma)} \mathrm{e}^{-\sigma s}[Pa_1 F(Z(s+r)) + Qa_2 G(Y(s+r))]ds, \tag{85}$$

where

$$u(t) = -\mathrm{e}^{-\sigma t}[a_1 Y(t) + a_2 Z(t)] + \mathrm{e}^{\sigma r} \int_t^{t+r} \mathrm{e}^{-\sigma s}[Pa_1 Z(s) + Qa_2 Y(s)]ds.$$

Using that $a_1 > 0$, $a_2 < 0$, $P < 0$, $Q < 0$, and inequalities (82), it follows that the integral in the right hand side of Eq. 85 is positive, implying $u(t) \ge u(T(\sigma))$ for all $t \le T(\sigma)$.

Proposition 17 *The function u is positive for all $t \in \mathbb{R}$. In particular, $u(t) > u(T(\sigma)) \stackrel{\text{def}}{=} c(\sigma) > 0$, for all $t < T(\sigma)$.*

Proof In view of the inequality at the end of the last paragraph, it is enough to prove that $u(t) > 0$ for $t \in \mathbb{R}$. Let $\Omega(s) \stackrel{\text{def}}{=} a_1 P Z(s) + a_2 Q Y(s)$. For all $s \in \mathbb{R}$, $\Omega(s) > 0$ because $a_1 > 0$, $a_2 < 0$, $P < 0$, $Q < 0$, $Y(s) > 0$ and $Z(s) < 0$ (the proof that these last strict inequalities hold for all $s \in \mathbb{R}$, under the hypothesis $f'(-\gamma_1) f'(\gamma_2) > 0$, is given in [6] propositions 2.6 and 2.7). Moreover, $\dot{\Omega}(s) = a_1 P \dot{Z}(s) + a_2 Q \dot{Y}(s) \geq 0$ because $\dot{Y}(s) \geq 0$ and $\dot{Z}(s) \leq 0$. Therefore,

$$\int_t^{t+r} \mathrm{e}^{-\sigma s}[P a_1 Z(s) + Q a_2 Y(s)] ds = \int_t^{t+r} \mathrm{e}^{-\sigma s} \Omega(s) ds \geq \Omega(t) \int_t^{t+r} \mathrm{e}^{-\sigma s} ds$$
$$= \Omega(t) \frac{\mathrm{e}^{-\sigma t}}{\sigma}(1 - \mathrm{e}^{-r\sigma}).$$

This and the definition of $u(t)$ imply

$$u(t) \geq -\mathrm{e}^{-\sigma t}[a_1 Y(t) + a_2 Z(t)] + \mathrm{e}^{\sigma r} \Omega(t) \frac{\mathrm{e}^{-\sigma t}}{\sigma}(1 - \mathrm{e}^{-r\sigma})$$
$$= \frac{\mathrm{e}^{-\sigma t}}{\sigma}\{Y(t)[-\sigma a_1 + (\mathrm{e}^{\sigma r} - 1) a_2 Q] + Z(t)[-\sigma a_2 + (\mathrm{e}^{\sigma r} - 1) a_1 P]\}.$$

From Eq. 84 we get $-\sigma a_1 + \mathrm{e}^{\sigma r} Q a_2 = a_1$ and $-\sigma a_2 + \mathrm{e}^{\sigma r} P a_1 = a_2$. Substituting these equations in the above inequality we obtain

$$u(t) \geq \frac{\mathrm{e}^{-\sigma t}}{\sigma}\{Y(t)[a_1 - a_2 Q] + Z(t)[a_2 - a_1 P]\}.$$

Again from Eq. 84 we get $(1+\sigma) a_1 = \mathrm{e}^{\sigma r} Q a_2$, and from proposition 16 we get $\sigma + 1 = H \mathrm{e}^{\sigma r}$ which imply $Q a_2 = H a_1$, and analogously we get $P a_1 = H a_2$. So substituting these equations in the inequality above we finally get

$$u(t) \geq \frac{\mathrm{e}^{-\sigma t}}{\sigma}\{Y(t) a_1 (1 - H) + Z(t) a_2 (1 - H)\} > 0$$

because $a_1 > 0$, $a_2 < 0$, $Y(t) > 0$, $Z(t) < 0$, and $0 < H < 1$. □

Now, we prove Lemma 8. From proposition 17 and the definition of $u(t)$ we get for $t < T(\sigma)$

$$\mathrm{e}^{\sigma r} \int_t^{t+r} \mathrm{e}^{-\sigma s}[P a_1 Z(s) + Q a_2 Y(s)] ds = \mathrm{e}^{\sigma r} \int_t^{t+r} \mathrm{e}^{-\sigma s} \Omega(s) ds$$
$$\geq \mathrm{e}^{-\sigma t}[a_1 Y(t) + a_2 Z(t)] + c > c$$

because $a_1 > 0$, $a_2 < 0$, $Y(t) > 0$, and $Z(t) < 0$. As in the proof of proposition 17 we have $\Omega(s) > 0$ and $\dot{\Omega}(s) \geq 0$ which together with the previous inequality imply

$$c \mathrm{e}^{-\sigma r} \leq \int_t^{t+r} \mathrm{e}^{-\sigma s} \Omega(s) ds \leq \Omega(t+r) \int_t^{t+r} \mathrm{e}^{-\sigma s} ds = \Omega(t+r) \frac{\mathrm{e}^{-\sigma t}}{\sigma}(1 - \mathrm{e}^{-r\sigma}),$$

from which we derive

$$\Omega(t+r) > \mathrm{e}^{\sigma t} \frac{c\sigma}{\mathrm{e}^{r\sigma} - 1}.$$

Let $C \stackrel{\text{def}}{=} \min\{-a_1 P, a_2 Q\} > 0$. Then

$$C(|Y(t)| + |Z(t)|) = C(Y(t) - Z(t)) \geq a_1 P Z(t) + a_2 Q Y(t) = \Omega(t).$$

The last two inequalities imply the inequality in Lemma 8.

Now, the asymptotic behavior in Theorem 3 and the fact that $\dot{\phi}(t) \geq 0$ and $\dot{\chi}(t) \leq 0$ imply that $\chi(t) < \gamma_2$ and $\phi(t) > -\gamma_1$, for all $t \in \mathbb{R}$. This and a result in [6] (Theorem 2.1) mentioned in the beginning of this section imply that $\dot{\phi}(t) > 0$ and $\dot{\chi}(t) < 0$ for all $t \in \mathbb{R}$.

Finally, we have to show the inequality (20) in Theorem 3. Equation 78, the definitions of $\mu, \nu, b_1, b_2, c_1, c_2$, and $A \stackrel{\text{def}}{=} f'(\gamma_2)$ and $B \stackrel{\text{def}}{=} f'(-\gamma_1)$, imply the following equations

$$\begin{aligned}
(1+\mu)b_1 &= -A\exp(\mu \underline{r})c_2,\\
(1+\mu)c_2 &= -B\exp(\mu \overline{r})b_1,\\
(1-\nu)b_2 &= -B\exp(-\nu \underline{r})c_1,\\
(1-\nu)c_1 &= -A\exp(-\nu \overline{r})b_2.
\end{aligned}$$

Dividing the first and the fourth equations of this system we get:

$$\ln\left(\frac{c_1 c_2}{b_1 b_2}\right) + \mu \underline{r} + \nu \overline{r} = \ln\left(\frac{1+\mu}{1-\nu}\right) > 0.$$

Finally, dividing the second and the third equations of this system we get the inequality (20):

$$\ln\left(\frac{b_1 b_2}{c_1 c_2}\right) + \mu \overline{r} + \nu \underline{r} = \ln\left(\frac{1+\mu}{1-\nu}\right) > 0.$$

7 Proof of Theorem 2

The proof of Theorem 2 will be given through a series of propositions and lemmas. The following proposition is concerned with the numbers η_i appearing in the Eq. 7.

