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On the Mayer Series of Two-Dimensional Yukawa Gas at Inverse Temperature in the Interval of Collapse

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Abstract

We prove a theorem on the minimal specific energy for a ± 1 charged particles system, interacting through a class of pair potential v, that may be stated as follows: suppose v may be represented by a scale mixtures of d-dimensional Euclid's hat. If the number of particles n is even, then their interacting energy U_n divided by n is minimized by a constant B at the configurations with total charge zero and all particles collapsed to a point; if n is odd, then the ratio $U_n/(n-1)$ is minimized by a constant $\bar{B} = B$ at the configurations with total charge ± 1 and all particles collapsed to a point. The theorem is then used to investigate the convergence of the Mayer series for a gas of ± 1 charged particles interacting through the two-dimensional Yukawa pair potential v for inverse temperatures in the collapse interval $[4\pi, 8\pi)$. The convergence is proved in the present paper up to the second threshold 6π using the decomposition of the Yukawa potential into scales of modified Bessel functions (standard) and into scale mixtures of Euclid's hat. Moreover, assuming that **(i)** neutral subclusters of size smaller than an odd number $k > 1$ do not collapse inside a cluster of size larger than k for β in the threshold interval $[8\pi(k-2)/(k-1), 8\pi k/(k+1))$ and **(ii)** they satisfy a technical condition, then the Mayer series, discarding the first even coefficients of order smaller than k, converges.

Keywords Two-dimensional Yukawa potential · Minimal specific energy · Scale mixtures of Euclid's hat · Stability · Mayer series · Collapse interval

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The d-dimensional Euclid's hat $h(|x|/s)$ at scale $s > 0$ is, except by a constant, the self-convolution of an indicator function of a ball in $\mathbb{R}^d$ of radius $s/2$ centered at origin.

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1 Introduction and Background of Tools and Methods

The present paper investigates a system of particles with ± 1 charges living in a two dimensional Euclidian space and interacting through the Yukawa pair potential $v(x) = (-\Delta + 1)^{-1}(0, x)$. Because of the Yukawa looks at short distances like the Coulomb potential, the two-dimensional Yukawa gas inherits the instabilities of the corresponding Coulomb system when the inverse temperatures β belongs to the interval $[4\pi, 8\pi)$, in which a sequence of collapses of neutral cluster of size $2n$ occurs at the thresholds $\beta_{2n} = 8\pi\,(1 - 1/2n), n \in \mathbb{N}$. It remains an open problem for this system to establish convergence of the Mayer series in powers of activity z , with the first even terms removed from the series, how many depending on $\beta \in [4\pi, 8\pi)$. It is our purpose to revisit this long standing problem.

Benfatto [3] and collaborators from the Italian school (see references therein) initiate a program using iterated Mayer series for pressure (and correlation functions) together with ideas from the work of Gopfert–Mack [20] and Imbrie [25]. Brydges and Kennedy [8] have also considered the Mayer expansion of the two-dimensional Yukawa gas in the context of the Hamilton–Jacobi equation. We adopt in present investigation their continuum scaling renormalization method, adding to that approach a new ingredient. The novelty is related with the (short-range) decomposition of the Yukawa potential into scales. Instead of the standard decomposition $v(x) = \int_{-\infty}^{0}\left(d\left(-\Delta + e^{-s}\right)^{-1}(0, x)/ds\right)ds$ (or the discrete version of it) considered in the previous work, we use the scale mixtures $v(x) = \int_{0}^{1} g(s)h(|x|/s)ds$ of Euclid's hat $h(r)$. Using a concept introduced by Basuev [1], we first prove a theorem that the minimal specific energy $e(v)$ and the constrained (to nonzero total charge) modified minimal specific energy $\bar{e}(v)$ are equal.

Our main theorem on specific energy when applied to the two-dimensional Yukawa gas (see paragraph Achievements and unresolved issues below) leads one step further in the investigation given by Guidi and one of the authors (see [22, Conjecture 2.3 and Remark 7.5]) towards the convergence of the Mayer series for β in the whole interval $[4\pi, 8\pi)$ of collapse.

