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MECHANICS

A Gauge-Invariant Lagrangian Determined by the *n*-Point Probability Density Function of a Vorticity Field of Wave Optical Turbulence

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Abstract—The geometric methods for Yang–Mills fields of gauge transformations are applied to find an invariant Lagrangian in the fiber bundle of the configuration of $2d$ space X of a turbulent flow determined by the n-point probability density function (PDF) f_n. The two-dimensional wave optical turbulence is considered in the case of an inverse cascade of turbulence energy transfer under external impacts in the form of white Gaussian noise and large-scale friction. The n-point PDF of the vorticity field satisfies the f_n-equation from the Lundgren–Monin–Novikov hierarchy, and the conditions of equation invariance under external action are found. A Lagrangian, which is invariant relative to the $H \subset G$ subgroup (a group of the gauge transformations in the fiber bundle of the space X), and the conserved currents are constructed.

Keywords: optical turbulence, gauge transformation, Lundgren–Monin–Novikov equations, invariant Lagrangian

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In this paper, we report the results of studying conformal invariance of the n-point statistics of zero vorticity lines. In [1], the methodology presented in [3–5] for calculating symmetry transformations of one-point statistics was used in proving the conformal invariance of n-point ($n > 1$) statistics of isolines $\mathbf{x}(l)$ in the case of two-dimensional hydrodynamic turbulence in the absence of external impacts and at zero viscosity. In [2], the results of [1] were applied to the two-dimensional wave optical turbulence, which is studied within the hydrodynamic approximation of the nonlinear Schrödinger equation (NSE) for the weight velocity field $\mathbf{u}$ [6] (optical phase gradient of the wave function). The result obtained is indicative of conformal invariance of the n-point statistics (probability measure) of the zero vorticity line $\mathbf{x}(l,t)$ or the contour of a cluster of optical vortices. The boundary of vortex clusters was observed in [7]; beyond the vortex kernels, the velocity field is $\mathbf{u} \approx \mathbf{v}$, where variation in the density (optical intensity of the wave function) from the background value $\rho = 1$ is insignificant. Experiments for two-dimensional quantum fluids (polaritons) demonstrate turbulent states and the occurrence of the inverse cascade in dissipative quantum fluids, along with the formation of vortex clusters. It is established that one can determine the contributions from the compressible and incompressible components of the velocity field $\mathbf{u} = \sqrt{\rho/2}\mathbf{v}$ with the weight function $\sqrt{\rho}$ (ρ and $\mathbf{v}$ are the density and velocity of the quantum fluid, respectively) to the kinetic energy of turbulence, which is necessary for the formation of vortex clusters due to the possibility of direct measurement of the quantum fluid phase based on the analogy between quantum fluids and optical systems. The results are applied to toroidal optical vortices. Experimental data on the formation of toroidal structures of rays in optics were presented in [8]. The justification was performed using the $3D$-linear Schrödinger equation (parabolic approximation of the NSE) with anomalous dispersion of the group velocity of the wave packet Ψ, which is invariant at conformal transformations of the complex line C, where two phase elements of Ψ are located. The equation is used within certain approximations, neglecting the nonlinear effects of propagation of optical waves and interaction with the random-wave background. The latter results in the fact that the justification of such structures should be performed within the statistical theory

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with analysis of the symmetry of statistical distributions of the vorticity field. Expansion of the symmetry to conformal invariance is the program proposed by A.M. Polyakov [9] for the $2d$-statistical theory of hydrodynamic turbulence. The purpose of this study is to find the invariance conditions for the f_n-equation from the Lundgren–Monin–Novikov (LMN) hierarchy under external action in the form of white Gaussian noise and large-scale friction and to construct the invariant Lagrangian, energy–momentum tensor, and infinite number of conserved currents in fiber bundles of the configuration $2d$ space X, which makes it possible to apply methods of the conformal field theory [10] for the statistical theory of turbulence.