Proposition 18 *For a given $\xi > 0$ the equation $\phi_\epsilon(t) = \chi_\epsilon(t-\xi)$ has a unique solution of the form*

$$t = t(\xi,\epsilon) = \frac{\xi \mu_2}{\mu_2 + \nu_2} + \frac{\epsilon}{\mu_2 + \nu_2}\ln\left(\frac{b_2}{c_2}\right) + \epsilon \mathcal{E}(1/\epsilon).$$

In the same way, for a given $\xi > 0$ the equation $\chi_\epsilon(t) = \phi_\epsilon(t-\xi)$ has a unique solution of the form

$$t = t(\xi,\epsilon) = \frac{\xi \nu_1}{\mu_1 + \nu_1} + \frac{\epsilon}{\mu_1 + \nu_1}\ln\left(\frac{c_1}{b_1}\right) + \epsilon \mathcal{E}(1/\epsilon).$$

$\mathcal{E}$ is defined in Eq. 9.

Proof From Theorem 1 we get that $\dot{\phi}_\epsilon > 0$ and $\dot{\chi}_\epsilon < 0$. So, if a solution to each equation in the proposition exists, then it is unique. The asymptotic formulas for ϕ and χ in Theorem 1 imply (definitions of $\mathcal{E}$ and $\mathcal{E}_-$ given in (9) and (10), respectively)

$$\begin{aligned}
\phi_\epsilon(t) &= -\gamma_1 + b_1\exp(\nu_1 t/\epsilon)[1 + \mathcal{E}_-(t/\epsilon)],\\
\phi_\epsilon(t) &= \gamma_2 - b_2\exp(-\nu_2 t/\epsilon)[1 + \mathcal{E}(t/\epsilon)];
\end{aligned} \tag{86}$$

$$\begin{aligned}
\chi_\epsilon(t) &= \gamma_2 - c_2\exp(\mu_2 t/\epsilon)[1 + \mathcal{E}_-(t/\epsilon)],\\
\chi_\epsilon(t) &= -\gamma_1 + c_1\exp(-\mu_1 t/\epsilon)[1 + \mathcal{E}(t/\epsilon)].
\end{aligned} \tag{87}$$

For $\epsilon < 1$ there exists α independent of ϵ, sufficiently small, $0 < \alpha < 1$, such that $\phi_\epsilon(\alpha\xi) < \chi_\epsilon(\alpha\xi - \xi)$ (for $t = \alpha\xi$) and $\phi_\epsilon(\xi - \alpha\xi) > \chi_\epsilon(-\alpha\xi)$ (for $t = \xi - \alpha\xi$). So, a solution t of the equation exists and $\alpha\xi < t < \xi - \alpha\xi$. Using the asymptotic expressions above we write $\phi_\epsilon(t) = \chi_\epsilon(t - \xi)$ as

$$b_2 \exp(-\nu_2 t/\epsilon)[1 + \mathcal{E}(t/\epsilon)] = c_2 \exp(\mu_2[t - \xi])[1 + \mathcal{E}_-([t - \xi]/\epsilon)].$$

Taking the ln of both sides we get

$$t = \frac{\xi\mu_2}{\mu_2 + \nu_2} + \frac{\epsilon}{\mu_2 + \nu_2} \ln\left(\frac{b_2}{c_2}\right) + \frac{\epsilon}{\mu_2 + \nu_2}\{\mathcal{E}(t/\epsilon) - \mathcal{E}_-([t - \xi]/\epsilon)\}.$$

So, the a priori bound $\alpha\xi < t < \xi - \alpha\xi$, implies the result in the statement of the proposition. □

Corollary 1 *The numbers η_i appearing in Eq. 7 are approximately given by,*

$$\eta_i = \theta_{i-1} + \frac{\mu_2}{\mu_2 + \nu_2}\delta_i + \frac{\epsilon}{\mu_2 + \nu_2} \ln\left(\frac{b_2}{c_2}\right) + \epsilon\mathcal{E}(1/\epsilon), \quad i \ odd,$$

and

$$\eta_i = \theta_{i-1} + \frac{\nu_1}{\mu_1 + \nu_1}\delta_i + \frac{\epsilon}{\mu_1 + \nu_1} \ln\left(\frac{c_1}{b_1}\right) + \epsilon\mathcal{E}(1/\epsilon), \quad i \ even.$$

We recall that $\delta_i = \theta_i - \theta_{i-1} > 0$ and $\theta_0 \stackrel{\text{def}}{=} -\epsilon r$.

Proof It is enough to notice that for i odd η_i is defined as the solution of $\phi_\epsilon(t - \theta_{i-1}) = \chi_\epsilon(t - \theta_i)$ and for i even as the solution of $\chi_\epsilon(t - \theta_{i-1}) = \phi_\epsilon(t - \theta_i)$. □

Before presenting the Lemma 9, which is the main point in the proof of Theorem 2, it is convenient to introduce some notation. We define

$$\tilde{\theta}_i = \theta_i + \epsilon r, \quad i = 0, 1, \ldots, n.$$

Notice that $\tilde{\theta}_0 = 0$ and $\tilde{\theta}_n = 1 + \epsilon r$. We also define

$$\begin{aligned} \Delta_i &= \phi_\epsilon(\eta_i - \tilde{\theta}_{i-1}) - \chi_\epsilon(\eta_i - \tilde{\theta}_i), \quad \text{for } i \text{ odd,} \\ \Delta_i &= \chi_\epsilon(\eta_i - \tilde{\theta}_{i-1}) - \phi_\epsilon(\eta_i - \tilde{\theta}_i), \quad \text{for } i \text{ even.} \end{aligned} \tag{88}$$

Lemma 9 *For a given $\delta \in A_n$, let x be the solution of Eq. 1 that satisfies the initial condition $x(t) = z(t+1)$, for $t \in [-1, 0]$, where z is the function given in (7). Then for $t \in [0, 1 + \eta_1]$*

$$\begin{aligned} x(t) &= \phi_\epsilon(t - \tilde{\theta}_i) + \sum_{j=0}^{i} \exp[-(t - \eta_j)/\epsilon]\Delta_j, \quad \eta_i \le t < \eta_{i+1}, \quad 0 \le i \le n, \quad i \ even, \\ x(t) &= \chi_\epsilon(t - \tilde{\theta}_i) + \sum_{j=0}^{i} \exp[-(t - \eta_j)/\epsilon]\Delta_j, \quad \eta_i \le t < \eta_{i+1}, \quad 1 \le i < n, \quad i \ odd, \end{aligned} \tag{89}$$

with the definitions $\Delta_0 \stackrel{\text{def}}{=} 0$, $\eta_0 \stackrel{\text{def}}{=} 0$, and $\eta_{n+1} \stackrel{\text{def}}{=} 1 + \eta_1$.

Proof For $t \in [0, 1]$ Eq. 1 with the above initial condition can be written as

$$\epsilon\dot{x}(t) = -x(t) + f(z(t)).$$

For $t \in [0, \eta_1)$, from Eq. 7, we get

$$\epsilon \dot{x}(t) = -x(t) + f(\phi_\epsilon(t + \epsilon r)).$$

Since $x(0) = 0$, this equation is solved by $x(t) = \phi_\epsilon(t)$ (see Eq. 4).

For $t \in [\eta_i, \eta_{i+1})$, $1 \le i < n$, i odd, Eq. 1 becomes

$$\epsilon \dot{x}(t) + x(t) = f(\chi_\epsilon(t - \theta_i)). \tag{90}$$

Notice that $x_p(t) = \chi_\epsilon(t - \tilde{\theta}_i) = \chi_\epsilon(t - \theta_i - \epsilon r)$ is a particular solution of this equation, for $t \in [\eta_i, \eta_{i+1})$. The Eq. 90 is a linear non-homogeneous equation for x, since the right hand side of the equation is given. Moreover, $x(t) = x_p(t) + \sum_{j=0}^{i} \exp[-(t - \eta_j)/\epsilon]\Delta_j$ is also a solution of this Eq. 90 for $t \in [\eta_i, \eta_{i+1})$, because each term $\exp[-(t - \eta_j)/\epsilon]\Delta_j$ is a solution of the linear homogeneous equation $\epsilon \dot{x}(t) + x(t) = 0$.

For $t \in [\eta_i, \eta_{i+1})$, $2 \le i \le n$, i even, Eq. 1 becomes

$$\epsilon \dot{x}(t) + x(t) = f(\phi_\epsilon(t - \theta_i)). \tag{91}$$

Now, $x_p(t) = \phi_\epsilon(t - \tilde{\theta}_i) = \phi_\epsilon(t - \theta_i - \epsilon r)$ is a particular solution of the Eq. 91 for $t \in [\eta_i, \eta_{i+1})$. Likewise the i odd case $x(t) = x_p(t) + \sum_{j=0}^{i} \exp[-(t - \eta_j)/\epsilon]\Delta_j$ is a solution of Eq. 91 for $t \in [\eta_i, \eta_{i+1})$.