In the present paper, the convergence of the Mayer series is proven up to the second threshold $\beta \in [0, 6\pi)$.

We shall now review the tools and methods employed in present investigation. We refer to the references for detail.

Decomposition of radial positive functions of positive type. Positive definite functions have arisen in many areas of (pure and applied) mathematics and physics (see [32] for an historical survey). A continuous function f defined in $\mathbb{R}^d$ is called positive definite (abbreviated as p. d.) if the $n \times n$ real matrix $\left[f\left(x_i - x_j\right)\right]_{1\le i,j\le n}$ is positive definite for $n \in \mathbb{N}$ arbitrary elements $x_1, \ldots, x_n$ of $\mathbb{R}^d$:

$$\sum_{1\le i,j\le n} \bar{z}_i z_j f\left(x_i - x_j\right) \ge 0\,, \qquad \forall z_1, \ldots, z_n \in \mathbb{C}\,. \tag{1.1}$$

The celebrate work of Bochner (see e.g [5]) characterizes these functions as follows: f (with $f(0) = 1$) is positive definite if, and only if, it is a Fourier-Stieltjes transform $\hat{\mu}(x) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}d\mu(\xi)$ of a probability Borel measure μ on $\mathbb{R}^d$. Although powerful, Bochner's theorem may be difficult to use in practice: how do we know that a given f satisfies (1.1)?

Even when explicit computation of Fourier transform is available, how do we represent f into a scale mixtures of suitable elementary function?

Recent investigations (see [19,23,24] and references therein) seek for concrete examples and easy checkable criteria of p. d. function. A particularly interesting subclass of p. d. functions, denoted in [24] by Ω_d^+, is provided by radial continuous functions: $f(x) = \varphi(|x|)$ for some positive continuous function φ on $\mathbb{R}_+$. A simple example of a function that vanishes out of a ball B_s on $\mathbb{R}^d$ of radius $s > 0$ centered at origin is given by the d-dimensional Euclid's hat

$$\frac{4}{1/(\gamma_d (s/2)^d)} \chi_{s/2} * \chi_{s/2}(x) \equiv h(|x|/s)$$

where $\chi_r(x) \equiv \chi_{B_r}(x)$ is the indicator function of B_r and $\gamma_d = \pi^{d/2}\Gamma(d/2+1)$ is the volume of a unit ball. Jaming, Matolcsi and Révész [24] have identified certain compactly supported functions, alike this one, as extrema rays of the cone Ω_d^+, playing the same role as the family $\left\{e^{i\xi\cdot x}\right\}$ for the Bochner's theorem. So, if φ is an extremum ray of Ω_d^+ then, by Choquet representation,

$$\int_0^\infty \varphi\left(|x|/s\right) d\nu(s) \tag{1.2}$$

is an element of Ω_d^+ for a suitable positive measure ν supported on the family of scales $\{\varphi\left(|x|/s\right)\}$ of φ.

Gneiting [19] and Hainzl-Seiringer [23], on the other hand, characterized the subclass $H_d \subset \Omega_d^+$, formed by scaling mixtures of d-dimensional Euclid's hat (an analogue of Polya's criterion for radial characteristic functions on $\mathbb{R}^d$ has been provided for $d \geq 2$ in [19]). Hainzl-Seiringer's representation however suffices to make our point in the present work. Let us start with the two-dimensional Yukawa potential, which is not an element of H_2 (see Remark 2.4), given by the Green's function $v(1/\sqrt{\kappa}, x) = (-\Delta+\kappa)^{-1}(0,x)$ (the resolvent kernel of the Laplacian operator $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$). Applying Fourier transform to solve for v, together with $v(1/\sqrt{\kappa}, x) = v(1, \sqrt{\kappa}x)$, yields (see e.g. [18, Sect. 7.2])