1. BASIC CONSTRUCTIONS

1.1. Statistical Model

Statistical samples are considered in the manifold X (possibly nonconnected and nondifferentiable). Statistical models are a set of probability measures $\{\mu_{\mathbf{x}} | \mathbf{x} \in X\}$ (parameterized by variable $\mathbf{x} \in X$) or probability density functions specified in some sample space $\mathcal{M}$ of the observed data ω. The statistical model of a physical problem consists in the choice of the probability measure. Let us consider an n-point sample $(\mathbf{x}_{(1)},\ldots,\mathbf{x}_{(n)}) \in X$, which corresponds to the observed data $(\omega_{(1)},\ldots,\omega_{(n)})$ and X as the configuration space of a $2d$ turbulent flow so that $\omega_{(i)}$ are values of the vorticity component $\Omega(\mathbf{x}_{(i)},t)$. The space of states of point $\mathbf{x}_{(i)}$ of the turbulent flow is a one-dimensional fiber bundle $\mathcal{M} \simeq R$ over X. The space of states of the n-point sample $(\mathbf{x}_{(1)},\ldots,\mathbf{x}_{(n)})$ is the direct product $\mathcal{M}^n = \mathcal{M}\times\ldots\times\mathcal{M} \simeq R^n$. Let us consider the set of probability measures $\{\mu_{\mathbf{x}} | \mathbf{x} \in X\}$ (parameterized x) at $\mathcal{M}$ and the standard Lebesgue measure ν defined in $\mathcal{M}$. It is assumed that the set of measures $\{\mu_{\mathbf{x}}\}$ is absolutely continuous relative to the measure ν. Then the map $\mu_{\mathbf{x}} \to (d\mu_{\mathbf{x}}/d\nu)^{1/2}$, where $d\mu_{\mathbf{x}}/d\nu$ is the Radon–Nikodim derivative, determines the embedding of the set $\{\mu_{\mathbf{x}} | \mathbf{x} \in X\}$ into unit sphere S of the Hilbert space $L^2(\mathcal{M},\nu)$. The derivative $d\mu_{\mathbf{x}}/d\nu = f$ is a probability density function (PDF) in S, the form of which will be specified below for the model of wave optical turbulence.

1.2. Gauge Transformations

In accordance with the gauge transformation theory, we choose a group of transformations (Lie group), which acts on the manifold X, and consider conformal maps of the $2d$ manifold X, which are an infinite-dimensional Lie pseudogroup defined on the complex line C, the action of which can be lifted to fiber bundle $\mathcal{P} = P_{\mathbf{x}_{(1)}}\times\ldots\times P_{\mathbf{x}_{(n)}}$ of the manifold X with the base $\mathbf{x}_{(j)}$, where the fiber $P_{\mathbf{x}_{(j)}}$ is the group of transformations G_j. In differential geometry, such fiber bundles are referred to as principle.

1.3. Wave Optical Turbulence

The statistical description of the set of optical vortices is based on the analogy between optical and hydrodynamic fields [12]. Within the hydrodynamic approximation of the NSE determined by the Euler equation for an ideal incompressible fluid, the equations for the multipoint PDFs f_n, $n = 1,\ldots,\infty$, the vorticity fields $\mathbf{w}$ are determined by the hierarchy of the LMN equations. We are interested in the properties of statistics that are independent of the properties of an external random force (i.e., invariance, when the turbulence is implemented on scales that are larger than the correlation length of the external force or in inverse cascades). To simulate nonlinear propagation of optical waves in terms of the scalar complex wave function $\Psi(X,Y,T)$ for the envelopes, we apply the NSE, which in dimensionless variables x, y, t, and ψ has the form

$$i\psi_t + \Delta\psi + \psi - |\psi|^2\psi = 0. \quad (1)$$

The Madelung transformation [11]

$$\psi = \sqrt{\rho}e^{i\phi}, \quad |\psi|^2 = \rho,$$

where $|\psi|^2$ is the optical intensity and ϕ is the wave function phase, sets the correspondence between the optical and hydrodynamic fields, and ρ and $\mathbf{v}$ satisfy the Euler equations for an inviscid polytropic gas with the adiabatic index $\gamma = 2$. Thus, the optical intensity is the density ρ, the optical phase gradient $\nabla\phi$ is the velocity $\mathbf{v}$, the nonlinear perturbation of the refractive index corresponds to pressure p, and the distance passed by the optical wave corresponds to time t. At the points $\mathbf{x_i} = (x_i, y_i) \in R^2$ where $\psi = 0$, the phase ϕ is uncertain and the vorticity is the distribution of the delta function $\delta(\mathbf{x} - \mathbf{x_i})$. The transition to the velocity $\mathbf{u} = \sqrt{\rho}\mathbf{v}$ (as in [6]), with approximation of $\mathbf{v}$ near $\mathbf{x_i}$ by the Pitaevskii vortex solution [13] with weight $\sqrt{\rho}$, allows one to pass to the localized and rapidly decaying vorticity field $\mathbf{w}$, which is still singular at $r = 0$ $(r = |\mathbf{x} - \mathbf{x_i})|)$ but is not the distribution of the delta function any longer. The Hamiltonian of the NSE

