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Mixture of two ultra cold bosonic atoms confined in a ring: stability and persistent currents

E T D Matsushita[1] and E J V de Passos

Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, C.E.P. 05389-970, São Paulo, São Paulo, Brazil

[1] The author to whom any correspondence should be addressed

E-mail: edutoshio@uol.com.br and edutoshio@uol.com.br

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Abstract

In this article we investigate the stability of quantized yrast states in a mixture of two distinguishable equal mass bosonic atoms confined in a ring. We focus in the study of energetic stability since the Bloch analysis and the Bogoliubov theory establishes that only energetically stable quantized yrast states are capable of sustain a persistent current. In the framework of the Bogoliubov theory we study the stability in two different cases chosen by physical considerations. In one case we analyze how the inter and intraspecies interaction strengths affect the stability of a selected quantized yrast state specified by the angular momentum per particle and the population imbalance. In the other case, for a fixed dynamics specified by given values of interaction strengths, we determine the stability of quantized yrast states as function of the angular momentum per particle and the population imbalance. We also examined the stability of the mixture in the rarefied limit and we found a critical value of the population imbalance which gives the size of the window of energetic stability and a critical value of angular momentum per particle which is an upper bound of the possible values of angular momentum per particle carried by energetically stable quantized yrast states.

1. Introduction

The properties of superfluidity of a system of ultra cold atoms confined in a ring has been extensively studied in recent years, both experimentally [1–5] and theoretically [6–8]. The experiments of references [1–5] have managed to create in the laboratory persistent currents in this system. At the theoretical side an analysis by Bloch [9] concluded that the occurrence of persistent currents is related to the stability of yrast states. As the angular momentum L_z commutes with the Ring Hamiltonian H the stationary states can be chosen to be simultaneous eigenstates of L_z and H. The state with the lowest energy for a given L, where L is an eigenvalue of L_z, is referred to as yrast state. According to Bloch only yrast states which are local minima of the yrast spectrum are capable of sustain a persistent current. Variational methods [7, 8], the Bogoliubov theory (when applicable) [6] and a truncated diagonalization of the Ring Hamiltonian [6] have been employed to calculate the yrast spectrum. The main conclusion from these calculations was that only yrast states with an integer angular momentum per particle $\frac{L}{N} = l$, with $|l| = 1, 2, \ldots$, is capable of sustain a persistent current. These states are known as quantized yrast (QY) states [6].

A non trivial extension of the previous case is to investigate persistent currents in systems which are a mixture of two ultra cold distinguishable atoms confined in a torus where the transverse component is so tight that the system is effectively one dimensional, the ring [6, 12, 15]. If A and B are the labels of the two species, we have a mixture of N_A atoms of specie A and N_B atoms of specie B, the total number of atoms being equal to $N = N_A + N_B$ and the total angular momentum equal to $L = L_A + L_B$. At fixed value of N the number of atoms of each species can be parameterized as $N_A = \frac{N}{2}(1 - f)$ and $N_B = \frac{N}{2}(1 + f)$ where f is the population imbalance, $|f| \leqslant 1$. The atoms interact via a contact interaction with strengths U_{AA}, U_{BB} and U_{AB}. Quantitative studies of this system have been done only by variational methods in the limit of equal [10–14] and unequal [15] interaction strengths.

In this article we investigate, in the framework of the Bogoliubov theory, the stability properties of current carrying QY states. In our analysis we consider equal intraspecies interaction strengths, $U_{AA} = U_{BB} \equiv U$ different from interspecies interaction strength U_{AB}. In the Bogoliubov theory the determination of the stability of an equilibrium state is based on the following criterion: (i) if at least one eigenvalue is complex the equilibrium state is dynamically unstable; (ii) if the energies of the elementary excitations are all real the equilibrium state is dynamically stable [18–20]; (iii) if the energies of the elementary excitations are all real and positive the equilibrium state is energetically stable [15, 20, 21]. This criterion is based on the following property [22]: The eigenvalues of the Bogoliubov-de Gennes diagonalization problem can be complex or real. When the eigenvalues are real they come in pairs of eigenvectors with opposite eigenvalues and opposite norm. To each of these pairs we associate an elementary excitation whose energy is the eigenvalue of the eigenvector with positive norm.

Since an energetically stable QY state is a local minimum and taking into consideration the Bloch analysis [9] we conclude that only the energetically stable QY states are capable of sustain a persistent current. We focus our investigation in the study of the energetic stability of the QY states. We solved the Bogoliubov-de Gennes equations of the model to find analytic expressions for the energies of the elementary excitations written in terms of the system parameters (U_{AB}, U, l, f). Two of these parameters, l and f, specify the QY state and the other two, U_{AB} and U, specify the dynamics. Based on the stability criterion we determine the inequalities that when satisfied define the domain of dynamical and energetic stability in the four dimensional space of system parameters [12, 15].

Nevertheless we found two cases where physical considerations reduce the study in the four dimensional space to a study in two dimensional planes spanned by a pair of system parameters with the other pair kept at fixed values. One is when we are studying the stability of a selected QY state. In this case we fix the values of l and f equal to the labels of the selected QY state which reduce the inequalities in the four dimensional space into inequalities in the $U_{AB} \times U$ plane. The stability diagram in this plane defines the domain of energetic stability and displays the stability as a function of U_{AB} and U. Measurements to determine experimentally the stability diagram is in principle feasible through the mechanism of Feshbach resonance. In the other case we are studying the stability at fixed dynamics. If we fix the values of U_{AB} and U, the inequalities in the four dimensional space reduce into inequalities in the $l \times f$ plane. However, from the viewpoint of physics, f is a bounded quantity, $|f| \leqslant 1$, and the physical values of f and l are restricted to the sector of the $l \times f$ plane, the sector of physical significance (SPS), defined by inequalities $|l| \geqslant 0$ and $|f| \leqslant 1$. Indeed in this work we consider only $l \neq 0$ states since a $l = 0$ state does not carry a current. This restricts the investigation to the region of the SPS defined by the inequalities $|l| \geqslant 1$ and $|f| \leqslant 1$. From now on we denoted this region as $\mathrm{SPS}_{|l|\geqslant 1}$. The QY states appear when we postulate a correspondence between points in the $|l| \geqslant 1$ region of SPS of coordinates (l, f) and the QY states whose labels are these coordinates. This correspondence extends to the stability properties. The regime of stability of a QY state is equal to the regime of stability of the corresponding point in the $\mathrm{SPS}_{|l|\geqslant 1}$. The domain of dynamical and energetic stability in the $\mathrm{SPS}_{|l|\geqslant 1}$ is determined by the intersection of the domains in the $l \times f$ plane of the above named inequalities and the domain of $\mathrm{SPS}_{|l|\geqslant 1}$ which displays the stability of the QY states as function of l and f. The experimental determination of the stability diagram requires the measurements of stability of all QY states which, in principle, can be done.