$$v(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\xi\cdot x}\frac{1}{\xi^2+1}d\xi = \frac{1}{2\pi}K_0(|x|) \tag{1.3}$$

where $v(x) \equiv v(1,x)$ and K_0 is the modified Bessel function of second kind of order 0. Hainzl-Seiringer's formula for this function reads

$$v(x) = \int_0^\infty h(|x|/s) g(s) ds \tag{1.4}$$

where $(h(0) = 1)$

$$h(w) = \frac{2}{\pi}\left(\arccos w - w\sqrt{1-w^2}\right), \qquad \text{if} \quad 0 < w \leq 1 \tag{1.5}$$

$h(w) = 0$ if $w > 1$ and

$$g(s) = \frac{-s}{4\pi}\int_s^\infty K_0'''(r)\frac{r}{\sqrt{r^2-s^2}}dr\,. \tag{1.6}$$

The scale mixture density $g(s)$ for the Yukawa potential in $d=1$ and 3 dimensions, and for the Coulomb potential in d-dimensions, are known in terms of elementary functions (see Examples 1 and 2 of [23]). For the Yukawa potential in 2-dimensions, however, $g(s)$ can only be written in term of Meijer G-functions (see [4] for an introduction): $t^2 G_{13}^{30}\left(t\,\middle|\,{}^{-3/2}_{-2,-1,0}\right) = G_{13}^{30}\left(t\,\middle|\,{}^{1/2}_{0,1,2}\right)$.

We observe that $h(w)$ is convex and its scale mixtures (1.4) preserves it. This useful property, as we shall see, distinguishes (1.4) from another common scaling decomposition of (1.3): together with $v(s,x) = \left(-\Delta + 1/s^2\right)^{-1}(0,x) = (1/2\pi)K_0(|x|/s)$, we write

$$v(x) = \int_0^1 \dot{v}(s;x)ds$$

by the fundamental theorem of calculus. Taking the derivative of $v(s;x)$ with respect to s, yields

$$v(x) = \int_0^1 \tilde{h}(|x|/s)\frac{1}{2\pi s}ds \tag{1.7}$$

where $\tilde{h}(w) = -wK_0'(w) = wK_1(w)$ and K_1 is the modified Bessel function of second kind of order 1. Like $h(w)$, $\tilde{h}(w)$ for $w \in \mathbb{R}_+$ decreases from $\tilde{h}(0) = 1$ to 0 monotonously as w increases but changes from concave to convex at w_0 (see Fig. 3). As the Yukawa potential $v(x)$ behaves like the Coulomb potential $(1/2\pi)\log|x|^{-1}$ at short distances, the mixture density $g(s)$ for both decompositions of v, (1.4) and (1.7), behave as $(2\pi s)^{-1}$ in a neighborhood of $s = 0$.

Gaussian Processes and renormalization group. Positive definite functions plays an important role on renormalization group (RG) methods in statistical physics. Brydges and collaborators [7] (see also [10]) coined a term "finite range decomposition" to the mixtures of scale (1.2) for some compactly supported radial extremal functions φ. They used a probabilistic argument as follows: breaking up the range of integration into disjoint union of intervals $I_j = [L^{-j}, L^{-j+1})$, $j \geq 1$ for $L > 1$ and $I_0 = [1,\infty)$, (1.4) may be seen as the "finite range" decomposition

$$\phi = \sum_{j\geq 0}\zeta_j \tag{1.8}$$

of a Gaussian process ϕ of mean $\mathbb{E}\phi(x) = 0$ and covariance $\mathbb{E}\phi(x)\phi(y) = v(x-y)$ into a family of independent Gaussian processes $\{\zeta_j\}$ of mean $\mathbb{E}\zeta_j = 0$ and covariance

$$\mathbb{E}\zeta_j(x)\zeta_j(y) = \int_{I_j} g(s)h(|x-y|/s)\,ds \equiv v_{I_j}(x-y)$$

(as the sum of covariances $\sum_{j\geq 0} v_{I_j}$ equals (1.4)). See [7,10] for extensions of finite range decomposition to a large class of positive definite functions on $\mathbb{R}^d$ and $\mathbb{Z}^d$.

