Categorical Constructions of Sets Valued on Semicartesian and Involutive Quantales

JOSÉ G. ALVIM*
University of São Paulo, São Paulo, Brazil
HUGO L. MARIANO†
University of São Paulo, São Paulo, Brazil
CAIO DE A. MENDES‡
University of São Paulo, São Paulo, Brazil

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This work mainly concerns the—here introduced—category of $\mathcal{Q}$-sets and their functional morphisms, for $\mathcal{Q}$ an involutive and semicartesian quantale. In particular, we describe in detail the limits and colimits of this complete and cocomplete category, and establish that it is locally presentable.

In the 1970s, the topos of sheaves over a locale/complete Heyting algebra $\mathbb{H}$, denoted as $\mathrm{Sh}(\mathbb{H})$, was described, alternatively, as a category of $\mathbb{H}$-sets [4]. More precisely, in [3], there were three categories, whose objects were locale valued sets, that are equivalent to the category $\mathrm{Sh}(\mathbb{H})$.

Initially separate from the world of sheaves, there was a non-commutative and non-idempotent generalization of locales called "quantales", introduced in the mid 1980s by C.J. Mulvey [6]. In the early 1990s, quantales show up in logic, and in the study of C^*-algebras. Sheaves over certain quantales have been considered in [1] and, recently, by A. Tenório, C. Mendes and H. Mariano in [7]. Categories of sheaves on *involutive quantale*—identified with some enriched categories—hearkening to the work of R. Walters [8], were revisited by H. Heymans, I. Stubbe, and P. Resende. Examples of involutive quantales aren't few: binary relations over any set, maximal spectra of non-commutative C^* algebras, quantales of ideals of a ring endowed with an involution, etc.

Categories of sheaves over (certain subclasses of) quantales, in the sense of categories of quantale valued sets, have been proposed over the years in attempts to expand the celebrated notion of $\mathbb{H}$-sets (for complete Heyting algebras) to the broader category of parameter algebras consisting of certain quantales. For instance, categories of sets valued on right-sided idempotent quantales, were considered by M. Coniglio, F. Miraglia, and U. Solitro in the latte 1990s, [5].

In this work we deal with involutive and semicartesian quantales. Semicartesianess means that the quantale admits projections ($a \otimes b \leq a, b$). The logical meaning of an involution on a quantale can be extracted by inspecting the involution-free general case and then adding the involution: non-commutative substructural logic. Non-commutative logic is *temporal* since the relative order of premises of an entailment are logically relevant. As such, the involution reflects, or rather imposes, a way to temporally rearrange terms while preserving entailment.

Definition: Let $\mathcal{Q}$ be an involutive semicartesian quantale. A right $\mathcal{Q}$-set is a pair $X = (|X|, \delta)$ where $|X|$ is a usual set and $\delta : |X| \times |X| \to \mathcal{Q}$ is a function such that, for all x, y and $z \in |X|$: $\delta(x, y)^* = \delta(y, x)$ (symmetry), $\delta(x, y) \otimes \delta(y, z) \leq \delta(x, z)$ (transitivity), $\delta(x, y) \otimes \delta(y, y) = \delta(x, y)$

*alvim@ime.usp.br
†hugomar@ime.usp.br
‡caio.mendes@alumni.usp.br

(local right identity). A left $\mathcal{Q}$-set satisfies the first two axioms and a symmetrically analogous version of the third axiom. We also define $\mathrm{E}x := \delta(x,x)$, the extent of x.

Our theory of semicartesian involutive $\mathcal{Q}$-sets describes objects that capture a certain notion of "becoming", where the involution will play a role in "reversing a process", in that $\delta^*(x,y) = \delta(y,x)$, like how homotopies are reversible. This "becoming" expression is perhaps overly poetic, since the $\mathcal{Q}$-set itself is static and has no dynamics, rather, it describes an "equality with direction".

Definition: Let $\mathcal{Q}$ a quantale and $X = (|X|, \delta_X), Y = (|Y|, \delta_Y)$ $\mathcal{Q}$-sets. A funcional morphism $f : X \to Y$ is a function $|f| : |X| \to |Y|$ such that: $\delta(x,y) \leq \delta(f(x), f(y))$ (increasing), $\mathrm{E}x = \mathrm{E}f(x)$ (extent preservation). The identity morphism and the morphism composition are defined as the usual function identity and function composition respectively. The corresponding category will be denoted by $\mathcal{Q}$-Set.

Concerning the category of $\mathcal{Q}$-sets:

1. We describe, in detail, the limits and colimits of this complete and cocomplete category;
2. We describe generators;
3. We prove that it is a κ-locally presentable category (where $\kappa = \max\{|\mathcal{Q}|^+, \aleph_0\}$);
4. We investigate if the category has some form of subobjets classifier;
5. We investigate a certain family of monoidal products defined over this category;
6. We discuss the issue of "change of basis" induced by appropriate morphisms between the parametrizing quantales involved in the definition of $\mathcal{Q}$-sets.

We will also discuss the important notions of *relational morphisms*, *singletons*, and *Scott-Completion*, that play an important role in the connection between the categories $\mathrm{Sh}(\mathbb{H})$ and $\mathbb{H}-sets$, in the vein of [2].

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