We do not know of any work that determines the effect of the inter and intraspecies interaction strengths in the stability of QY states and its dependence on the angular momentum per particle and population imbalance. In the literature we can find the theoretical work [15] which investigates the stability of mixtures in a ring. Both of us deduced inequalities that when satisfied define the domain of dynamical e energetic stabilities and study the stability in two dimensional planes spanned by pairs of system parameters. We differ from [15] not only by the subject under investigation but also by the way that the inequalities are handled. Reference [15] is mainly interested in establishing under what conditions mixtures of two Bose condensates where the occupied states are eigenvectors of the angular momentum operator with different eigenvalues, $l_A \neq l_B$, can be yrast states. On the other hand, in this article we investigate the stability of QY states where $l_A = l_B$. With respect to the inequalities, it is where appears one of the novelties of our work, that is, the choice of two dimensional planes based on physical considerations which led to descriptions with straightforward physical interpretation and experimental relevance. Differently from us, the choice of planes in [15] is not based on physical considerations which led to descriptions lacking physical and experimental meaning.

This article is organized as follows: In section 2 we describe the system under consideration and we define the states that will be identified with the QY states compatible with the Bogoliubov theory. We discussed briefly our method to solve the Bogoliubov-de Gennes equations of the model and we found analytic expressions for the energy of the elementary excitations written in terms of the system parameters. Based on the stability criterion, we determine the inequalities that when satisfied are the necessary and sufficient conditions for QY states to be dynamically and energetically stable. In section 3 we present the stability diagram in the $U_{AB} \times U$ plane to show that the energetic phase boundary is the positive branch of a hyperbola and how the inter and intraspecies interaction strengths affect the stability. The stability in the rarefied limit of the minority component was also examined. In section 4 we present the stability diagram in the $SPS_{|l|\geqslant 1}$ and we investigate the consequences of the occurrence of two critical quantities: $f_{\rm crit}(l)$ and $l_{\rm crit}$. In section 5 we present our conclusions.

2. Bogoliubov stability criterion

In our system, a two-component gas is confined in a tight toroidal trap of radius R and cross-section S, the ring. The two-component gas is a mixture of N_A atoms of specie A and N_B atoms of specie B with $N = N_A + N_B$ fixed. The Ring Hamiltonian, in units of $\frac{\hbar^2}{2MR^2}$, in second quantization reads [12, 15]

$$H = \sum_s \sum_m m^2 a^\dagger_{s,m} a_{s,m} + \frac{1}{2} \sum_{s,s'} \sum_{m_i} U_{ss'} a^\dagger_{s,m_1} a^\dagger_{s',m_2} a_{s,m_3} a_{s',m_4} \delta_{m_1+m_2, m_3+m_4} \tag{1}$$

where $a_{s,m}$ ($a^\dagger_{s,m}$) is the bosonic annihilation (creation) operator of an atom of specie $s = A, B$ in an eigenstate of l_z with eigenvalue m and $U_{ss'} = \frac{4Ra_{ss'}}{S}$ with $2\pi U_{ss'}$ being the effective interaction strength between atoms of species s and s' confined in a ring, in units of $\frac{\hbar^2}{2MR^2}$, where $a_{ss'}$ is the respective s–wave scattering length [12, 15].

In a mean field theory, a current carrying QY state is a mixture of two Bose condensed states where N_A atoms of specie A and N_B atoms of specie B occupy the same eigenstate of l_z of eigenvalue l and wave function $\phi_l(\theta) = \frac{1}{\sqrt{2\pi}} e^{il\theta}$ and angular momentum equal to $L = N_A l + N_B l = Nl$ with $|l| \geqslant 1$. In the Bogoliubov theory the QY state is identified with the vacuum of the shifted operators $c_{s,m}$ defined by $a_{s,m} = c_{s,m} + z_{s,l}\delta_{m,l}$ which is a coherent state of the annihilation operator of atoms in the occupied states [16]. The c–numbers $z_{s,l}$ are determined by imposing that the vacuum is an equilibrium state which requires that the mean value of the Hamiltonian in the vacuum [17],

$$\langle H \rangle = \sum_s l^2 |z_{s,l}|^2 + \frac{1}{2} \sum_{s,s'} U_{ss'} |z_{s,l}|^2 |z_{s'l}|^2, \tag{2}$$

is stationary with respect to the variations of the $z_{s,l}$ subject to the number-conserving constraints $N_s = |z_{s,l}|^2$ for $s = A, B$ leading to the equations

$$\mu_A = l^2 + N_A U_{AA} + N_B U_{AB} \tag{3a}$$

$$\mu_B = l^2 + N_B U_{BB} + N_A U_{AB}. \tag{3b}$$

The equations (3a) and (3b) are invariant by a phase change of the $z_{s,l}$ and therefore they can be chosen real and equal to $z_{A,l} = \sqrt{N_A}$ and $z_{B,l} = \sqrt{N_B}$.

The next step is to write the Grand-Hamiltonian $\mathcal{H} \equiv H - \sum_s \mu_s N_s$ as a normal order expansion with respect to the shifted operators, $\mathcal{H} = \sum_{i=0}^4 \mathcal{H}_i$, where the term $\mathcal{H}_i$ involves i shifted operators. The dynamics in the neighborhood of an equilibrium state is described by an Effective Hamiltonian which is the normal ordered expansion of the Grand-Hamiltonian up to the second order in the shifted operators. Since $\mathcal{H}_0$ is a constant and $\mathcal{H}_1$ identically zero (stationary condition) the Effective Hamiltonian reduces to the quadratic term $\mathcal{H}_2$ given by

$$\mathcal{H}_2 = \sum_q \sum_{s,s'} [(q^2 + 2ql)\delta_{s,s'} + U_{ss'}\sqrt{N_s N_{s'}}] c^\dagger_{s,l+q} c_{s',l+q} + \frac{1}{2} \sum_q \sum_{s,s'} U_{ss'} \sqrt{N_s N_{s'}} (c_{s,l+q} c_{s',l-q} + c^\dagger_{s',l-q} c^\dagger_{s,l+q}) \tag{4}$$

with $m = l + q$ where q is the transferred angular momentum (relative angular momentum). An inspection of the coupling structure of (4) shows that only pairs $(l \pm q)_A$ and $(l \pm q)_B$ are coupled which allow us to express $\mathcal{H}_2$ as direct sum of a doublet and quadruplets specified by the magnitude of the transferred angular momentum q,

