TILTING MODULES

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1. Introduction

Let A be a finite dimensional algebra over a field k. Let mod$A$ denote the category of finitely generated right A-modules and by ind$A$ the subcategory of mod$A$ of the indecomposable A-modules. Our main objective is the study of ind$A$.

If $T \in$mod$A$, then $B = (\text{End}_A T)^{op}$ is also a finite dimensional k-algebra. Depending on the conditions imposed on T, one can get useful informations on mod$B$ from the informations one has on mod$A$, and consequently, on B from A. In the study of the relations between mod$A$ and mod$B$ we can consider the following functor

$$F = \text{Hom}_A(T, -): \text{mod}A \longrightarrow \text{mod}B$$

Theorem 1. *The above functor is an equivalence if and only if T is a progenerator of modA.*

It is worthwhile to mention that, as a consequence of Morita theorem, we have the following: if $A_A = P_1^{n_1} \oplus \cdots \oplus P_t^{n_t}$ with $P_i \neq P_j$ whenever $i \neq j$ and $T = P_1 \oplus \cdots \oplus P_t$, then mod$A$ and mod$(\text{End}_A T)^{op}$ are equivalent. Therefore, we can restrict our study to the algebras $A_A = P_1 \oplus \cdots \oplus P_t$ with $P_i \neq P_j$ if $i \neq j$. Such algebras are called *basic.*

On the other hand, in the case given by the above theorem, the module T is *too good.* The overall strategy is then to look for some weaker conditions in the A-module T in order to produce an (endomorphism) algebra B as far as possible of A, but with mod$A$ and mod$B$ still having something in common. The so-called tilting modules suit nicely for this purpose, and this will be the main subject of this note. From now on, all algebras are basic, indecomposable and finite dimensional over an algebraically closed field k.

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2. Tilting modules

Let A be an algebra. The notion of tilting modules came from a search of conditions on a module T that helps to understand the algebra $(\mathrm{End}_A T)^{op}$ from A. Let us recall this notion, which arises mainly from the work of Brenner-Butler-Happel-Ringel [5, 8].

Definition 1. An A-module T is called *tilting* if satisfies:
(i) The projective dimension of T ($\mathrm{pd}T$) does not exceed one;
(ii) $\mathrm{Ext}^1_A(T,T) = 0$;
(iii) There exists a short exact sequence

$$0 \longrightarrow A_A \longrightarrow T'_A \longrightarrow T''_A \longrightarrow 0$$

with T' and T'' direct sums of direct summands of T, that is, they belong to $\mathrm{add}T$.

Examples. Let $A_A = P_1 \oplus \cdots \oplus P_t$ be an algebra.
(1) The module $T = P_1 \oplus \cdots \oplus P_t$ is clearly a tilting module.
(2) Suppose A is hereditary and let $T = DA = \mathrm{Hom}_k(A,k)$. Then, T is the sum of the indecomposable injective A-modules. Therefore, $\mathrm{Ext}^1_A(T,T) = 0$. Also, $\mathrm{pd}T \leq 1$ because A is hereditary. Consider now the following short exact sequence

$$0 \longrightarrow A_A \stackrel{\iota}{\longrightarrow} T_1 \longrightarrow Coker(\iota) \longrightarrow 0$$

where ι is the injective envelope of A. Then $T_1 \in \mathrm{add}T$, and since $\mathrm{Coker}\iota$ is a quotient of T_1, it also belongs to $\mathrm{add}T$. Therefore, T is a tilting module.
(3) Let A be the matrix algebra

$$\begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

The indecomposable A-modules are $P_1 = e_{11}A$, $P_2 = e_{22}A$, $P_3 = e_{33}A = S_3$, $S_1 = P_1/\mathrm{rad}P_1$, $S_2 = P_2/\mathrm{rad}P_2$ and P_1/S_3, where e_{ii} is the matrix with 1 in the coordinate (i,i) and 0 in the other coordinates. Observe that P_i is the indecomposable projective associated to the row i and S_i is the simple module associated to P_i. It is not difficult to see that $T = P_1 \oplus S_1 \oplus S_3$ is a tilting module.