So, we have shown that the functions given in Eq. 89 solve Eq. 1 in the intervals (η_i, η_{i+1}). If we show that these solutions glue continuously at the points $\eta_1, \ldots, \eta_n$ then they will also glue differentially, since the differential equation is satisfied in a neighborhood of η_i, and the Lemma 9 will be proved. At η_i, $1 \le i < n$, the function x (89) for i odd satisfies

$$\begin{aligned} x(\eta_i) - \lim_{t \to \eta_{i-}} &= \chi_\epsilon(\eta_i - \tilde{\theta}_i) + \sum_{j=0}^{i} \exp[-(\eta_i - \eta_j)/\epsilon]\Delta_j \\ &\quad -\{\phi_\epsilon(\eta_i - \tilde{\theta}_{i-1}) + \sum_{j=0}^{i-1} \exp[-(\eta_i - \eta_j)/\epsilon]\Delta_j \\ &= \chi_\epsilon(\eta_i - \tilde{\theta}_i) - \phi_\epsilon(\eta_i - \tilde{\theta}_{i-1}) + \Delta_i = 0, \end{aligned}$$

due to the definition of Δ_i, Eq. 88. In the same way we show that at η_i, $2 \le i \le n$, i even, the continuity of x holds. □

Proposition 19 *The Δ_i, $i = 1, \ldots, n$, defined in (88) satisfy the following estimates. For i odd, $1 \le i < n$,*

$$\Delta_i = \exp\left[\frac{-\mu_2\nu_2}{\mu_2 + \nu_2}\frac{\delta_i}{\epsilon}\right](k_1 + \mathcal{E}(1/\epsilon))$$

where

$$k_1 = -b_2 \exp\left[\frac{-\nu_2}{\mu_2 + \nu_2}\ln(b_2/c_2) + \nu_2 r\right] + c_2 \exp\left[\frac{\mu_2}{\mu_2 + \nu_2}\ln(b_2/c_2) - \mu_2 r\right] < 0. \tag{92}$$

For i even, $2 \le i \le n$,

$$\Delta_i = \exp\left[\frac{-\mu_1\nu_1}{\mu_1 + \nu_1}\frac{\delta_i}{\epsilon}\right](k_2 + \mathcal{E}(1/\epsilon))$$

where

$$k_2 = -b_1 \exp\left[\frac{\nu_1}{\mu_1 + \nu_1}\ln(c_1/b_1) - \nu_1 r\right] + c_1 \exp\left[\frac{-\mu_1}{\mu_1 + \nu_1}\ln(c_1/b_1) + \mu_1 r\right] > 0. \tag{93}$$

Proof For i odd, from corollary 1, and the definitions $\tilde{\theta}_i = \theta_i + \epsilon r$ and $\delta_i = \theta_i - \theta_{i-1} > 0$, we get

$$\eta_i - \tilde{\theta}_{i-1} = \eta_i - \theta_{i-1} - \epsilon r = \frac{\mu_2}{\mu_2+\nu_2}\delta_i + \epsilon\left[\frac{1}{\mu_2+\nu_2}\ln(b_2/c_2) - r\right] + \epsilon\mathcal{E}(1/\epsilon),$$

and

$$\eta_i - \tilde{\theta}_i = \eta_i - \delta_i - \theta_{i-1} - \epsilon r = \frac{-\nu_2}{\mu_2+\nu_2}\delta_i + \epsilon\left[\frac{1}{\mu_2+\nu_2}\ln(b_2/c_2) - r\right] + \epsilon\mathcal{E}(1/\epsilon).$$

These expressions, the definition of Δ_i (Eq. 88), the fact that $\eta_i - \tilde{\theta}_{i-1} > 0$ and $\eta_i - \tilde{\theta}_i < 0$ for ϵ small, and the asymptotic expressions for ϕ_ϵ and χ_ϵ given in Eqs. 86 and 87, respectively, imply the estimate for Δ_i in statement of the proposition 19 (i odd). Now, consider the function $G(t) \stackrel{\text{def}}{=} \phi_\epsilon(t-\theta_{i-1}) - \chi_\epsilon(t-\theta_i)$. This function satisfies $\dot{G}(t) > 0$, $G(\eta_i) = 0$, and therefore $G(\eta_i - \epsilon r) = \Delta_i < 0$. This shows that $k_1 < 0$. The results for η_i with i even are obtained in the same way. □

Proposition 20 *For* $i = 1, 2, \ldots, n$ *the following holds*

$$\sum_{j=0}^{i} \exp[\eta_j/\epsilon]\Delta_j = \exp[\eta_i/\epsilon]\Delta_i(1 + \mathcal{E}(1/\epsilon)).$$

Proof For j odd, from corollary 1 and proposition 19 we get

$$\begin{aligned}
\exp[\eta_j/\epsilon]\Delta_j &= \exp\left[\frac{\theta_{j-1}}{\epsilon} + \frac{\mu_2}{\mu_2+\nu_2}\frac{\delta_j}{\epsilon} + \frac{1}{\mu_2+\nu_2}\ln\left(\frac{b_2}{c_2}\right) + \mathcal{E}(1/\epsilon)\right] \\
&\quad \times \exp\left[\frac{-\mu_2\nu_2}{\mu_2+\nu_2}\frac{\delta_j}{\epsilon}\right](k_1 + \mathcal{E}(1/\epsilon)) \\
&= B_j \exp\left(\frac{1}{\epsilon(\mu_2+\nu_2)}[\mu_2(1-\nu_2)\theta_j + \nu_2(1+\mu_2)\theta_{j-1}]\right) \\
&\stackrel{\text{def}}{=} B_j \exp[a_j/\epsilon],
\end{aligned} \tag{94}$$

where $B_j < 0$ is a function of ϵ that remains bounded and strictly negative as $\epsilon \to 0$. Analogously, for j even,

$$\begin{aligned}
\exp[\eta_j/\epsilon]\Delta_j &= \exp\left[\frac{\theta_{j-1}}{\epsilon} + \frac{\nu_1}{\mu_1+\nu_1}\frac{\delta_j}{\epsilon} + \frac{1}{\mu_1+\nu_1}\ln\left(\frac{c_1}{b_1}\right) + \mathcal{E}(1/\epsilon)\right] \\
&\quad \times \exp\left[\frac{-\mu_1\nu_1}{\mu_1+\nu_1}\frac{\delta_j}{\epsilon}\right](k_2 + \mathcal{E}(1/\epsilon)) \\
&= B_j \exp\left(\frac{1}{\epsilon(\mu_1+\nu_1)}[\nu_1(1-\mu_1)\theta_j + \mu_1(1+\nu_1)\theta_{j-1}]\right) \\
&\stackrel{\text{def}}{=} B_j \exp[a_j/ep],
\end{aligned} \tag{95}$$

where $B_j > 0$ is a function of ϵ that remains bounded and strictly positive as $\epsilon \to 0$. So,

$$\sum_{j=0}^{i} \exp[\eta_j/\epsilon]\Delta_j = \sum_{j=0}^{i} \exp[a_j/\epsilon]B_j,$$

where a_j are quantities that do not depend on ϵ, and $B_j(\epsilon)$ are continuous and remain bounded and strictly different from zero as $\epsilon \to 0$. Therefore, if $a_J > a_j$, $j = 1, \ldots, i\}$ (recall that $\Delta_0 = 0$) then $\sum_{j=0}^{i} \exp[a_j/\epsilon]B_j = \exp[a_J/\epsilon]B_J(1 + \mathcal{E}(1/\epsilon))$. So, for $1 \leq j \leq i$ we must find the largest exponent amongst those in Eqs. 94 and 95. From Theorem 1 we have that $0 < \mu_1 < 1$ and $0 < \nu_2 < 1$, so the constants multiplying θ_j and θ_{j-1} in Eqs. 94 and 95 are positive. For i odd, this, and the fact that $\theta_k < \theta_{k+1}$, imply $a_i > a_j$, for all j odd with $1 \leq j < i$, and $a_{i-1} > a_j$, for all j even with $2 \leq j < i-1$. So, $\max\{a_j : 1 \leq j \leq i-2\} < \max\{a_{i-1}, a_i\}$. The same happens for i even. Now, for i odd, using that $\theta_i > \theta_{i-1} > \theta_{i-2}$, we get