$$\mathcal{H}_2 = \mathcal{H}_2^{(0)} + \sum_{q>0} \mathcal{H}_2^{(q)} \tag{5}$$

with

$$\mathcal{H}_2^{(0)} = \sum_{s,s'} h_{s,l;s',l} c^\dagger_{s,l} c_{s',l} + \frac{1}{2} \Delta_{s,l;s',l} (c_{s,l} c_{s',l} + c^\dagger_{s',l} c^\dagger_{s,l}) \tag{6}$$

$$\mathcal{H}_2^{(q)} = \sum_{\substack{s,s' \\ \lambda,\lambda'}} \left[h_{s,l+\lambda q; s',l+\lambda' q} c^\dagger_{s,l+\lambda q} c_{s',l+\lambda' q} + \frac{1}{2} \Delta_{s,l+\lambda q; s',l+\lambda' q} (c_{s,l+\lambda q} c_{s',l+\lambda' q} + c^\dagger_{s',l+\lambda' q} c^\dagger_{s,l+\lambda q}) \right] \tag{7}$$

where $\lambda,\ \lambda' = \pm 1$ give the sign of the transferred angular momentum and

$$h_{s,l+\lambda q;s',l+\lambda' q} = [(q^2 + 2\lambda ql)\delta_{s,s'} + U_{ss'}\sqrt{N_s N_{s'}}]\delta_{\lambda,\lambda'} \tag{8}$$

$$\Delta_{s,l+\lambda q;s',l+\lambda' q} = U_{ss'}\sqrt{N_s N_{s'}}\delta_{\lambda,-\lambda'}. \tag{9}$$

The energies and the composition of the elementary excitations are found solving the Bogoliubov-de Gennes (BdG) equations which correspond to solve the eigenvalue problem of a non-hermitian matrix $\eta\mathcal{M}\mathcal{V} = E\mathcal{V}$ with $\mathcal{M} = \begin{bmatrix} h & \Delta \\ \Delta & h \end{bmatrix}$ and $\eta = \mathrm{diag}(1, -1)$. In a quadruplet, the eigenvalue problem reduces to the diagonalization of 8×8 matrix which has been done for $l = 0$ leading to a double degenerate spin and density modes whose excitation energies are [12, 14, 15]

$$E_d = \sqrt{\frac{1}{2}[c_{AA} + c_{BB} + \sqrt{(c_{AA} + c_{BB})^2 - 4(c_{AA}c_{BB} - c_{AB}^2)}\,]} \tag{10a}$$

$$E_s = \sqrt{\frac{1}{2}[c_{AA} + c_{BB} - \sqrt{(c_{AA} + c_{BB})^2 - 4(c_{AA}c_{BB} - c_{AB}^2)}\,]} \tag{10b}$$

where

$$c_{AA} \equiv q^2(q^2 + 2U_{AA}N_A), \qquad c_{BB} \equiv q^2(q^2 + 2U_{BB}N_B) \qquad \text{and} \qquad c_{AB} \equiv 2q^2U_{AB}\sqrt{N_A N_B}. \tag{10c}$$

An inspection of the BdG eigenvalue problem shows that the $l = 0$ and $l \neq 0$ cases differ by the presence of a l–dependent shift $\pm 2ql$, [15]. The eigenvalue problem for the shifted eigenvalues $E = E_{\mathrm{shift}} \pm 2ql$ becomes independent of l and equal to the $l = 0$ case. Therefore the excitation energies for $l \neq 0$ are non degenerate and equal to $E_d \pm 2ql$ and $E_s \pm 2ql$. Besides knowing that in the $l = 0$ eigenvalue problem the eigenvectors with positive norm have positive eigenvalues, from the above named properties we conclude that eigenvectors of the $l \neq 0$ eigenvalue problem with positive shifted eigenvalues have positive norm. The doublet diagonalization gives two zero energy modes which does not affect the stability of the QY states and will be ignored from now on.

2.1. Dynamical stability criterion

According to the Bogoliubov theory an equilibrium state is dynamically stable if all the excitation energies are real. The existence of at least one complex energy is sufficient to guarantee the dynamical instability of the corresponding equilibrium state. As $E_s < E_d$ we see that the energies of the quadruplet with magnitude of transferred angular momentum q are real if E_s^2 is real and positive. From (10a) and (10b), this condition is satisfied if

$$c_{AA} + c_{BB} > 0, \qquad c_{AA}c_{BB} - c_{AB}^2 > 0 \qquad \text{and} \qquad (c_{AA} + c_{BB})^2 - 4(c_{AA}c_{BB} - c_{AB}^2) > 0. \tag{11}$$

The last inequality is always satisfied since it is equal to $(c_{AA} - c_{BB})^2 + 4c_{AB}^2 > 0$. The other two inequalities of (11) can be cast into the form

$$\begin{aligned} &(D1) \qquad\qquad q^2 + U_{AA}N_A + U_{BB}N_B > 0 \\ &(D2)\ \frac{q^4}{4} + \frac{q^2}{2}(U_{AA}N_A + U_{BB}N_B) + N_A N_B(U_{AA}U_{BB} - U_{AB}^2) > 0 \end{aligned} \tag{12}$$

When these inequalities are satisfied the energies of this quadruplet are real. However we need to find the conditions under which the energies of all quadruplets are real. It is easily seen that the polynomial in the inequality (D1) is an increased function of q whereas the inequality in (D2) is an increased function of q in the domain defined by (D1). Thus if they are satisfied for $q = q_{\min} = 1$ then they are satisfied for all $q > q_{\min}$. We conclude that the equilibrium state is dynamically stable if [12, 15]

$$\begin{aligned} &(D1) \qquad\qquad 1 + U_{AA}N_A + U_{BB}N_B > 0 \\ &(D2)\ \frac{1}{4} + \frac{1}{2}(U_{AA}N_A + U_{BB}N_B) + N_A N_B(U_{AA}U_{BB} - U_{AB}^2) > 0 \end{aligned} \tag{13}$$

Notice that these inequalities do not depend on the angular momentum per particle l of the QY state, consequently the QY states are all dynamically stable or all unstable.