$$H = \int[|\nabla\psi|^2 + 1/2(|\nabla\psi|^2 - 1)^2]d\mathbf{x} \quad (2)$$

in hydrodynamic variables $\mathbf{u}$ and ρ can be written as

$$H = H_K + H_0, \quad H_K = \frac{1}{2}\int u^2 dx,$$
$$H_0 = \frac{1}{2}\int[(\rho - 1)^2 + 2|\nabla\sqrt{\rho}|^2]d\mathbf{x}, \quad u = |\mathbf{u}|, \tag{3}$$

where H_K coincides with the Hamiltonian of an ideal incompressible fluid. As was shown in [6], the Hamiltonian H_K is dominant in the expansion of H on motion scales of $\sim\xi$ of the vortex core radius, where pressure p changes significantly and $\rho \approx 1$ (background density value) at $r \gg \xi$. The field divergence $\mathbf{u}$ (i.e., $\gamma(x) = \nabla \cdot \mathbf{u}$) on these motion scales is $\gamma(\mathbf{x}) \ll 1$ (see [6]). Thus, the hydrodynamic approximation of the NSE on motion scales of $\sim\xi$ is determined by the Euler equation for an ideal incompressible fluid.

1.4. Statistical Description of the Optical Vorticity Field

Equation f_n from the LMN hierarchy is considered under external forcing (white Gaussian noise and large-scale Ekman friction), which leads to statistical stationarity of the PDF. The following designations are used: $f_n(\mathbf{x}_{(1)}, \omega_{(1)}, \ldots, \mathbf{x}_{(n)}, \omega_{(n)})$ is the n-point PDF ($n = 1, \ldots$) and $\omega_{(i)}$ ($i = 1, \ldots, n$) is the value of the vorticity component $\Omega(\mathbf{x}_{(i)}, t)(\equiv \Omega_{(i)})$ at the point $\mathbf{x}_{(i)}$. A superscript indicates the vector component. Using $\mathbf{x}_{(i)}$, we introduce complex variables $z_{(i)} = x^1_{(i)} + ix^2_{(i)}$, or $z_{(i)} = x_{(i)} + iy_{(i)}$. An arbitrary f_n-equation from the LMN hierarchy in complex variables takes the form [14]

$$\sum_{j=1}^{n} \mathrm{Re}(\nabla_{z_{(j)}} \cdot [\langle \mathcal{U}(z_{(j)}, \bar{z}_{(j)}) | \{\omega_{(l)}, z_{(l)}, \bar{z}_{(l)}\}\rangle]) f_n = \mathcal{F}, \tag{4}$$

$$\mathcal{F} = \beta \frac{\partial}{\partial \omega_{(n)}}(\omega_{(n)} f_n) - \frac{1}{2}\sum_{j=1}^{n} Q(x_{(n)} - x_{(j)}) \frac{\partial^2}{\partial \omega^2_{(j)}} f_n, \tag{5}$$
$$\beta = \text{const.}$$

The first term is the friction damping (Ekman friction), through which the interaction energy shifts to larger scales of the inverse cascade. The second term is the excitation of the system by white Gaussian noise with a short correlation length: $Q(x_{(n)} - x_{(j)})$ is the external impact amplitude, $j = 1, \ldots, n$ and $n = 1, \ldots, \infty$. Re indicates the real part of a complex number. The velocity vector is

$$\langle \mathcal{U}(z_{(j)}, \bar{z}_{(j)}, t) | \{\omega_{(l)}, z_{(l)}, \bar{z}_{(l)}\}\rangle$$
$$= \langle u(\mathbf{x}_{(j)}, t) | \omega_{(l)}, \mathbf{x}_{(l)}\rangle + i\langle v(\mathbf{x}_{(j)}, t) | \omega_{(l)}, \mathbf{x}_{(l)}\rangle, \tag{6}$$