$$a_i = \frac{1}{\epsilon(\mu_2+\nu_2)}[\mu_2(1-\nu_2)\theta_i + \nu_2(1+\mu_2)\theta_{i-1}] > \frac{\theta_{i-1}}{\epsilon},$$
$$a_{i-1} = \frac{1}{\epsilon(\mu_1+\nu_1)}[\nu_1(1-\mu_1)\theta_{i-1} + \mu_1(1+\nu_1)\theta_{i-2}] < \frac{\theta_{i-1}}{\epsilon},$$

which implies that $a_i > a_{i-1}$. For i even, using that $\theta_i > \theta_{i-1} > \theta_{i-2}$, we get

$$a_i = \frac{1}{\epsilon(\mu_1+\nu_1)}[\nu_1(1-\mu_1)\theta_i + \mu_1(1+\nu_1)\theta_{i-1}] > \frac{\theta_{i-1}}{\epsilon},$$
$$a_{i-1} = \frac{1}{\epsilon(\mu_2+\nu_2)}[\mu_2(1-\nu_2)\theta_{i-1} + \nu_2(1+\mu_2)\theta_{i-2}] < \theta_{i-1}\epsilon,$$

which implies that $a_i > a_{i-1}$. □

Lemma 10 *Let x be the solution of Eq. 1 that is given in Lemma 9. It satisfies $x(0) = 0$ and, for ϵ sufficiently small, x has n zeroes in the interval $(0, 1+\eta_1)$, $x(\theta_i') = 0$, with $0 < \theta_1' < \theta_2', \ldots, \theta_n' < 1 + \eta_1$.*

For i odd, $1 \leq i < n$, θ_i' satisfies the following estimate:

$$\theta_i' = \theta_i + \epsilon r - \sigma_i \quad \text{with} \quad \sigma_i = \epsilon \exp\left[-\frac{b\delta_i}{\epsilon}\right](k_3 + \mathcal{E}(1/\epsilon)), \tag{96}$$

where (recall that $\delta_i = \theta_i - \theta_{i-1} > 0$),

$$b \stackrel{\text{def}}{=} \frac{\nu_2(1+\mu_2)}{\mu_2+\nu_2} > 0, \quad k_3 = \frac{k_1}{\dot{\chi}(0)} \exp\left[\frac{1}{\mu_2+\nu_2}\ln(b_2/c_2) - r\right] > 0,$$

and $k_1 < 0$ (Eq. 92, proposition 19).

For i even, $2 \leq i \leq n$, θ_i' satisfies the following estimate:

$$\theta_i' = \theta_i + \epsilon r - \sigma_i, \quad \text{with} \quad \sigma_i = \epsilon \exp\left[-\frac{a\delta_i}{\epsilon}\right](k_4 + \mathcal{E}(1/\epsilon)), \tag{97}$$

where

$$a = \frac{\mu_1(1+\nu_1)}{\mu_1+\nu_1} > 0, \quad k_4 = \frac{k_2}{\dot{\phi}(0)} \exp\left[\frac{1}{\mu_1+\nu_1}\ln(c_1/b_1) - r\right] > 0,$$

and $k_2 > 0$ (Eq. 93, proposition 19). We define $\theta_0' = 0$ and $\sigma_0 = 0$.

Proof Let us consider the case of i odd. From Lemma 9, θ_i' is the solution of the following equation

$$x(t) = \chi_\epsilon(t - \tilde{\theta}_i) + \sum_{j=0}^{i} \exp[-(t-\eta_j)/\epsilon]\Delta_j. \tag{98}$$

Let us define

$$T = \frac{t - \tilde{\theta}_i}{\epsilon}, \quad \text{and} \quad R(\epsilon) = \sum_{j=0}^{i} \exp[-(\tilde{\theta}_i - \eta_j)/\epsilon]\Delta_j = 0.$$

Then Eq. 98 can be written as

$$e^T x(t) = e^T \chi(T) + R \stackrel{\text{def}}{=} F(T, R) = 0,$$

where $\chi(T) = \chi_\epsilon(\epsilon T)$ is the function given in Theorem 1. Function $F(T, R)$ satisfies $F(0, 0) = 0$ (because $\chi(0) = 0$), $\partial_T F(0, 0) = \dot{\chi}(0)$, and $\partial_R F(0, 0) = 1$. So, the solution of $F(T, R)$ is given by $T = -[R/\dot{\chi}(0)] + \mathcal{O}(R^2)$, and using that $\epsilon T = \theta_i' - \tilde{\theta}_i = \theta_i' - \theta_i - \epsilon r$, we get

$$\theta_i' = \theta_i + \epsilon r - \epsilon \frac{R(\epsilon)}{\dot{\chi}(0)}(1 + \mathcal{O}(R)). \tag{99}$$

Now, from corollary 1 and propositions 19 and 20 we obtain

$$\begin{aligned} R(\epsilon) &= \exp(-\tilde{\theta}_i/\epsilon) \sum_{j=0}^{i} \exp[\eta_j/\epsilon]\Delta_j = \exp[-(\tilde{\theta}_i - \eta_i)/\epsilon]\Delta_i(1 + \mathcal{E}(1/\epsilon)) \\ &= \exp\left[\frac{-\nu_2(1+\mu_2)}{\mu_2+\nu_2}\frac{\delta_i}{\epsilon}\right] \exp\left(\frac{1}{\mu_2+\nu_2}\ln\left(\frac{b_2}{c_2}\right) - r\right)(k_1 + \mathcal{E}(1/\epsilon)). \end{aligned} \tag{100}$$

Equations 100, 99, the expression (92) for k_1, and the fact that $\dot{\chi}(0) < 0$, imply that θ_i' is given by Eq. 96 in the statement of Lemma 10, for i odd.

For i even, from Lemma 9, θ_i' is the solution of

$$x(t) = \phi_\epsilon(t - \tilde{\theta}_i) + \sum_{j=0}^{i} \exp[-(t - \eta_j)/\epsilon]\Delta_j = 0.$$

The same reasoning used in the case of i odd leads to the following expression

$$\theta_i' = \theta_i + \epsilon r - \epsilon \frac{R(\epsilon)}{\dot{\phi}(0)}(1 + \mathcal{O}(R)), \tag{101}$$

where R is given by Eq. 100 (same expression as in the case i odd). Using corollary 1, and propositions 19 and 20, $R(\epsilon)$ can be written as

$$\begin{aligned} R(\epsilon) &= \exp(-\tilde{\theta}_i/\epsilon) \sum_{j=0}^{i} \exp[\eta_j/\epsilon]\Delta_j = \exp[-(\tilde{\theta}_i - \eta_i)/\epsilon]\Delta_i(1 + \mathcal{E}(1/\epsilon)) \\ &= \exp\left[\frac{-\mu_1(1+\nu_1)}{\mu_1+\nu_1}\frac{\delta_i}{\epsilon}\right] \exp\left(\frac{1}{\mu_1+\nu_1}\ln\left(\frac{c_1}{b_1}\right) - r\right)(k_2 + \mathcal{E}(1/\epsilon)). \end{aligned} \tag{102}$$

This equation, the Eq. 101, the expression (93) for k_2, and the fact that $\dot{\phi}(0) > 0$, imply that θ_i' is as given by Eq. 97 in the statement of Lemma 10, for i even. □

The initial condition that generates the solution x in Lemma 9 is in the set X_n (see the paragraph after Eq. 10 for the definition of X_n and other function sets used below). Lemma 10 implies that, for ϵ sufficiently small, the first zero of x after $t = 1$ is given by $\theta_n' < 1 + \eta_1$.