When a statistical system is represented by the expectation $\mathbb{E}\mathscr{Z}$ of a functional $\mathscr{Z}(\phi)$, the decomposition (1.8) of the Gaussian field ϕ can be used to integrate out each ζ_j at a time. Let $\mathbb{E}^{(j)}$ denote the expectation with respect the Gaussian field ζ_j. The renormalization group is a method of calculating the expectation $\mathbb{E}\mathscr{Z}$ through the sequence of maps $\mathscr{Z}_j \longmapsto \mathscr{Z}_{j+1} = \mathbb{E}^{(j+1)}\mathscr{Z}_j$ starting from $\mathscr{Z}_0 = \mathscr{Z}$. The limit $\lim_{j\to\infty}\mathscr{Z}_j$, supposing it exists, is obtained provided $\mathscr{Z}_j \longmapsto \mathscr{Z}_{j+1}$ is amenable to be analyzed as a dynamical system depending on parameters in the initial condition. For instance, the decomposition (1.8) of ϕ into finite range fields ζ_j corresponding to the Yukawa potential (1.4), the limit $j \to \infty$ drives the statistical system into the short scaling limit $s \to 0$ for which the potential diverges logarithmically. For an infinitely many-particle system with ± 1 charges, the existence of $\lim_{j\to\infty}\mathscr{Z}_j$ expresses the thermodynamical stability of the system. We shall come back to this issue below.

Hamilton–Jacobi equation and majorant method. Under the Kac–Siegert transformation (see e.g. [9,15]), the grand partition function for the two-dimensional Yukawa gas of particles with ± 1 charges can be written as the expectation $\mathbb{E}\mathscr{Z}_0$ (with respect to the Gaussian field ϕ) of

$$\begin{aligned}\mathscr{Z}_0(\phi) &= \exp(\mathscr{V}_0(\phi)) \ , \\ \mathscr{V}_0(\phi) &= z\int_{\mathbb{R}^2} :\cos\sqrt{\beta}\phi(x):_v \, dx \\ &= \sum_{\sigma\in\{-1,1\}}\int_{\mathbb{R}^2} dx\ z : e^{i\sqrt{\beta}\sigma\phi(x)} :_v \end{aligned} \tag{1.9}$$

where the parameters β and z are, respectively, the inverse temperature and activity and $:\cdot:_v$ indicates Wick ordering with respect to the potential v. In the present work, we shall adopt the continuum scale decomposition (1.3) instead of (1.8). The induced RG dynamics is thus generated by a Hamilton–Jacobi equation as proposed in [8] by Brydges and Kennedy. Let us expand these ideas in some detail. A scale-dependent-interaction $v : \mathbb{R}_+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is introduced replacing (1.4) by a mixture supported in a finite interval $[t_0, t]$ of scales

$$v(t,x) = \int_{t_0}^{t} h(|x|/s)g(s)ds \tag{1.10}$$

where $t_0 > 0$ is a cutoff of the short scale distances and, since g and h are continuous, we have $\lim_{t\searrow t_0} v(t,x) \equiv 0$. The renormalization group is now given by a convolution mapping $(t,\phi) \longmapsto \mathscr{Z}(t,\phi) = \mathbb{E}^{(t)}\mathscr{Z}_0(+\cdot)$ with initial data $\mathscr{Z}(t_0,\phi) = \mathscr{Z}_0()$, where $\mathbb{E}^{(t)}$ denotes the expectation with respect the Gaussian field ζ with covariance $v(t, x-y)$. Formally, $\mathscr{Z}(t,\phi)$ satisfies the Cauchy problem of a "heat equation"

$$\frac{\partial\mathscr{Z}}{\partial t} = \frac{1}{2}\Delta_{\dot{v}}\mathscr{Z} \ , \qquad \lim_{t\searrow t_0}\mathscr{Z}(t,\phi) = \mathscr{Z}_0()$$