$$\langle u(\mathbf{x}_{(j)}, t) | \{\omega_{(l)}, \mathbf{x}_{(j)}\}\rangle$$
$$= \int d\mathbf{x}_{(n+1)} d\omega_{(n+1)} \omega_{(n+1)} \alpha^1(\mathbf{x}_{(l)} - \mathbf{x}_{(n+1)}) \tag{7}$$
$$\times \frac{f_{n+1}(\mathbf{x}_{(n+1)}, \omega_{(n+1)}, \{\mathbf{x}_{(l)}, \omega_{(l)}\}, t)}{f_n(\{\mathbf{x}_{(l)}, \omega_{(l)}\}, t)},$$

$$\langle v(\mathbf{x}_{(j)}, t) | \{\omega_{(l)}, \mathbf{x}_{(l)}\}\rangle$$
$$= \int d\mathbf{x}_{(n+1)} d\omega_{(n+1)} \omega_{(n+1)} \alpha^2(\mathbf{x}_{(j)} - \mathbf{x}_{(n+1)}) \tag{8}$$
$$\times \frac{f_{n+1}(\mathbf{x}_{(n+1)}, \omega_{(n+1)}, \{\mathbf{x}_{(l)}, \omega_{(l)}\}, t)}{f_n(\{\mathbf{x}_{(l)}, \omega_{(l)}\}, t)}.$$

The PDF class is determined by the conditions of normalization, coincidence, and separation of PDFs [14] on the corresponding scales.

2. CONFORMAL GAUGE TRANSFORMATIONS OF THE VORTICITY FIELD STATISTICS

2.1. Infinitesimal Operator

The Gauge transformation is function $g(\mathbf{x})$ taking values in the G group acting in the space of fiber bundles over X. The action of the Lie pseudogroup of the conformal transformations of X can be lifted to fiber bundle $\mathcal{P}$ of the manifold X with the base $\mathbf{x}_{(j)}$. Fiber $P_{\mathbf{x}_{(j)}}$ is the Lie transformation group G_j (Lie algebra $\mathfrak{s}_j$) determined by the infinitesimal operator $S_{(j)}$, $j = 1, \ldots, n$ [1]:

$$S_{(j)} = \xi^1 \frac{\partial}{\partial x^1_{(1)}} + \xi^2 \frac{\partial}{\partial x^2_{(1)}} + \xi^3 \frac{\partial}{\partial \omega_{(1)}} + \ldots$$
$$+ \xi^{3n-2} \frac{\partial}{\partial x^1_{(n)}} + \xi^{3n-1} \frac{\partial}{\partial x^2_{(n)}} + \xi^{3n} \frac{\partial}{\partial \omega_{(n)}} + \eta^1_{(n)} \frac{\partial}{\partial f_n}$$
$$+ \xi^{3n+1} \frac{\partial}{\partial x^1_{(n+1)}} + \xi^{3n+2} \frac{\partial}{\partial x^2_{(n+1)}}$$
$$+ \xi^{3n+3} \frac{\partial}{\partial \omega_{(n+1)}} + \eta^2_{(n)} \frac{\partial}{\partial f_{n+1}}, \tag{9}$$

j indexes ξ^{3n+1}, ξ^{3n+2}, and ξ^{3n+3}. The coordinates of the infinitesimal operator are determined by the formulas

$$\xi^1 = c^{11}(\mathbf{x}_{(1)}) x^1_{(1)} + c^{12}(\mathbf{x}_{(1)}) x^2_{(1)} + d^1(\mathbf{x}_{(1)}), \tag{10}$$

$$\xi^2 = c^{21}(\mathbf{x}_{(1)}) x^1_{(1)} + c^{22}(\mathbf{x}_{(1)}) x^2_{(1)} + d^2(\mathbf{x}_{(1)}), \tag{11}$$

$$\xi^3 = [6c^{11}(\mathbf{x}_{(1)})]\omega_{(1)}, \tag{12}$$

$$\ldots \quad \ldots \quad \ldots$$

$$\xi^{3n-2} = c^{11}(\mathbf{x}_{(n)}) x^1_{(n)} + c^{12}(\mathbf{x}_{(n)}) x^2_{(n)} + d^1(\mathbf{x}_{(n)}), \tag{13}$$