Indeed, $\theta'_{n-1} = \theta_{n-1} + \epsilon r + \mathcal{E}(1/\epsilon)$ and $\theta_{n-1} < 1$, so $\theta'_{n-1} < 1$ for ϵ sufficient small. Then, using that $\theta'_n = \theta_n + \epsilon r + \mathcal{E}(1/\epsilon)$, $\theta_n = 1$, and from corollary 1,

$$\eta_1 - \epsilon r = -\epsilon r + \theta_0 + \frac{\mu_2}{\mu_2 + \nu_2}\delta_1 + \mathcal{O}(\epsilon) = -2\epsilon + \frac{\mu_2}{\mu_2 + \nu_2}\delta_1 + \mathcal{O}(\epsilon) > 0,$$

we get that $1 < \theta'_n < 1 + \eta_1$, for ϵ sufficiently small. So, according to the definition of $\overline{X}_n$ in Sect. 2, the initial condition of Lemma 9 is in $\overline{X}_n$ with $T = \theta'_n$. Then Lemma 10 implies that T is given by the expression (11) as stated in Theorem 2. From the definition of $F_n(x) = \psi_T(x)$ we get that $F_n(x)(t) = x(t + \theta'_n)$, for $t \in [-1, 0]$, where x is the solution given in Lemma 9. Notice that for ϵ sufficiently small the zeroes of $F_n(x)$ are located at $-1 + \hat{\theta}_1 < -1 + \hat{\theta}_2 < \cdots < -1 + \hat{\theta}_{n-1} < 1 - \hat{\theta}_n = 0$, where $\hat{\theta}_i = 1 - \theta'_n + \theta'_i$. Now, if we define $\hat{\theta}_0 = -\epsilon r = \theta_0$ as before and $\delta'_i = \hat{\theta}_i - \hat{\theta}_{i-1} = \theta'_i - \hat{\theta}'_{i-1}$ then, from Lemma 10, we get the Eqs. 13 in item (iii) of Theorem 2.

The following lemma completes the proof of item (ii) of Theorem 2.

Lemma 11 *Let*

$$\beta \stackrel{\text{def}}{=} \min\left\{\frac{\mu_2\nu_2}{\mu_2+\nu_2}\delta_1, \frac{\mu_1\nu_1}{\mu_1+\nu_1}\delta_2, \frac{\mu_2\nu_2}{\mu_2+\nu_2}\delta_3, \ldots, \frac{\mu_2\nu_2}{\mu_2+\nu_2}\delta_n\right\}$$

Then

$$\sup_{t\in[-1,0]} |F_n(x)(t) - P_n \circ F_n(x)(t)| = ||F_n(x) - P_n \circ F_n(x)||_0$$
$$\leq \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon)), \quad (103)$$

where $k_5 = \max\{|k_1|, |k_2|\}$, and k_1 and k_2 are given in proposition 19 (Eqs. 92, 93).

Proof Let θ'_i be the zeroes of x as given in Lemma 10 (Eqs. 99 and 101). Using proposition 20 we can write the solution $x(t)$ (89) (Lemma 9) as

$$\begin{aligned} x(t) &= \phi_\epsilon(t - \tilde{\theta}_i) + \exp[-(t - \eta_i)/\epsilon]\Delta_i(1 + \mathcal{E}(1/\epsilon)), \quad \eta_i \leq t \leq \eta_{i+1}, \quad i \text{ even},\\ x(t) &= \chi_\epsilon(t - \tilde{\theta}_i) + \exp[-(t - \eta_i)/\epsilon]\Delta_i(1 + \mathcal{E}(1/\epsilon)), \quad \eta_i \leq t \leq \eta_{i+1}, \quad i \text{ odd}, \end{aligned} \quad (104)$$

for $i = 1, 2, \ldots, n$, where $\eta_{n+1} \stackrel{\text{def}}{=} \theta'_n$.

To prove Lemma 11, we have to compare $x(t)$ to the following function

$$\begin{aligned} \hat{z}(t) &= \phi_\epsilon(t - (\theta'_n - 1 - \epsilon r)) = \phi_\epsilon(t + \sigma_n), \quad \text{for} \quad -\sigma_n \leq t \leq \eta'_1,\\ \hat{z}(t) &= \chi_\epsilon(t - \theta'_1), \quad \text{for} \quad \eta'_1 < t \leq \eta'_2,\\ \hat{z}(t) &= \phi_\epsilon(t - \theta'_2), \quad \text{for} \quad \eta'_2 < t \leq \eta'_3,\\ &\ldots\\ \hat{z}(t) &= \phi_\epsilon(t - \theta'_n), \quad \text{for} \quad \eta'_n < t \leq \theta'_n = 1 + \epsilon r - \sigma_n, \end{aligned} \quad (105)$$

where the numbers η'_i are such that $\hat{z}$ is continuous. Notice that $P_n \circ F_n(x)(t) = \hat{z}(t + \theta'_n)$. As in corollary 1 the numbers η'_i are approximately given by

$$\eta'_i = \theta'_{i-1} + \frac{\mu_2}{\mu_2+\nu_2}\delta'_i + \frac{\epsilon}{\mu_2+\nu_2}\ln\left(\frac{b_2}{c_2}\right) + \epsilon\mathcal{E}(1/\epsilon), \quad i \text{ odd},$$

and

$$\eta_i' = \theta_{i-1}' + \frac{\nu_1}{\mu_1+\nu_1}\delta_i + \frac{\epsilon}{\mu_1+\nu_1}\ln\left(\frac{c_1}{b_1}\right) + \epsilon\mathcal{E}(1/\epsilon), \quad i \text{ even},$$

where in the above expressions we shall use $\theta_0' = -\sigma_n$. From these expressions, corollary 1, and Lemma 10 we get

$$\eta_i' = \eta_i + \epsilon r + \epsilon\mathcal{E}(1/\epsilon), \quad \text{for} \quad i = 1, \ldots, n.$$

In particular, for ϵ sufficiently small $\eta_i' > \eta_i$. Thus, for $t \in [\eta_i', \eta_{i+1}]$, i even, $2 \le i \le n$, with $\eta_{n+1} \stackrel{\text{def}}{=} \theta_n'$, we get from Eq. 105

$$x(t) - \hat{z}(t) = \phi_\epsilon(t - \tilde{\theta}_i) - \phi_\epsilon(t - \theta_i') + \exp[-(t-\eta_i)/\epsilon]\Delta_i(1 + \mathcal{E}(1/\epsilon)).$$

Let $|\Delta_{max}| = \max\{|\Delta_j|, j = 1, 2, \ldots, n\}$. From the definitions of β and k_5 we get $|\Delta_{max}| \le k_5 \exp^{-\beta/\epsilon}$. Then using proposition 19 and Lemma 10 we get that for any $t \in \mathbb{R}$ the following inequality holds:

$$\begin{aligned} |\phi_\epsilon(t+\sigma_i) - \phi_\epsilon(t)| &= |\sigma_i| \left| \int_0^1 \dot{\phi}_\epsilon(t + s\sigma_i)ds \right| \le \frac{|\sigma_i|}{\epsilon} \max_{t\in\mathbb{R}} |\dot{\phi}(t)| \stackrel{\text{def}}{=} \frac{|\sigma_i|}{\epsilon} k_6 \\ &= |\Delta_{max}| \frac{|\sigma_i|}{|\Delta_{max}|\epsilon} k_6 = |\Delta_{max}|\mathcal{E}(1/\epsilon). \end{aligned} \tag{106}$$

So, for $t \in [\eta_i', \eta_{i+1}]$, with i even, we get

$$|x(t) - \hat{z}(t)| \le |\Delta_{max}|\mathcal{E}(1/\epsilon) + |\Delta_i| \le |\Delta_{max}|(1 + \mathcal{E}(1/\epsilon)) \le k_5 \exp^{-\beta/\epsilon}(1 + \mathcal{E}(1/\epsilon)) .$$

In the same way we prove that $|x(t) - \hat{z}(t)| \le k_5 \exp^{-\beta/\epsilon}(1 + \mathcal{E}(1/\epsilon))$, for $t \in [\eta_i', \eta_{i+1}]$, with i odd. Now, for $t \in [-\sigma_n, \eta_1]$ inequality (106) implies

$$|x(t) - \hat{z}(t)| = |\phi_\epsilon(t) - \phi_\epsilon(t + \sigma_n)| \le |\Delta_{max}|\mathcal{E}(1/\epsilon).$$