The next theorem is due to Brenner-Butler.

Theorem 2. (Brenner-Butler) *Let A be an algebra, T_A be a tilting module and $B = \mathrm{End}T$. Then*
(i) ${}_BT$ is a tilting module and $A \cong \mathrm{End}_B T$, canonically.
(ii) The functors $\mathrm{Hom}_A(T,-)$ and $- \otimes_B T$ induce mutually inverse equivalences between the full subcategories

$$\mathcal{T} = \{M_A |\ Ext^1_A(T,M) = 0\} \quad and \quad \mathcal{Y} = \{N_B |\ Tor^B_1(N,T) = 0\}$$

while the functors $Ext_A^1(T,-)$ *and* $Tor_1^B(-,T)$ *induce mutually inverse equivalences between the full subcategories*

$$\mathcal{F} = \{M_A |\ Hom_A(T,M) = 0\} \quad and \quad \mathcal{X} = \{N_B |\ N \otimes_B T) = 0\}.$$

In other words, this theorem says that there are subcategories $\mathcal{F}$ and $\mathcal{T}$ of mod$A$ which are equivalent to $\mathcal{X}$ and $\mathcal{Y}$ of mod$B$, respectively. Clearly, in our strategy to get informations on mod$B$ from mod$A$, it is important that the subcategories $\mathcal{X}$ and $\mathcal{Y}$ were somehow *significant* in mod$B$. This will occur if we impose some conditions on A. We shall see it in the next section.

3. Tilted algebras

Let A be now a hereditary k-algebra and $T \in$ mod$A$ be a tilting module. In this case, the algebra $B =$End$T$ is called *tilted.* The advantage in working in this context was shown by Brenner-Butler-Happel-Ringel [5, 8]. They have shown the following results. From now on we shall use the notation established in the Brenner-Butler theorem.

(i) Each object of mod$B$ belongs to either $\mathcal{X}$ or $\mathcal{Y}$ (in this case, we say that $(\mathcal{X},\mathcal{Y})$ is a splitting torsion theory). Then, the representation type of B is, in some sense, simplier than of A.

(ii) There exists an B-module X, with the same number of indecomposable non-isomorphic summands of the number of simple A-modules, such that: (a) the modules from $\mathcal{X}$ are generated by X, that is, for each $M \in \mathcal{X}$, there exists an epimorphism from a power X^r of X to M; and (b) the modules from $\mathcal{Y}$ are cogenerated by X, that is, for each $M \in \mathcal{Y}$ there exists a monomorphism from M to a power X^s of X.

Example. Let A and T be as in example 3. Then

$$\mathrm{B} = (\mathrm{End}_A T)^{op} = \begin{pmatrix} k & k & 0 \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

Observe that ind$B$ has 5 non-isomorphic indecomposable objects. We also have (using the notation from example 3):
$\mathcal{F} = \{S_2\}$
$\mathcal{T} = \{P_1, P_2, P_3, S_1\}$
$\mathcal{X} = \{e_{11}B/rad(e_{11}B)\}$
$\mathcal{Y} = \{e_{33}B, e_{22}B, e_{22}B/rad(e_{22}B), e_{11}B\}$

For more details on tilting theory we refer the reader to [1].

4. Homological Properties

Let B be a tilted algebra. What kind of homological properties on B one can expect, knowing that this algebra is defined from a hereditary algebra? In [8], Happel-Ringel have shown that the global dimension of B is at most two. Moreover, if $M \in \mathcal{X}$ then the injective dimension of M (id$M$) is at most one and if $M \in \mathcal{Y}$ then the projective dimension of M (pd$M$) is at most one. Consequently, we have the following proposition. We say that a property holds for almost all modules if it fails only for a finite number of non-isomorphic indecomposable modules.