$$\xi^{3n-1} = c^{21}(\mathbf{x}_{(n)}) x^1_{(n)} + c^{22}(\mathbf{x}_{(n)}) x^2_{(n)} + d^2(\mathbf{x}_{(n)}), \tag{14}$$

$$\xi^{3n} = [6c^{11}(\mathbf{x}_{(n)})]\omega_{(n)}, \tag{15}$$

$$\xi^{3n+1} = c^{11}(\mathbf{x}_{(j)}) x^1_{(n+1)} + c^{12}(\mathbf{x}_{(j)}) x^2_{(n+1)} + d^1(\mathbf{x}_{(j)}), \tag{16}$$

$$\xi^{3n+2} = c^{21}(\mathbf{x}_{(j)}) x^1_{(n+1)} + c^{22}(\mathbf{x}_{(j)}) x^2_{(n+1)} + d^2(\mathbf{x}_{(j)}), \tag{17}$$

$$\xi^{3n+3} = [2c^{11}(\mathbf{x}_{(j)})]\omega_{(n+1)}, \tag{18}$$

$k = 1, \ldots, n$; c^{ls} satisfies the relations $c^{11} = c^{22}$ and $c^{12} = -c^{21}$; c^{11}, c^{12}, and each pair ξ^1, ξ^2, ..., ξ^{3n-2}, ξ^{3n-1}

are arbitrary conjugate harmonic functions. Coordinates $\eta^1_{(n)}$ and $\eta^2_{(n)}$ can be written as

$$\eta^1_{(n)} = a^{00}_{(n)}(t, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(n)}) f_n, \tag{19}$$

$$a^{00}_{(n)} = -\left(\frac{\partial\xi^1}{\partial x^1_{(1)}} + \frac{\partial\xi^2}{\partial x^2_{(1)}} + \ldots + \frac{\partial\xi^{3n-2}}{\partial x^1_{(n)}} + \frac{\partial\xi^{3n-1}}{\partial x^2_{(n)}}\right), \tag{20}$$

$$\eta^2_{(n)} = a^{00}_{(n+1)}(t, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(n+1)}) f_{(n+1)}, \tag{21}$$

$$\begin{aligned} a^{00}_{(n+1)} = -\Bigg(&\frac{\partial\xi^1}{\partial x^1_{(1)}} + \frac{\partial\xi^2}{\partial x^2_{(1)}} + \frac{\partial\xi^4}{\partial x^1_{(2)}} \\ &+ \frac{\partial\xi^5}{\partial x^2_{(2)}} + \ldots + \frac{\partial\xi^{3n+1}}{\partial x^1_{(n+1)}} + \frac{\partial\xi^{3n+2}}{\partial x^2_{(n+1)}}\Bigg). \end{aligned} \tag{22}$$

The infinitesimal operator $S_{(j)}$ generates the Lie (pseudo-)group G_j (G_j and G_k $(j \neq k)$ are isomorphic), which invariantly transforms characteristic Eq. (4) (see [1])

$$\frac{d}{ds}\mathbf{X}_{n(j)}(s) = \langle \mathbf{u}(\mathbf{x}_{(j)}, t) | \omega_{(l)}, \mathbf{x}_{(l)} \rangle\Big|_{\{\omega_{(l)}, \mathbf{x}_{(l)}\} = \{\Omega_{(l)}(s), \mathbf{X}_{n(l)}(s)\}} \tag{23}$$

with zero vorticity $\Omega_{(l)}(s) = 0$, $l = 1, \ldots, n$, and Eq. (4) along $\mathbf{X}_{n(j)}(s)$. We define $G = G_1 \times \ldots \times G_n$ as the direct product of Lie groups G_j; G is a Lie (pseudo-)group again. Under the action of G transformations, the f_n-equation remains invariant only along $\mathbf{X}_{n(j)}(s)$, $j = 1, \ldots, n$, with $\Omega_{(l)}(s) = 0$; G retains the PDF class [2]. Furthermore, the left-hand side of (4) along $\mathbf{X}_{n(j)}(s)$ is transformed as

$$\begin{aligned} &\sum_{j=1}^{n} \mathrm{Re}(\nabla_{z^*_{(j)}} \cdot [\langle U^*(z^*_{(j)}, \bar{z}^*_{(j)}) | \{\omega^*_{(l)}, z^*_{(l)}, \bar{z}^*_{(l)}\}\rangle]) f^*_n \\ &= \gamma \sum_{j=1}^{n} \mathrm{Re}(\nabla_{z_{(j)}} \cdot [\langle U(z_{(j)}, \bar{z}_{(j)}) | \{\omega_{(l)}, z_{(l)}, \bar{z}_{(l)}\}\rangle]) f_n, \end{aligned} \tag{24}$$