2.2. Energetic stability criterion

According to the Bogoliubov theory an equilibrium state is energetically stable if all the excitation energies are real and positive. As $E_d > E_s$, the energies of a quadruplet q are real and positive if $E_s > 2ql$ which implies that

$$c_{AA} + c_{BB} - 8q^2l^2 > 0 \qquad \text{and} \qquad (c_{AA} - 4q^2l^2)(c_{BB} - 4q^2l^2) - c_{AB}^2 > 0. \tag{14}$$

The inequalities (14) can be cast into the form

$$\begin{aligned} &(E1) \qquad q^2 - 4l^2 + U_{AA}N_A + U_{BB}N_B > 0 \\ &(E2)\ \frac{(q^2 - 4l^2)^2}{4} + \frac{q^2 - 4l^2}{2}(U_{AA}N_A + U_{BB}N_B) + N_A N_B(U_{AA}U_{BB} - U_{AB}^2) > 0 \end{aligned} \tag{15}$$

The analysis of these inequalities is analogous to the previous case. Therefore it follows that the equilibrium state is energetically stable if [12, 15]

$$\begin{aligned} &(E1) \qquad 1 - 4l^2 + U_{AA}N_A + U_{BB}N_B > 0 \\ &(E2)\ \frac{(1 - 4l^2)^2}{4} + \frac{1 - 4l^2}{2}(U_{AA}N_A + U_{BB}N_B) + N_A N_B(U_{AA}U_{BB} - U_{AB}^2) > 0 \end{aligned} \tag{16}$$

Different from the dynamical stability conditions, the energetic stability conditions depend on the angular momentum per particle l of the QY state.

To give a preview of the kind of analysis that these inequalities will be subjected, consider the limit of equal interaction strengths discussed in References [12, 15]. Taking $U_{AA} = U_{BB} = U_{AB} = U$ in the inequalities (26), it is easily seen that they are incompatible and consequently there is no energetically stable QY states with $|l| \geqslant 1$ [10, 12, 15]. On the other hand, from inequalities (13), we see that the QY states are dynamically stable if $UN > -\frac{1}{2}$.

3. Stability of a selected QY state as a function of the interaction strengths U_{AB} and U

In this section we discuss the properties of energetic stability of a selected QY state of angular momentum per particle equal to l in a mixture of population imbalance equal to f, in function of the intra and interspecies interaction strengths, u and u_{AB}, where $u \equiv NU$ and $u_{AB} \equiv NU_{AB}$. As already pointed out we will not consider the $l = 0$ QY state since it does not carry a current.

3.1. Dynamical stability

In terms of the system parameters the inequalities (13) take the form

$$\begin{aligned} &(D1) \qquad u + 1 > 0 \\ &(D2) \left[u + \frac{1}{1 - f_{\rm p}^2}\right]^2 - u_{AB}^2 - \left[\frac{f_{\rm p}}{1 - f_{\rm p}^2}\right]^2 > 0 \end{aligned} \tag{17}$$

where $f_{\rm p}$ and $l_{\rm p}$ are the labels of the selected QY state. The next step is to determine the domains of these two inequalities. The inequality (D2) is a second order polynomial in u of roots equal to

$$u_{\pm} = -\frac{1}{1 - f_{\rm p}^2} \pm \sqrt{u_{AB}^2 + a^2(f_{\rm p})} \tag{18}$$

where $a(f_{\rm p})$ is the semi-major axis given by

$$a(f_{\rm p}) = \left|\frac{f_{\rm p}}{1 - f_{\rm p}^2}\right|. \tag{19}$$

The curves $u = u_{\pm}$ are, respectively, the positive and negative branches of the hyperbola that arises when we take the equal sign in (D2). The inequality (D2) can be expressed as $(u - u_+)(u - u_-) > 0$ and since $u_+ > u_-$ it reduces to $u - u_+ > 0$ or $u - u_- < 0$ which splits (17) into two disjoint inequalities: (a) (D1) $u > -1$; (D2) $u - u_+ > 0$ and (b) (D1) $u > -1$; (D2) $u - u_- < 0$. The boundaries of these inequalities are the branches of the hyperbola referred above. Concerning the inequality (D1) we see that its domain is the semi-plane $u > -1$ and the boundary the straight line $u = -1$. To determine the intersection between (D1) and (D2) notice that the inequalities $u > -1$ and $u - u_- < 0$ are incompatible, therefore the negative branch is discarded. On the other hand the domain of $u - u_+ > 0$ is immersed in the domain of $u > -1$. Consequently their intersection is $u - u_+ > 0$ itself. Thus, the dynamical phase boundary is the positive branch $u = u_+$ and the dynamically stable domain is the internal region of this branch, $u > u_+$.

The dynamical phase boundary has a parametric dependence on $f_{\rm p}$. When $f_{\rm p} = 0$, this curve are the straight lines $u = -1 + |u_{AB}|$. For $f_{\rm p} \neq 0$, it is an increased function of $|u_{AB}|$ with a minimum at $u_{AB} = 0$, the value of u at the minimum is equal to $u_{\min} = -\frac{1}{1 + f_{\rm p}}$. At $f_{\rm p} = 1$, u is independent of u_{AB} and equal to the straight line $u = -\frac{1}{2}$.

3.2. Energetic stability

The dynamically stable domain can be split as the union of two disjoint domains: the energetically stable and unstable domains. Besides we can define two limits: one when dynamical stability is equivalent to energetic stability and the other when dynamical stability is equivalent to energetic instability. Having this in mind, to start our discussion we write the inequalities (16) in the form

$$(E1)\qquad u-(4l_{\mathrm{p}}^2-1)>0$$
$$(E2)\left[u-\frac{4l_{\mathrm{p}}^2-1}{1-f_{\mathrm{p}}^2}\right]^2-u_{AB}^2-\left[\frac{(4l_{\mathrm{p}}^2-1)f_{\mathrm{p}}}{1-f_{\mathrm{p}}^2}\right]^2>0 \tag{20}$$

A comparison between inequalities (17) and (20) shows that its analysis goes through identical steps. Therefore we can assert that the energetic phase boundary is the positive branch of the hyperbola that arises when we take the equal sign in (E2), $u=u_+$, the energetically stable domain being the internal region of this branch, $u>u_+$ with

$$u_+=\frac{4l_{\mathrm{p}}^2-1}{1-f_{\mathrm{p}}^2}+\sqrt{u_{AB}^2+a^2(l_{\mathrm{p}},f_{\mathrm{p}})} \tag{21}$$

where $a(l_{\mathrm{p}}, \mathrm{f_p})$ is the semi-major axis given by

$$a(l_{\mathrm{p}},f_{\mathrm{p}})=\left|\frac{(4l_{\mathrm{p}}^2-1)f_{\mathrm{p}}}{1-f_{\mathrm{p}}^2}\right|. \tag{22}$$