Proposition 3. *Let B be a tilted algebra. Then B is of finite representation type (that is, there are only finitely many non-isomorphic indecomposable B-modules) if and only if pd$M$ = id$M$ = 2 for almost all indecomposable modules.*

In [2, 3], in a joint work with I. Assem, we have studied other aspects of this question. We shall comment in the sequel some results form there.

Let A be an algebra. Given two indecomposable A-modules X and Y, and $i \geq 1$, we denote by $\mathrm{rad}^i(X,Y)$ the vector space generated by the morphisms from X to Y which are composite of i non-invertible morphisms and by $\mathrm{rad}^\infty(X,Y)$ the intersection of all $\mathrm{rad}^i(X,Y)$, $i \geq 1$.

Let now A be a hereditary algebra and T be a tilting A-module. If $\mathrm{rad}^\infty(-,T) = 0$, then the algebra B =End$T$ is called *concealed*. In many aspects, the concealed algebras are those which are closer to hereditary algebras. We have proven in [2] the following result.

Theorem 4. *Let A be a representation-infinite algebra. The following are equivalent:*
(i) A is concealed;
(ii) $\mathrm{rad}^\infty(-,A) = 0$ and $\mathrm{rad}^\infty(DA,-) = 0$;
(iii) pd$M$ = 1 and id$M$ = 1 for almost all indecomposable modules M.

In [3], we have also characterize those tilted algebras which satisfy one of the following properties: (a) pd$M$ =2 and id$M$ =1 for almost all indecomposable modules; and (b) pd$M$=1 and id$M$=2 for almost all indecomposable modules. These characterizations are too technicals and we shall not discuss it here.

5. Generalized tilting module

We shall now comment briefly on a generalisation of the notion of tilting module, due to Miyashita [9].

Definition 2. An A-module T is called a *generalized tilting module* provided:
(i) pd$T < \infty$;
(ii) $\mathrm{Ext}^i_A(T,T) = 0$ for all $i > 0$.

(iii) There exists a long exact sequence

$$0 \longrightarrow {}_AA \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_m \longrightarrow 0$$

with $T_i \in \mathrm{add}(T)$, for all $i = 0, 1, \cdots, m$.

Let T be a generalized tilting module. In general, the relations between mod$A$ and mod$(\mathrm{End}_A T)^{op}$ are not as good as in the situation when pd$T \leq 1$, which is the situation we have been considering.

Let now M be an A-module satisfying conditions (i) and (ii) of definition 2 above (we say that M is a generalized partial tilting module). If there exists a module X such that $M \oplus X$ is a generalized tilting module, then we shall call such an X a *complement* to M. If pd$M \leq 1$, then M has always a complement [4]. However, in general, there are generalized partial tilting modules without complements, as shown by Ricark-Schofield [10]. In a joint work with Happel and Unger [7], we have given a sufficient condition on M to have a complement. Recall that a subcategory $\mathcal{C}$ of mod$A$ is said to be *contravariantly finite* provided for each $X \in$mod$A$ there exists a morphism $Y \longrightarrow X$ with $Y \in \mathrm{add}\mathcal{C}$ such that the induced morphism $\mathrm{Hom}_A(C, Y) \longrightarrow \mathrm{Hom}_A(C, X)$ is surjective for all $C \in \mathcal{C}$. We have proven the following.

Theorem 4. *Let M be a generalized partial tilting module. If*

$$\mathcal{C}_M = \{X \in modA : pdX < \infty \text{ and } Ext^i_A(X, M) = 0 \text{ for all } i > 0\}$$

is contravariantly finite, then M has a complement.

As a consequence of this result one can prove (see [6]) the following.

Theorem 5. *Let A be an algebra such that $rad^\infty(DA, -) = 0$. If M is a generalized partial tilting module satisfying $rad^\infty(M, -) = 0$, then there exists a complement X of M satisfying $rad^\infty(X, -) = 0$.*

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