$$\gamma = \prod_{j=1}^{n} |F_{z_{(j)}}|^{-2}, \tag{25}$$

accordingly, the first term on the right-hand side of (4)

$$\beta \frac{\partial}{\partial \omega^*_{(n)}} (\omega^*_{(n)} f^*_n) = \gamma\beta \frac{\partial}{\partial \omega_{(n)}} (\omega_{(n)} f_n). \tag{26}$$

Let us consider the second term in F in the transformed variables:

$$\begin{aligned} &\frac{1}{2}\sum_{j=1}^{n} Q^*(x^*_{(n)} - x^*_{(j)}) \frac{\partial^2}{\partial \omega^{*2}_{(j)}} f^*_n \\ &= \frac{1}{2}\sum_{j=1}^{n} Q^*(x^*_{(n)} - x^*_{(j)}) |F_{z_{(j)}}|^{-4} \gamma \frac{\partial^2}{\partial \omega^2_{(j)}} f_n. \end{aligned} \tag{27}$$

The invariance of (4) requires the following condition on the transformation $Q(x_{(n)} - x_{(j)})$:

$$Q^*(x^*_{(n)} - x^*_{(j)}) = |F_{z_{(j)}}|^4 Q\big(x_{(n)} - x_{(j)}\big). \tag{28}$$

The form of $S_{(j)}$ indicates that the infinitesimal operator

$$\begin{gathered} T_{(j)} = \xi^{3j-2}\frac{\partial}{\partial x^1_{(j)}} + \xi^{3j-1}\frac{\partial}{\partial x^2_{(j+1)}} + \eta^1_{(n)}\frac{\partial}{\partial f_n}, \\ \eta^1_{(n)} = -2\xi^{3j-2}_{x^1_{(j)}} \end{gathered} \tag{29}$$

generates the Lie subalgebra $\mathfrak{t}_j \subset \mathfrak{s}_j$ and the Lie (pseudo-)group H_j acting in $K_{\mathbf{x}_{(j)}} = C \times S$:

$$\begin{gathered} z^*_{(j)} = F(z_{(j)}), \quad dz^*_{(j)} = F_{z_{(j)}} dz_{(j)}, \quad d\bar{z}^*_{(j)} = \bar{F}_{\bar{z}_{(j)}} d\bar{z}_{(j)}, \\ f^*_n = |F_{z_{(j)}}|^{-2} f_n, \quad |F_{z_{(j)}}|^2 = F_{z_{(j)}} \bar{F}_{\bar{z}_{(j)}}, \end{gathered} \tag{30}$$

where $F = U + iV$; U and V are conjugate harmonic functions (i.e., F is conformal map); $F_{z_{(j)}}$ is the derivative with respect to $z_{(j)}$; and the group parameter a is omitted in the designations. Correspondingly, $H = H_1 \times \ldots \times H_n$ is the Lie (pseudo-)group in $\mathcal{K} = K_{\mathbf{x}_{(1)}} \times \ldots \times K_{\mathbf{x}_{(n)}} \subset P$. Let us find the representation of the Lie algebra $\mathfrak{t}_j$. In complex variables, the operator $T_{(j)}$ takes the form

$$\begin{aligned} T_{(j)} = &F(z_{(j)})\frac{\partial}{\partial z_{(j)}} + \bar{F}(\bar{z}_{(j)})\frac{\partial}{\partial \bar{z}_{(j)}} \\ &- F_{z_{(j)}}(z_{(j)}) f_n \frac{\partial}{\partial f_n} - \bar{F}_{\bar{z}_{(j)}}(\bar{z}_{(j)}) f_n \frac{\partial}{\partial f_n}. \end{aligned} \tag{31}$$