In figure 1 we present the stability diagrams in the $u_{AB}\times u$ plane for selected QY states. We select mixtures with equal population ($f_{\mathrm{p}}=0$), a moderate imbalance ($f_{\mathrm{p}}=0.50$) and a rarefied minority component ($f_{\mathrm{p}}=0.98$). For each value of f_{p} we take $l_{\mathrm{p}}=1, 2, 3$. The graphs in each panel display the domain of energetic stability (green area), the domain of energetic or dynamical instability (red area) and the energetic phase boundary (orange curve) which exhibits the stability of selected QY states as a function of the inter and intraspecies interaction strengths. The figure 1 shows that energetic phase boundary depends on the parameters l_{p} and f_{p} through the dependence of the hyperbola parameters on these quantities. Indeed for an equal population mixture $f_{\mathrm{p}}=0$ the energetic phase boundary are the straight lines $u=4l_{\mathrm{p}}^2-1+|u_{AB}|$ which coincides with the asymptotes of the hyperbola at $f_{\mathrm{p}}=0$. For $f_p\neq 0$ the energetic phase boundary is an increased function of $|u_{AB}|$ with minimum at $u_{AB}=0$ and the value of u at the minimum equal to $u_{\min}(l_{\mathrm{p}},f_{\mathrm{p}})=\frac{4l_{\mathrm{p}}^2-1}{1-f_{\mathrm{p}}}$. An inspection of these graphs in the $f_{\mathrm{p}}\to 1$ limit (the third row in figure 1), shows that u is nearly independent of u_{AB} and equal to $u_{\min}(\mathrm{l_p},f_{\mathrm{p}})$ which coincides with $f_p\to 1$ limit of (21). This consideration reveals that $f_p=1$ is an asymptote of the energetic phase boundary in the rarefied limit which implies that in the mixture we always have a seed of the minority component.

To better our understanding of the rarefied limit, notice that in the previous discussion it was shown that in this limit the energetic phase boundary is independent of u_{AB}. This suggests the interpretation that, in this limit, we have a mixture of two non-mutually interacting gases: a majority component and a rarefied minority component. Of course the condition of energetic stability of the mixture reduces to conditions of energetic stability for each component (16) (a) $u>\frac{4l_{\mathrm{p}}^2-1}{2n_A}$ and (b) $u>\frac{4l_{\mathrm{p}}^2-1}{2n_B}$. As $n_A\ll n_B$ we see that (b) is automatically satisfied if (a) is satisfied. Therefore the condition of energetic stability is given by $u>\frac{4l_{\mathrm{p}}^2-1}{1-f_{\mathrm{p}}}$ since $n_A=\frac{1-f_{\mathrm{p}}}{2}$. Note that the boundary $u=\frac{4l_{\mathrm{p}}^2-1}{1-f_{\mathrm{p}}}$ coincides with the $f_{\mathrm{p}}\to 1$ limit of the curve $u=u_+$ in the $u_{AB}\times u$ plane. Therefore we conclude that the stability of the mixture when $f_{\mathrm{p}}\to 1$ is dominated by the stability of the minority component.

The conclusion from all this is that energetically stable two-component QY states does not evolve continuously to states of one component when $f_{\mathrm{p}}\to 1$ since we always have a seed of the minority component [10, 12]. As a reinforcement of the exactness of this conclusion note that at $f_{\mathrm{p}}=1$, the inequalities (16) are incompatible which means that $f_{\mathrm{p}}=1$ is not defined in the energetic stable domain [12].

4. Stability of the QY states at fixed dynamics

In this section, our task is mainly to determine for fixed values of u and u_{AB}, $u=u_{\mathrm{p}}$ and $u_{AB}=u_{AB\mathrm{p}}$, the energetically stable QY states. The intersection of the domains of (13) and (16) in the $l\times f$ plane with the SPS defines, respectively, the domains of dynamical and energetic stabilities in this sector of the $l\times f$ plane. In our