So, to finish the proof we have to show that the inequality (103) in the Lemma 11 is valid for $t \in \cup_{i=1}^n [\eta_i, \eta_i']$. Let us do it for $t \in [\eta_i, \eta_i']$ with i odd, $1 < i \le n-1$. The cases $i = 1$ and i even are proven in a similar way. From Eq. 105, and the expression for $\hat{z}$, for $t \in [\eta_i, \eta_i']$, we get

$$x(t) - \hat{z}(t) = G(t) + \exp[-(t-\eta_i)/\epsilon]\Delta_i(1 + \mathcal{E}(1/\epsilon)), \quad \text{where}$$
$$G(t) \stackrel{\text{def}}{=} \chi_\epsilon(t - \tilde{\theta}_i) - \phi_\epsilon(t - \theta_{i-1}').$$

Now, using the definition of η_i', that $\dot{\chi}_\epsilon < 0$, $\theta_i' = \tilde{\theta}_i - \sigma_i$, and $\sigma_i > 0$ we get

$$0 = \chi_\epsilon(\eta_i' - \tilde{\theta}_i + \sigma_i) - \phi_\epsilon(\eta_i' - \theta_{i-1}') < \chi_\epsilon(\eta_i' - \tilde{\theta}_i) - \phi_\epsilon(\eta_i' - \theta_{i-1}' = G(\eta_i').$$

This, $\eta_i < \eta_i'$, and $\dot{G}(t) < 0$ imply that $G(\eta_i) > 0$ and $0 < G(t) < G(\eta_i)$ for $t \in [\eta_i, \eta_i']$. From proposition 19, $\Delta_i < 0$ for $i > 1$ odd, implying

$$x(t) - \hat{z}(t) = G(t) + \exp[-(t-\eta_i)/\epsilon]\Delta_i(1 + \mathcal{E}(1/\epsilon)) < G(t) < G(\eta_i).$$

Thus from Eq. 106, from the definition (88) of Δ_i, and for $t \in [\eta_i, \eta_i']$, $i > 1$ odd, we get

$$\begin{aligned} x(t) - \hat{z}(t) < G(\eta_i) &= \chi_\epsilon(\eta_i - \tilde{\theta}_i) - \phi_\epsilon(\eta_i - \tilde{\theta}_{i-1}) + \phi_\epsilon(\eta_i - \tilde{\theta}_{i-1}) \\ &\quad -\phi_\epsilon(\eta_i - \tilde{\theta}_{i-1} + \sigma_{i-1}) < \Delta_i + |\Delta_{max}|\mathcal{E}(1/\epsilon) < k_5 \exp^{-\beta/\epsilon}(1 + \mathcal{E}(1/\epsilon)). \end{aligned}$$

Finally, for $t \in [\eta_i, \eta_i']$ with $i > 1$ odd, using that $G(t) > 0$ we get:

$$\begin{aligned} x(t) - \hat{z}(t) &= G(t) + \exp[-(t-\eta_i)/\epsilon]\Delta_i(1+\mathcal{E}(1/\epsilon)) \\ &> \exp[-(t-\eta_i)/\epsilon]\Delta_i(1+\mathcal{E}(1/\epsilon)) > \Delta_i(1+\mathcal{E}(1/\epsilon)) \\ &> -k_5 \exp^{-\beta/\epsilon}(1+\mathcal{E}(1/\epsilon)). \end{aligned}$$

□

The last part of Theorem 2 that has to be proven is its statement (i). This statement is a consequence of the following Lemma 12.

Lemma 12 *Let $\psi_t(x)(s) = x(t+s)$ and $\varphi_t(x)(s) = \tilde{z}(t+s)$, $s \in [-1, 0]$, where $x(t)$, $t \in [-1, 1+\eta_1]$ is the solution given in Lemma 9 and $\tilde{z}(t) = z(t+1)$, $t \in \mathbb{R}$, where z is the periodic extension of the function in Eq. 7. Then*

$$\sup_{t\in[-1,1+\eta_1]} |x(t) - \tilde{z}(t)| \le \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon)),$$

where $\eta_1 > 0$ is given in corollary 1 and β and k_5 are given in Lemma 11.

Proof From Lemma 9 the difference $x(t) - \tilde{z}(t)$ is zero for $t \in [-1, \eta_1]$. In the interval $[\eta_1, 1+\eta_1+\epsilon r]$ the function $\tilde{z}$ is given by

$$\begin{aligned} &\tilde{z}(t) = \phi_\epsilon(t-\tilde{\theta}_i) \quad \eta_i + \epsilon r \le t < \eta_{i+1} + \epsilon r, \;\; 0 \le i \le n, \;\; i \text{ even}, \\ &\tilde{z}(t) = \chi_\epsilon(t-\tilde{\theta}_i) + \sum_{j=0}^{i} \quad \eta_i + \epsilon r \le t < \eta_{i+1} + \epsilon r, \;\; 1 \le i < n, \;\; i \text{ odd}, \end{aligned}$$

where we use $\eta_0 + \epsilon r \stackrel{\text{def}}{=} 0$ and $\eta_{n+1} \stackrel{\text{def}}{=} 1 + \eta_1$. Thus, from Lemma 9 and proposition 20, for $t \in [\eta_i + \epsilon r, \eta_{i+1}]$, $i = 1, 2, \ldots, n$, we get

$$\begin{aligned} |x(t) - \tilde{z}(t)| &= \left|\sum_{j=0}^{i} \exp[-(t-\eta_j)/\epsilon]\Delta_j\right| = |\exp[-(t-\eta_i)/\epsilon]\Delta_i(1+\mathcal{E}(1/\epsilon))| \\ &\le \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon)). \end{aligned}$$

For $t \in [\eta_i, \eta_i + \epsilon r]$, $1 \le i \le n$, i odd,

$$x(t) - \tilde{z}(t) = G(t) + \sum_{j=0}^{i} \exp[-(t-\eta_j)/\epsilon]\Delta_j, \quad \text{where } G(t) \stackrel{\text{def}}{=} \chi_\epsilon(t-\tilde{\theta}_i) - \phi_\epsilon(t-\tilde{\theta}_{i-1}).$$

Then estimates similar to those made in the proof of Lemma 11 for a similar function G appearing there give the inequality $|x(t) - \tilde{z}(t)| \le \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon))$. The case $t \in [\eta_i, \eta_i + \epsilon r]$, $2 \le i < n$, i even, is proved likewise. □

8 Proof of Theorem 4

The proof of Theorem 4 will be given through a series of propositions and lemmas. The propositions and lemmas in this section have analogues in the previous section. Moreover, the proofs use the same ideas. So, most of the proofs in this section will be omitted.

Proposition 21 *For a given* $\xi > 0$ *the equation* $\phi_\epsilon(t) = \chi_\epsilon(t-\xi)$ *has a unique solution of the form*

$$t = t(\xi,\epsilon) = \frac{\xi\mu}{\mu+\nu} + \frac{\epsilon}{\mu+\nu}\ln\left(\frac{b_2}{c_2}\right) + \epsilon\mathcal{E}(1/\epsilon).$$

In the same way, for a given $\xi > 0$ *the equation* $\chi_\epsilon(t) = \phi_\epsilon(t-\xi)$ *has a unique solution of the form*

$$t = t(\xi,\epsilon) = \frac{\xi\mu}{\nu+\mu} + \frac{\epsilon}{\nu+\mu}\ln\left(\frac{c_1}{b_1}\right) + \epsilon\mathcal{E}(1/\epsilon),$$

μ, ν *satisfying (20).*

Proof The proof is similar to that of proposition 18. □

Corollary 2 *The numbers* η_i, $i = 1, 2, \ldots, 2n$ *appearing in Eq. 23 are approximately given by,*

$$\begin{aligned} \eta_i &= \theta_{i-1} + \frac{\mu}{\mu+\nu}\delta_i + \frac{\epsilon}{\mu+\nu}\ln\left(\frac{b_2}{c_2}\right) + \epsilon\mathcal{E}(1/\epsilon), \quad i \text{ odd}, \\ \eta_i &= \theta_{i-1} + \frac{\mu}{\nu+\mu}\delta_i + \frac{\epsilon}{\nu+\mu}\ln\left(\frac{c_1}{b_1}\right) + \epsilon\mathcal{E}(1/\epsilon), \quad i \text{ even}, \end{aligned} \tag{107}$$

where $\delta_i = \theta_i - \theta_{i-1} > 0$, $i = 1, 2, \ldots 2n$ *and* $\theta_0 \stackrel{\text{def}}{=} -\epsilon\bar{r}$.