The transformations $z_{(j)} \mapsto z_{(j)} + \epsilon(z_{(j)})$ and $\bar{z}_{(j)} \mapsto \bar{z}_{(j)} + \bar{\epsilon}(\bar{z}_{(j)})$ determine the representation of the operator $T_{(j)}$. Infinitesimal holomorphic transformations $z_{(j)}$ and $\bar{z}_{(j)}$ can be written as

$$\begin{aligned} z^*_{(j)} &= z_{(j)} + \epsilon(z_{(j)}) = z_{(j)} + F(z_{(j)})\delta s, \\ \bar{z}^*_{(j)} &= \bar{z}_{(j)} + \bar{\epsilon}(\bar{z}_{(j)}) = \bar{z}_{(j)} + \bar{F}(\bar{z}_{(j)})\delta s. \end{aligned} \tag{32}$$

We use the Laurent expansion

$$\epsilon(z_{(j)}) = -\sum_{n=-\infty}^{\infty} \varepsilon_n z^{n+1}_{(j)}, \quad \bar{\epsilon}(\bar{z}_{(j)}) = -\sum_{n=-\infty}^{\infty} \bar{\varepsilon}_n \bar{z}^{n+1}_{(j)}.$$

Each harmonic in the series generates the transformations $z_{(j)} \to z^*_{(j)} \equiv z_{(j)} - \varepsilon_n z^{n+1}_{(j)}$ and $\bar{z}_{(j)} \to \bar{z}^*_{(j)} \equiv \bar{z}_{(j)} - \bar{\varepsilon}_n \bar{z}^{n+1}_{(j)}$ and the corresponding infinitesimal operators $l_n = -z^{n+1}_{(j)} \dfrac{d}{dz_{(j)}}$ and $\bar{l}_n = -\bar{z}^{n+1}_{(j)} \dfrac{d}{d\bar{z}_{(j)}}$, where ε_n and $\bar{\varepsilon}_n$ are transformation parameters, which make a basis in the infinite-dimensional conformal Lie algebra (two copies of the Witt algebra): the basis of the operator $T_{(j)}$ is $k_n \oplus \bar{k}_n$, $n \in Z$,

$$k_n = -z_{(j)}^{n+1}\frac{d}{dz_{(j)}} - (n+1)z_{(j)}^n f_n \frac{d}{f_n},$$
$$\bar{k}_n = -\bar{z}_{(j)}^{n+1}\frac{d}{d\bar{z}_{(j)}} - (n+1)\bar{z}_{(j)}^n f_n \frac{d}{df_n}. \tag{33}$$

Thus, the algebra generated by the infinitesimal operator $T_{(j)}$ ($\mathfrak{t}_j \subset \mathfrak{s}_j$) is a linear shell over C basis elements of $k_n \oplus \bar{k}_n$ and the Lie bracket: $[k_n, k_m]f = k_n k_m f - k_m k_n f = (n-m)k_{n+m}$, $[\bar{k}_n, \bar{k}_m]f = (n-m)\bar{k}_{n+m}f$, $[k_n, \bar{k}_m]f = 0$.

2.2. Lagrangian

Invariants of transformation groups comprise a part of the physical process. A geometric object, which is behind the measurement procedure, is a Lagrangian manifold with an action integral (Lagrangian) [15]. Central moments or differential forms determine the differential invariants of an admitted transformation group [15]. Let us consider the fiber $K_{\mathbf{x}_{(j)}}$ and the differential form $dz_{(j)}d\bar{z}_{(j)} = 1/2\left(dz_{(j)} \otimes d\bar{z}_{(j)} + d\bar{z}_{(j)} \otimes dz_{(j)}\right)$ [16]. With allowance for (30), simple calculations show that the infinitesimal operator $T_{(j)}$ generates the invariant $dl_{(j)}^2 = f_n dz_{(j)}d\bar{z}_{(j)}$ of the Lie (pseudo-)group H_j. The trajectory action integral (Lagrangian) $\gamma_{(j)}$, parameterized by the parameter $\tau_{(j)}$ (its length in the metric $dl_{(j)}^2$), is determined by the formula [16]

$$\mathscr{E}_{(j)} = \int\limits_{\gamma_{(j)}} f_n^{-1}(\tau^2_{(j)x^1_{(j)}} + \tau^2_{(j)x^2_{(j)}})d\tau_{(j)},$$
$$\tau^2_{(j)x^1_{(j)}} + \tau^2_{(j)x^2_{(j)}} = f_n, \tag{34}$$

or

$$\mathscr{E}_{(j)} = 4\int\limits_{\gamma_{(j)}} (f_n)^{-1}\left(z_{(j)}\bar{z}_{(j)}\right)\partial z_{(j)}\tau_{(j)}(z_{(j)}, \bar{z}_{(j)})$$
$$\times \partial_{\bar{z}_{(j)}}\tau_{(j)}\left(z_{(j)}, \bar{z}_{(j)}\right)d\tau_{(j)}. \tag{35}$$