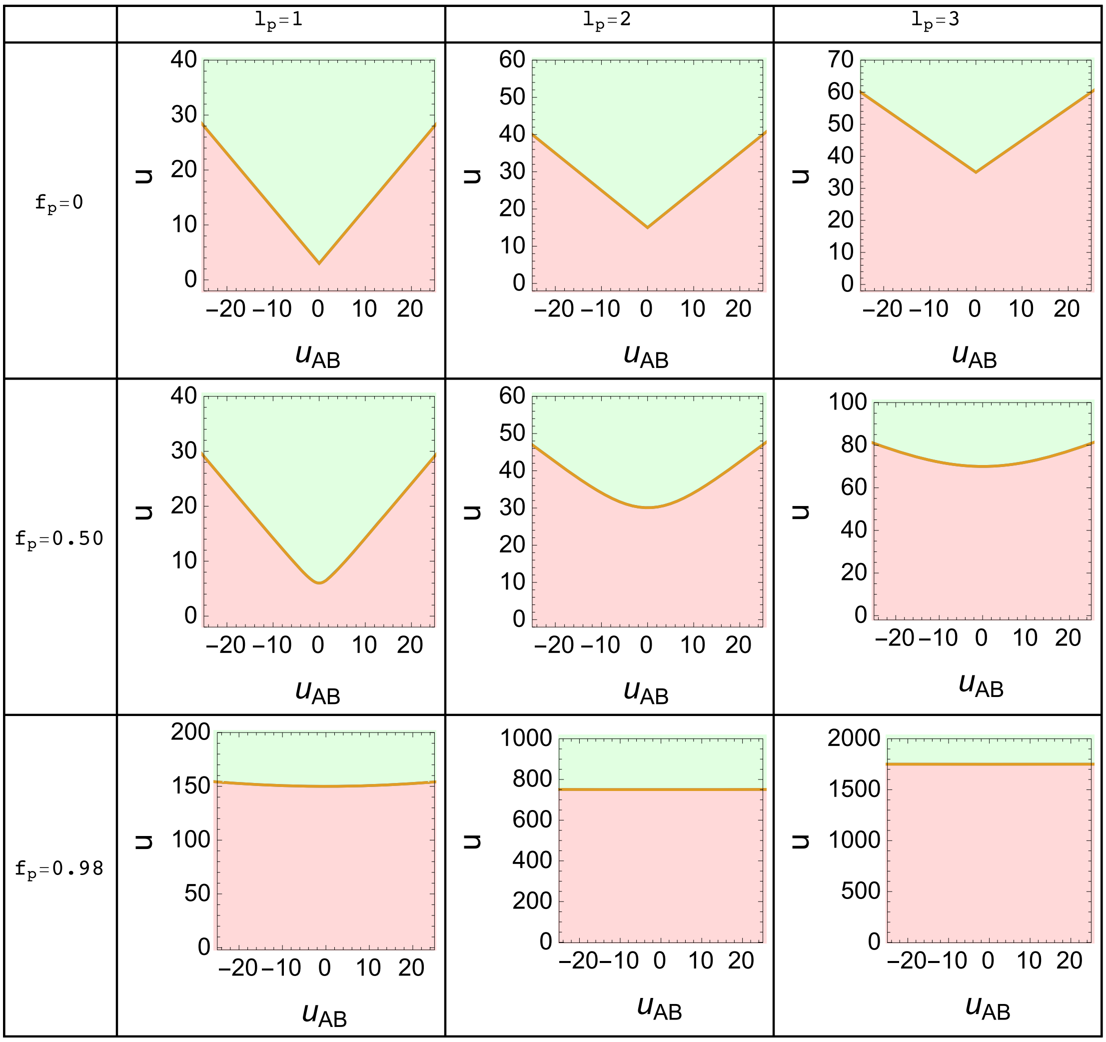

Figure 1. Stability diagrams in the $u_{AB} \times u$ plane for different values of population imbalance $f_{\rm p}$ (rows) and angular momentum per particle $l_{\rm p}$ (columns). Each pair of values $l_{\rm p}$ and $f_{\rm p}$ selects a QY state. The green area is the region in the $u_{AB} \times u$ plane where the QY state is energetically stable, consequently, capable to sustain persistent currents. The red area is the domain of energetic or dynamical instability. The orange curve is the energetic phase boundary which is the positive branch of an hyperbola. These graphs exhibit the dependence of the phase boundary on the parameters $f_{\rm p}$ and $l_{\rm p}$. Indeed when $f_{\rm p}$ starts to increase, for a fixed value of $l_{\rm p}$, the region where the phase boundary is nearly constant increases and, in the $f_{\rm p} \to 1$ limit, u is a constant equal to $u_{\min}(l_{\rm p}, f_{\rm p}) = \frac{4l_{\rm p}^2 - 1}{1 - f_{\rm p}}$. Notice that $u = u_+$, equation (21), grows very fast when $f_{\rm p} \to 1$.

work we consider only $l \neq 0$ states since a $l = 0$ state does not carry a current which restricts our study of stability to the $\mathrm{SPS}_{|l| \geqslant 1}$, the $|l| < 1$ region is not subject of our analysis.

4.1. Dynamical Stability

The inequalities (13) can be written as

$$(D1)\;\; u_{\rm p} > -1$$
$$(D2)\;\; f^2 < p(0) \tag{23}$$

where

$$p(0) = \frac{(u_{\rm p} + 1)^2 - u_{AB{\rm p}}^2}{u_{\rm p}^2 - u_{AB{\rm p}}^2} \quad \text{with} \quad u_{\rm p}^2 - u_{AB{\rm p}}^2 > 0. \tag{24}$$

As $p(0) > 1$, the inequalities (23) are equivalent to

$$(D1)\quad u_{\rm p} > -1$$
$$(D2)\; |f| < \sqrt{p(0)} \tag{25}$$

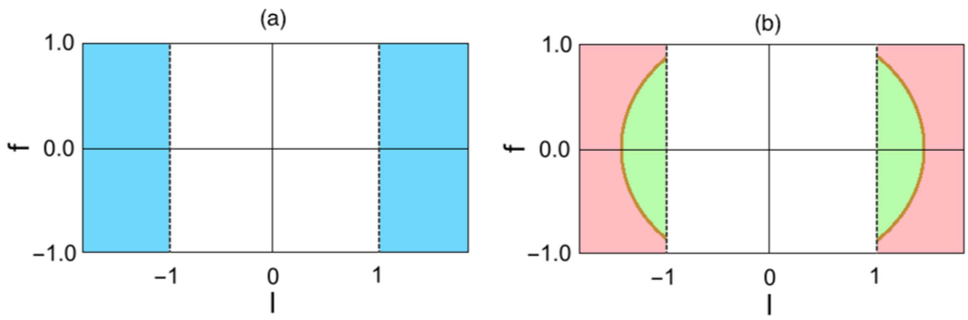

Figure 2. (a) Dynamically stable domain (blue area) in the SPS$_{|l|\geqslant 1}$. (b) Energetically stable domain (green area) in the SPS$_{|l|\geqslant 1}$. The red area is the energetically unstable domain. The figures above are not in scale to facilitate the visualization of our procedure to determine the stability of the mixture. The white area is the domain SPS$_{|l|<1}$.

The dynamically stable domain is given by the intersection of (25) with the SPS$_{|l|\geqslant 1}$. As $p(0) > 1$ we see that the SPS$_{|l|\geqslant 1}$ is immersed in the domain of (D2). Therefore their intersection is the SPS$_{|l|\geqslant 1}$ itself, consequently all QY states with $|l| \geqslant 1$ are dynamically stable. The dynamically stable domain is shown in figure 2(a).

4.2. Energetic stability

We write the inequalities (16) in the form

$$(E1)\ l^2 < \frac{u_p + 1}{4}$$
$$(E2)\quad f^2 < p(l) \tag{26}$$

with

$$p(l) = \frac{(4l^2 - u_p - 1)^2 - u_{ABp}^2}{u_p^2 - u_{ABp}^2}. \tag{27}$$

The first step is to determine the domain of (26) in the $l \times f$ plane. f^2 is a positive quantity therefore $p(l)$ cannot be negative. Since $p(l)$ is a quadratic polynomial in l^2, its four roots are $l = \pm l_+$ and $l = \pm l_-$ where

$$l_\pm = \sqrt{\frac{1}{4}(u_p + 1 \pm |u_{ABp}|)}. \tag{28}$$

Once we know the roots, the signs of $p(l)$ are easily determined. We find that $p(l) > 0$ in the intervals $|l| < l_-$ and $|l| > l_+$ with a gap in the interval $l_- < |l| < l_+$. Thus (26) splits into two disjoint inequalities: (a) (E1) $l^2 < \frac{u_p+1}{4}$, (E2) $f^2 < p(l)$, (E3) $|l| > l_+$ and (b) (E1) $l^2 < \frac{u_p+1}{4}$, (E2) $f^2 < p(l)$, (E3) $|l| < l_-$. The inequalities (E3) are respectively equivalent to $l^2 - \frac{u_p+1}{4} > \frac{|u_{ABp}|}{4}$ and $l^2 - \frac{u_p+1}{4} < -\frac{|u_{ABp}|}{4}$ which show that (E1) and (E3) in (a) are incompatible inequalities and it is discarded. On the other hand, in (b) the domain $|l| < l_-$ is immersed in the domain of (E1), therefore, the intersection is $|l| < l_-$ itself which reduces (26) to