In the following we just prove the statements in Theorem 4 related to the initial condition $x \in Y_n \cap W_n$, which is associated to $\delta \in A_n$. The statements related to the initial condition $x \in X_n \cap W_n$ are proved in the same way. Before presenting the Lemma 13, which is the main point in the proof of Theorem 4, it is convenient to introduce some notation. We define

$$\begin{aligned} \tilde{\theta}_i &= \theta_i + \epsilon\underline{r}, \quad i = 1, 3, 5, \ldots, n, \\ \tilde{\theta}_i &= \theta_i + \epsilon\bar{r}, \quad i = 2, 4, 6, \ldots, n-1, \end{aligned}$$

with $\tilde{\theta}_0 = 0$ and

$$\begin{aligned} \Delta_i &= \chi_\epsilon(\eta_i - \tilde{\theta}_{i-1}) - \phi_\epsilon(\eta_i - \tilde{\theta}_i) \quad i = 1, 3, 5, \ldots, n, \\ \Delta_i &= \phi_\epsilon(\eta_i - \tilde{\theta}_{i-1}) - \chi_\epsilon(\eta_i - \tilde{\theta}_i) \quad i = 2, 4, 6, \ldots, n-1. \end{aligned} \tag{108}$$

Lemma 13 *For a given* $\delta \in A_n$, *let* x *be the solution of Eq. 1 that satisfies the initial condition* $x(t) = z(t+1)$ *(notice that the initial condition is in* $Y_n \cap W_n$*), for* $t \in [-1, 0]$, *where* z *is the function given in (23). Then on the interval* $t \in [0, 1+\eta_1]$

$$\begin{aligned} x(t) &= \phi_\epsilon(t-\tilde{\theta}_i) + \sum_{j=0}^{i}\exp[-(t-\eta_j)/\epsilon]\Delta_j \quad \eta_i \le t < \eta_{i+1}, \quad 1 \le i \le n, \quad i \text{ odd}, \\ x(t) &= \chi_\epsilon(t-\tilde{\theta}_i) + \sum_{j=0}^{i}\exp[-(t-\eta_j)/\epsilon]\Delta_j \quad \eta_i \le t < \eta_{i+1}, 0 \le i < n, \quad i \text{ even}, \end{aligned} \tag{109}$$

with the definitions $\Delta_0 \stackrel{\text{def}}{=} 0$, $\eta_0 \stackrel{\text{def}}{=} 0$, *and* $\eta_{n+1} \stackrel{\text{def}}{=} 1 + \eta_1$.

Proof The proof is similar to that of Lemma 9. □

Proposition 22 *The* Δ_i, $i = 1, \ldots, n$, *defined in* (108) *satisfy the following estimates. For i odd,* $1 \le i \le n$,

$$\Delta_i = \exp\left[\frac{-\mu\nu}{\mu+\nu}\frac{\delta_i}{\epsilon}\right](k_1 + \mathcal{E}(1/\epsilon))$$

where

$$k_1 = c_1 \exp\left[\frac{-\nu}{\mu+\nu}\ln(b_2/c_2) + \nu\bar{r}\right] - b_1 \exp\left[\frac{\mu}{\mu+\nu}\ln(b_2/c_2) - \mu\underline{r}\right] > 0. \quad (110)$$

For i even, $2 \le i < n$,

$$\Delta_i = \exp\left[\frac{-\nu\mu}{\nu+\mu}\frac{\delta_i}{\epsilon}\right](k_2 + \mathcal{E}(1/\epsilon))$$

where

$$k_2 = c_2 \exp\left[\frac{\mu}{\nu+\mu}\ln(c_1/b_1) - \mu\bar{r}\right] - b_2 \exp\left[\frac{-\nu}{\nu+\mu}\ln(c_1/b_1) + \nu\underline{r}\right] < 0. \quad (111)$$

Proof The inequalities for Δ_i and the expressions for k_1 and k_2 are obtained in the same way as those in proposition 19. The proofs that $k_1 > 0$ and $k_2 < 0$ are different. After some algebraic manipulations of the expressions for k_1 and k_2 (Eqs. 110, 111) in the statement of the proposition 22 can be written as

$$k_1 = c_1 \exp\left[\frac{-\nu}{\mu+\nu}\ln(b_2/c_2) + \nu\bar{r}\right]\left\{1 - \exp\left[-\ln\left(\frac{c_1c_2}{b_1b_2}\right) - \mu\underline{r} - \nu\bar{r}\right]\right\}$$

and

$$k_2 = c_2 \exp\left[\frac{\mu}{\nu+\mu}\ln(c_1/b_1) - \mu\bar{r}\right]\left\{1 - \exp\left[\ln\left(\frac{b_1b_2}{c_1c_2}\right) + \mu\bar{r} + \nu\underline{r}\right]\right\}.$$

These two inequalities, and the inequality (20) in Theorem 3, imply that $k_1 > 0$ and $k_2 < 0$. □

Proposition 23 *For* $i = 1, 2, \ldots, n$ *the following holds*

$$\sum_{j=0}^{i} \exp[\eta_j/\epsilon]\Delta_j = \exp[\eta_i/\epsilon]\Delta_i(1 + \mathcal{E}(1/\epsilon)).$$

Proof The proof is similar to that of proposition 20. □

Lemma 14 *Let x be the solution of Eq. 1 that is given in Lemma 13. It satisfies* $x(0) = 0$ *and, for* ϵ *sufficiently small, x has n zeroes in the interval* $(0, 1 + \eta_1)$, $x(\theta_i') = 0$, *with* $0 < \theta_1' < \theta_2', \ldots, \theta_n' < 1 + \eta_1$.

For i odd, $1 \le i \le n$, θ_i' *satisfies the following estimate:*

$$\theta_i' = \theta_i + \epsilon\underline{r} - \sigma_i \quad \text{with} \quad \sigma_i = \epsilon \exp\left[-\frac{a\delta_i}{\epsilon}\right](k_3 + \mathcal{E}(1/\epsilon)),$$

where

$$a \overset{\text{def}}{=} \frac{\nu(1+\mu)}{\mu+\nu} > 0, \quad k_3 = \frac{k_1}{\dot{\phi}(0)} \exp\left[\frac{1}{\mu+\nu}\ln(b_2/c_2) - \underline{r}\right] > 0,$$

and $k_1 > 0$ *is given in Eq. 110 (proposition 22).*

For i even, $2 \le i < n$, θ'_i satisfies the following estimate:

$$\theta'_i = \theta_i + \epsilon\bar{r} - \sigma_i, \quad \text{with} \quad \sigma_i = \epsilon \exp\left[-\frac{a\delta_i}{\epsilon}\right](k_4 + \mathcal{E}(1/\epsilon)),$$

where a is given above and

$$k_4 = \frac{k_2}{\dot{\chi}(0)} \exp\left[\frac{1}{\nu+\mu}\ln(c_1/b_1) - \bar{r}\right] > 0,$$

and $k_2 < 0$ is given in Eq. 111 (proposition 22).

Proof The proof is similar to the proof of Lemma 10. □

The initial condition that generates the solution x (Eq. 109) in Lemma 13 is in the set $Y_n \cap W_n$. Lemma 14 implies that, for ϵ sufficiently small, the first zero of x after $t = 1$ is given by $\theta'_n < 1 + \eta_1$. The argument for this is similar to that in the previous section right after the proof of Lemma 10. Moreover, for ϵ sufficiently small, Eq. 1 gives

$$\dot{x}(\theta') = -x(\theta') + f(x(\theta' - 1)) = f(x(\epsilon\underline{r} - \sigma_n)) = f(x(\epsilon\underline{r} + \mathcal{E}(1/\epsilon))) > 0$$

because $0 < \epsilon\underline{r} + \mathcal{E}(1/\epsilon) < \eta_1$, $x(\epsilon\underline{r} + \mathcal{E}(1/\epsilon)) < 0$, and for a negative feedback equation $xf(x) < 0$. So, $x(t + \theta')$, $t \in [-1, 0]$, is in X_n and the initial condition x is in $\overline{Y}_n$. This implies that the T appearing in item (ii) of Theorem 4 is given by $T = \theta'_n$ and Lemma 14 implies that T has the expression given in Theorem 4. From the definition of $F_{Y_n}(x) = \psi_T(x)$ we get that $F_{Y_n}(x)(t) = x(t + \theta'_n)$, for $t \in [-1, 0]$, where x is the solution (109) given in Lemma 13. Notice that for ϵ sufficiently small, the zeroes of $F_{Y_n}(x)$ are located at $-1 + \hat{\theta}_1 < -1 + \hat{\theta}_2 < \cdots < -1 + \hat{\theta}_{n-1} < 1 - \hat{\theta}_n = 0$, where $\hat{\theta}_i = 1 - \theta'_n + \theta'_i$. Now, if we define $\hat{\theta}_0 = -\epsilon\underline{r}$ and $\delta'_i = \hat{\theta}_i - \hat{\theta}_{i-1} = \theta'_i - \hat{\theta}'_{i-1}$ then from Lemma 14 we get all the equations in item (iii) of Theorem 4. The following lemma completes the proof of item (ii) of Theorem 2.