The Lie (pseudo-)group H_j invariantly transforms (34) and, accordingly, (35). To prove the invariance of (34), it is sufficient to find symmetries of the equation (eikonal equation) (see (34))

$$\tau^2_{(j)x^1_{(j)}} + \tau^2_{(j)x^2_{(j)}} = f_n. \tag{36}$$

The widest group of transformations (36) consists of equivalence transformations, which was calculated in [17]. The subalgebra of the equivalence transformation algebra, which retains the $\tau_{(j)}$ invariant, has the form

$$Y_{(j)} = \Phi(x^1_{(j)}, x^2_{(j)})\frac{\partial}{\partial x^1_{(j)}} + \Psi(x^1_{(j)}, x^2_{(j)})\frac{\partial}{\partial x^2_{(j)}}$$
$$- 2\Phi_{x^1_{(j)}}(x^1_{(j)}, x^2_{(j)})f_n\frac{\partial}{\partial f_n}, \tag{37}$$

where Φ and Ψ are arbitrary conjugate functions. Thus, $T_{(j)}$ and $Y_{(j)}$ coincide.

For Lagrangian (34), the energy–momentum tensor [16] has the following components:

$$T_{11} = 2\tau^2_{(j)x^1_{(j)}} - f_n, \quad T_{22} = 2\tau^2_{(j)x^2_{(j)}} - f_n,$$
$$T_{12} = T_{21} = 2\tau_{(j)x^1_{(j)}}\tau_{(j)x^2_{(j)}}. \tag{38}$$

Here, $T = \{T_{ik}\}$ is a traceless tensor because $T_{11} + T_{22} = 2(\tau^2_{(j)x^1_{(j)}} + \tau^2_{(j)x^2_{(j)}} - f_n) = 0$. Therefore, $T_{z_{(j)}\bar{z}_{(j)}} = T_{\bar{z}_{(j)}z_{(j)}} = 0$, which suggests $\partial_{\bar{z}_{(j)}}T_{z_{(j)}z_{(j)}} = \partial_{z_{(j)}}T_{\bar{z}_{(j)}\bar{z}_{(j)}} = 0$. Therefore, the tensor has only two nonzero components: $T(z_{(j)}) = T_{z_{(j)}z_{(j)}}(z_{(j)}) = 1/4\left(T_{11} - 2iT_{12} - T_{22}\right)$ and $\bar{T}(\bar{z}_{(j)}) = T_{\bar{z}_{(j)}\bar{z}_{(j)}}(\bar{z}_{(j)}) = 1/4(T_{11} + 2iT_{12} - T_{22})$. In addition, $T_{z_{(j)}z_{(j)}} = (\partial_{z(j)}\tau_{(j)})^2$ and $T_{\bar{z}_{(j)}\bar{z}_{(j)}} = (\partial_{\bar{z}_{(j)}}\tau_{(j)})^2$. As a result, H_j retains an infinite number of the currents $j_{z_{(j)}} = T_{z_{(j)}z_{(j)}}\varepsilon_n z_{(j)}^{n+1}$ and $j_{\bar{z}_{(j)}} = T_{\bar{z}_{(j)}\bar{z}_{(j)}}\bar{\varepsilon}_n \bar{z}_{(j)}^{n+1}$.

Thus, the Lagrangian and the current in the fiber bundle $\mathscr{K}$ are defined as the vectors $\mathbf{E} = (\mathscr{E}_{(1)}, \ldots, \mathscr{E}_{(n)})$ and $\mathbf{j} = (j_{z_{(1)}}, \ldots, j_{z_{(n)}})$, $\bar{\mathbf{j}} = (j_{\bar{z}_{(1)}}, \ldots, j_{\bar{z}_{(n)}})$, $n = 1, \ldots, \infty$. The Lie pseudo-group H invariantly transforms $\mathbf{E}$ and retains the currents. One can interpret H_j as the equivalence transformation of eikonal equation (36) retaining the ray length.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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