$$(E1)\quad |l| < l_-$$
$$(E2)\ f^2 < p(l) \tag{29}$$

As $p(l) > 0$ in the interval $|l| < l_-$, (E2) is equivalent to $|f| < \sqrt{p(l)}$. The boundaries of this inequality are the curves $f = \sqrt{p(l)}$ and $f = -\sqrt{p(l)}$ where the last one is the reflection with respect to the l–axis of $f = \sqrt{p(l)}$. In the interval $|l| < l_-$, $f = \sqrt{p(l)}$ is a decreased function of $|l|$ vanishing at the extrema, $f(\pm l_-) = 0$, with a maximum at $l = 0$. The value at the maximum equal to $\sqrt{p(0)}$. Introducing the quantities

$$l_{crit} \equiv l_- \qquad \text{and} \qquad f_{crit}(l) \equiv \sqrt{p(l)} \tag{30}$$

the inequalities (29) can be written as

$$(E1)\quad |l| < l_{crit}$$
$$(E2)\ |f| < f_{crit}(l). \tag{31}$$

(31) are the inequalities whose intersection with SPS$_{|l|\geqslant 1}$ defines the domain of energetic stability. Its boundary in the $l \times f$ plane is the closed curve

$$|f| = f_{crit}(l) \qquad \text{if } |l| < l_{crit} \text{ and } 0 \leqslant |f| < \sqrt{p(0)} \tag{32}$$

whose domain is the internal region of this curve.

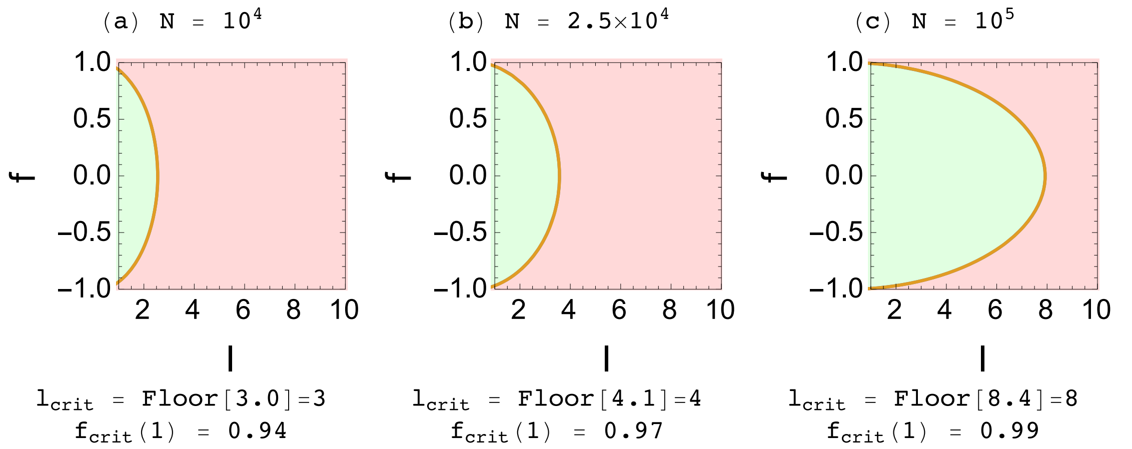

Figure 3. Stability diagrams in the $\mathrm{SPS}_{|l|\geqslant 1}$ for $U_{\mathrm{p}} = 0.05$, $U_{AB\mathrm{p}} = 0.95U_{\mathrm{p}}$ and for different values of N. We do not present the boundaries for $l < -1$ since they are equal to the reflection with respect to the f axis of the boundaries for $l > 1$. The energetically stable QY states are those whose corresponding points in $\mathrm{SPS}_{|l|\geqslant 1}$ are localized in the energetically stable domain (green area). An inspection of the graphs for fixed values of N shows that $f_{\mathrm{crit}}(l)$ is a decreased function of l whose largest value occur at $l = 1$ vanishing at $l = l_{\mathrm{crit}}$. In the graphs we see that $f_{\mathrm{crit}}(1)$ is weakly dependent of N and nearly equal to one. Concerning l_{crit}, the graphs show that it is a increased function of N. As consequence, there is a value of N such that $l_{\mathrm{crit}} = 1$ and below this value of N all energetically stable QY states disappear.

The energetically stable domain is defined by the intersection of (31) with the $\mathrm{SPS}_{|l|\geqslant 1}$. Taking into account that $\sqrt{p(1/2)} = 1$ and that $\sqrt{p(l)}$ is a decreased function of $|l|$, we see that one is an upper bound of $f_{\mathrm{crit}}(l)$, $|l| \geqslant 1$. Thus, the energetic phase boundary is the curve

$$|f| = f_{\mathrm{crit}}(l) \qquad \text{if } 1 \leqslant |l| < l_{\mathrm{crit}} \text{ and } 0 \leqslant |f| < f_{\mathrm{crit}}(1) \tag{33}$$

with domain being the internal region of this curve. The stability diagram in the $\mathrm{SPS}_{|l|\geqslant 1}$ is shown in figure 2(b). The energetically stable QY states are those whose corresponding points in the $\mathrm{SPS}_{|l|\geqslant 1}$ are in the energetic stable domain, figure 2(b).

To display the stability diagram in the $\mathrm{SPS}_{|l|\geqslant 1}$ we need to specify the values of u_{p} and $u_{AB\mathrm{p}}$. The strength $u_{ss'} = NU_{ss'}$ is proportional to N which is the total number of atoms in the mixture and $U_{ss'}$ is the interaction strength between atoms of species s and s', in units of $\frac{\hbar^2}{2MR^2}$. Thus even when the states of the mixture are unchanged we can change the values of $u_{ss'}$ by varying N. Taking roughly into account that in the experiments the interaction strengths are nearly equal, we fix the values of the intra and interspecies interaction strengths equal to $U_{\mathrm{p}} = 0.05$ and $U_{AB\mathrm{p}} = 0.95U_{\mathrm{p}}$, in units of $\frac{\hbar^2}{2MR^2}$, to calculate the stability diagram for different values of N, see figure 3. An inspection of these graphs shows that, for fixed values of N, $f_{\mathrm{crit}}(l)$ is a decreased function of $|l|$ where the size of the window of stability, $0 \leqslant |f| < f_{\mathrm{crit}}(l)$, is largest for $|l| = 1$ diminishing when $|l|$ increases and vanishing at $|l| = l_{\mathrm{crit}}$. These graphs also show that $f_{\mathrm{crit}}(1)$ is weakly dependent on N and nearly equal to one. Concerning l_{crit}, these graphs reveals that l_{crit} is a increased function of N which is equal to Floor[8.4] = 8 at $N = 10^5$, panel (c),[2]. If N decreases, l_{crit} also decreases and at $N = 10^4$, which is one order of magnitude smaller than the $N = 10^5$ mixture, l_{crit} is equal to Floor[3.0] = 3. This means that at this value of N, only QY states with $|l| = 1$ can be energetically stable. If we further lower the value of N, we can reach a point where all the energetically stable QY states disappear which happens when $N < 1.2 \times 10^3$, two orders of magnitude smaller than the $N = 10^5$ mixture.