Lemma 15 *Let*

$$\beta \stackrel{\text{def}}{=} \frac{\mu\nu}{\mu+\nu} \min\{\delta_1, \delta_2, \delta_3, \ldots, \delta_n\}.$$

Then

$$\sup_{t\in[-1,0]} |F_{Y_n}(x)(t) - P_{Y_n} \circ F_{Y_n}(x)(t)| \le \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon)),$$

where $k_5 = \max\{|k_1|, |k_2|\}$, and k_1 and k_2 are given in Eqs. 110, 111 (proposition 22).

Proof The proof of this lemma is similar to that of Lemma 11, excepting the proof that $\eta'_i > \eta_i$. So we shall only give the proof of this inequality. Here η'_i is defined as the roots of the following equations:

$$0 = \phi_\epsilon(\eta'_i - \theta'_i) - \chi_\epsilon(\eta'_i - \theta'_{i-1}) \quad i = 1, 3, \ldots, n,$$
$$0 = \chi_\epsilon(\eta'_i - \theta'_i) - \phi_\epsilon(\eta'_i - \theta'_{i-1}) \quad i = 2, 4, \ldots, n-1,$$

where $\theta'_0 \stackrel{\text{def}}{=} -\sigma_n$. Using proposition 21 we get that:

$$\eta'_i = \theta'_{i-1} + \frac{\mu}{\mu+\nu}\delta'_i + \frac{\epsilon}{\mu+\nu}\ln\left(\frac{b_2}{c_2}\right) + \epsilon\mathcal{E}(1/\epsilon), \quad i = 2, 4, \ldots, n-1,$$
$$\eta'_i = \theta'_{i-1} + \frac{\mu}{\mu+\nu}\delta_i + \frac{\epsilon}{\mu+\nu}\ln\left(\frac{c_1}{b_1}\right) + \epsilon\mathcal{E}(1/\epsilon), \quad i = 1, 3, \ldots, n,$$

where $\delta'_i = \theta'_i - \theta'_{i-1}$. Then using corollary 2 and Lemma 14, after some manipulation we get

$$\begin{aligned} \eta'_i &= \eta_i + \frac{\epsilon}{\mu+\nu}\left[ln\left(\frac{b_2 b_1}{c_2 c_1}\right) + +\nu\underline{r} + \mu\overline{r}\right] + \epsilon\mathcal{E}(1/\epsilon) \\ &= \eta_i + n + \epsilon\mathcal{E}(1/\epsilon), \quad i = 2, 4, \ldots, n-1 \\ \eta'_i &= \eta_i + \frac{\epsilon}{\mu+\nu}\left[ln\left(\frac{c_2 c_1}{b_2 b_1}\right) + +\nu\overline{r} + \mu\underline{r}\right] + \epsilon\mathcal{E}(1/\epsilon) \\ &= \eta_i + n + \epsilon\mathcal{E}(1/\epsilon), \quad i = 1, 3, \ldots, n \end{aligned} \tag{112}$$

Then the last inequality in Theorem 3 implies that $\eta_i < \eta'_i$, for $i = 1, 2, \ldots, n$. Now, let x be the solution (109) in Lemma 13, and $\hat{z}$ be the analogue of that function $\hat{z}$ (Eq. 105) used in the proof of Lemma 11. To prove the above inequality, it is equivalent to show that $|x(t) - \hat{z}(t)| \leq \exp\left[-\frac{\beta}{\epsilon}\right](k_5 + \mathcal{E}(1/\epsilon))$, for $t \in [\theta' - 1, \theta']$. As in the proof of Lemma 11 we split the proof of this inequality into two parts. First we show that the inequality holds inside the intervals $t \in [\eta'_i, \eta_{i+1}]$. This is done in the same way as in the proof of Lemma 11. Then we must show the inequality holds for $t \in [\eta_i, \eta'_i]$. This last part has one point that is not a straightforward modification of the corresponding part in the proof of Lemma 11. This point is highlighted in the proof given below.

Let $t \in [\eta_i, \eta'_i]$ with i even, $2 \leq i \leq n-1$. The case i odd is proven in a similar way. In this case the function to be bounded is

$$\begin{aligned} x(t) - \hat{z}(t) &= G(t) + \exp[-(t-\eta_i)/\epsilon]\Delta_i(1 + \mathcal{E}(1/\epsilon)), \\ \text{where} \quad G(t) &\stackrel{\text{def}}{=} \chi_\epsilon(t - \tilde{\theta}_i) - \phi_\epsilon(t - \theta'_{i-1}). \end{aligned}$$

Using the same arguments used in the proof of Lemma 11, we show that $0 < G(t) < G(\eta_i)$. Now, the only significant difference with the proof of Lemma 11 is how to bound $G(\eta_i)$ (in this case $G(\eta_i)$ is not by definition approximately Δ_i). From the expression (107) for η_i, i even, in corollary 2, we get

$$\begin{aligned} G(\eta_i) = \chi_\epsilon &\left[\frac{-\nu}{\nu+\mu}\delta_i + \frac{\epsilon}{\nu+\mu}\ln\left(\frac{c_1}{b_1}\right) - \epsilon\overline{r} + \epsilon\mathcal{E}(1/\epsilon)\right] + \\ &-\phi_\epsilon\left[\frac{\mu}{\nu+\mu}\delta_i + \frac{\epsilon}{\nu+\mu}\ln\left(\frac{c_1}{b_1}\right) - \epsilon\underline{r} + \sigma_{i-1} + \epsilon\mathcal{E}(1/\epsilon)\right]. \end{aligned}$$

As $\phi_\epsilon(x) = \phi(x/\epsilon)$, $\chi_\epsilon(x) = \chi(x/\epsilon)$, and $\sigma_{i-1} = \epsilon\mathcal{E}(1/\epsilon)$ (from Lemma 14) we get

$$\begin{aligned} G(\eta_i) = \chi &\left[\frac{-\nu}{\nu+\mu}\frac{\delta_i}{\epsilon} + \frac{1}{\nu+\mu}\ln\left(\frac{c_1}{b_1}\right) - \overline{r} + \mathcal{E}(1/\epsilon)\right] + \\ &-\phi\left[\frac{\mu}{\nu+\mu}\frac{\delta_i}{\epsilon} + \frac{1}{\nu+\mu}\ln\left(\frac{c_1}{b_1}\right) - \underline{r} + \mathcal{E}(1/\epsilon)\right]. \end{aligned}$$

Finally, using the asymptotic expressions (19) for ϕ and χ provided in Theorem 3, we get $G(\eta_i) = -\Delta_i(1 + \mathcal{E}(1/\epsilon)) > 0$. The rest of the proof is similar to that of Lemma 11. □

The last part of Theorem 4 that has to be proved is its statement i. Its proof follows closely the proof of Lemma 12, the existing differences being similar to those encountered in the proof of Lemma 15, therefore they can be handled in the same way (in particular, the identity $\eta_{n+i} = \eta'_i + \mathcal{E}(1/\epsilon)$, that appears in Eq. 112, is useful at this point). So the proof of part i of Theorem 4 will be omitted.

Finally, there is the case where the initial condition $x \in X_n \cap W_n$. The proof of this case is essentially the same as that outlined above. In most parts it is enough to exchange "i odd" by "i even" and vice versa.

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