The Reference [5] is the first experimental work to study persistent currents in a two-component Bose gas consisting of atoms of ^{87}Rb in two different hyperfine states, $F = 1, m_F = 1$ and $F = 1, m_F = 0$, confined in a tight toroidal trap. In this work they select QY states with $l = 3$ and measure the stability of these states as a function of the population imbalance. They found that there is an $f_{\mathrm{crit}}(l)$ such that only QY states with $|f| > f_{\mathrm{crit}}(l)$ are stable. In other words, $f_{\mathrm{crit}}(l)$ is a lower bound of the possible values of $|f|$. They conclude saying that large $|f|$ is fundamentally stable and small $|f|$ fundamentally unstable. In our work we found just the opposite, that is, there is $f_{\mathrm{crit}}(l)$ such that only QY states with $|f| < f_{\mathrm{crit}}(l)$ are energetically stable. In other words, $f_{\mathrm{crit}}(l)$ is an upper bound of the possible values of $|f|$. We conclude that small $|f|$ is fundamentally energetically stable and large $|f|$ fundamentally energetically unstable, just the opposite of the conclusion of Reference [5]. Notice that our disagreement is of qualitative nature which is worse than quantitative one. We do not know how to explain this disagreement. However we differ in what characterize the onset of the instability. In our work we

[2] Floor[x] is a function of a continuous variable x defined by Floor[x] $= n$ if $n \leqslant x < n + 1$ with n being the largest integer less or equal to x. In our case, the prescription to pull out integer values from continuous values of l is $[l] = \mathrm{Floor}\left[l + \frac{1}{2}\right]$.

are probing the stability of small oscillations of the mixture in the neighborhood of an equilibrium state. In other words, we are probing the stability of its normal modes. In this case an unstable normal mode is responsible for the onset of the instability. In the experimental determination of the stability we should follow the time evolution of a state constructed by the action of a weak perturbation of the equilibrium state. On the other hand, Reference [5] characterizes the onset of instability by the time of occurrence of the first phase slip, a criterion whose physical content is completely different from ours. An experimental work that probes the stability of small oscillations would settle these matters.

5. Conclusions

Our analysis of the stability diagram in the $U_{AB} \times U$ plane reveals how the inter and intraspecies interaction strengths affect the stability of a selected QY state specified by f_{p} and l_{p}. In particular it shows that the dynamical and energetic phase boundaries are positive branches of hyperbolas. In the dynamical case the hyperbola has center at $\left(0, -\frac{1}{1-f_{\mathrm{p}}^2}\right)$ and semi-major axes $a = \left|\frac{f_{\mathrm{p}}}{1-f_{\mathrm{p}}^2}\right|$ whereas in the energetic case has center at $\left(0, \frac{4l_{\mathrm{p}}^2-1}{1-f_{\mathrm{p}}^2}\right)$ and semi-major axes $a = \left|\frac{(4l_{\mathrm{p}}^2-1)f_{\mathrm{p}}}{1-f_{\mathrm{p}}^2}\right|$. The experimental confirmation of these predictions are in principle feasible through the mechanism of Feshbach resonance. The stability in the rarefied limit ($f_{\mathrm{p}} \to 1$) was also examined. Our study revealed that, in this limit, the energetic phase boundary is independent of interspecies interaction strength suggesting that we have a mixture of two non-mutually interacting gases: majority and rarefied minority components. The conclusion from this analysis is that when $f_{\mathrm{p}} \to 1$, energetically stable two-component QY states does not evolve continuously to states of one component once there is always a seed of the minority component. Besides we have shown that the stability is dominated by the minority component.

Equally well, for a fixed dynamics, the stability diagram in the $\mathrm{SPS}_{|l|\geqslant 1}$ determines the stability of the QY states as a function of l and f. As in the previous case we found analytic expressions for the dynamical and energetic phase boundaries. An inspection of the stability diagram in the $\mathrm{SPS}_{|l|\geqslant 1}$ reveals that (a) All QY states are dynamically stable; (b) Exist a l_{crit} in the sense that there is none energetically stable QY state with $|l| > l_{\mathrm{crit}}$. In other words, l_{crit} is an upper bound of the possible values of $|l|$ carried by an energetically stable QY state; (c) l_{crit} is an increased function of $u_{\mathrm{p}} - |u_{AB\mathrm{p}}|$, therefore it decreases when $u_{\mathrm{p}} - |u_{AB\mathrm{p}}|$ decreases, reaching the value $l_{\mathrm{crit}} = 1$. This happens when $u_{\mathrm{p}} - |u_{AB\mathrm{p}}| = 3$. Below this value all the energetically stable QY states disappear; (d) There is a $f_{\mathrm{crit}}(l)$ in the sense that, for a given l, only QY states with f in the interval $0 \leqslant |f| < f_{\mathrm{crit}}(l)$ are energetically stable. As $f_{\mathrm{crit}}(l)$ is a decreased function of $|l|$, its largest value occur at $|l| = 1$, diminishing when $|l|$ increases and vanishing when $|l| = l_{\mathrm{crit}}$.

The experimental confirmation of these predictions requires the measurements of the stability of all QY states which can be done. An example of this kind of measurement was performed in the experiment described in Reference [5] where they select QY states with $l = 3$ and measure the stability as a function of the population imbalance. In other words they go along the straight line $l = 3$ in the $\mathrm{SPS}_{|l|\geqslant 1}$. The conflict between our work and Reference [5] about the nature of the critical value of population imbalance can be clarified by an experiment designed to determine the stability of the normal modes of the mixture.

In summary, the analysis of the properties of the stability diagrams in these two planes shows its straightforward physical interpretation and experimental relevance which emphasizes the importance of a choice of planes guided by physical considerations.

ORCID iDs

E T D Matsushita https://orcid.org/0000-0003-3154-7693

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