where $\dot{v}(t,x) := \partial v/\partial t(t,x) = g(t)h(|x|/t)$, by the fundamental theorem of calculus, is the Euclid's hat weighted by the scale density $g(t)$ and $\Delta_{\dot{v}}$ is the "Laplacian" operator

$$\Delta_{\dot{v}}\mathscr{Z} = \int_{\mathbb{R}^2\times\mathbb{R}^2} dxdy\dot{v}(t,x-y)\frac{\delta^2\mathscr{Z}}{\delta\phi(x)\delta\phi(y)} \ . \tag{1.11}$$

Writing $\mathscr{Z}(t,\phi) = \exp(\mathscr{V}(t,\phi))$, the heat equation turns into a nonlinear equation for $\mathscr{V}$:

$$\frac{\partial\mathscr{V}}{\partial t} = \frac{1}{2}\Delta_{\dot{v}}\ \mathscr{V} + \frac{1}{2}(\nabla\mathscr{V},\nabla\mathscr{V})_{\dot{v}} \ , \qquad \lim_{t\searrow t_0}\mathscr{V}(t,\phi) = \mathscr{V}_0() \tag{1.12}$$

where $\Delta_{\dot{v}}$ acts as in (1.11) and

$$(\nabla\mathscr{V},\nabla\mathscr{V})_v = \int_{\mathbb{R}^2\times\mathbb{R}^2} dxdy\dot{v}(t,x-y)\frac{\delta\mathscr{V}}{\delta\phi(x)}\frac{\delta\mathscr{V}}{\delta\phi(y)} \ . \tag{1.13}$$

In [8], the authors considered the random field ϕ on $\mathbb{Z}^d$ instead, for which the functional derivative $\int dx\ v(x)\ \delta/\delta\phi(x)\ \mathscr{V}(\phi) = \lim_{\varepsilon\to 0}(\mathscr{V}(\phi+\varepsilon v) - \mathscr{V}(\phi))/\varepsilon$ becomes partial derivative $\partial/\partial\phi_x$ with respect to the variable $\phi_x \in \mathbb{R}$ at site x. Inserting the Taylor expansion (multi-index formula):

$$\mathscr{V}(t,\phi) = \sum_{n\geq 1}\sum_{\alpha:|\alpha|=n}\frac{1}{\alpha!}\frac{\partial^\alpha\mathscr{V}}{\partial\phi^\alpha}(t,0)\ \phi^\alpha$$

into an integral equation equivalent to (1.12), a system of equations for derivatives of $\mathscr{V}$ (by collecting order by order terms), together with an appropriate norm, is used to majorize

$\mathscr{V}(t,\phi)$ by the solution $\nu(t,\varphi)$ of a first order PDE equation in two independent real variables (t,φ), φ playing the role of chemical potential. The local existence and uniqueness of the Cauchy problem (1.12) are then proved in ref. [8] (see Theorem 2.2 and Proposition 2.6 therein) for a domain in plane (t,z) with $z=e^{\varphi}$ (β may be included as well). Quoting the authors, these results are "the precise version of the Mayer expansion".

Brydges and Kennedy have also provided an equivalent system of ordinary differential equations for the Ursell functions (Lemma 3.3 of [8]) which replaces (1.12) defined on $\mathbb{Z}^d$ and can be used for systems of point particles in $\mathbb{R}^d$. If $(\Omega,\mathscr{B},d\varrho(\zeta))$ denotes the finite measure space on $\{-1,1\}\times\mathbb{R}^2$ corresponding to the possible states of a single particle (we united σ and x into $\zeta=(\sigma,x)$), the solution of (1.12) may be represented formally as

$$\mathscr{V}(t,\phi)=\sum_{n\geq 1}\frac{1}{n!}\int d^n\varrho\,\psi_n^c(t,\zeta_1,\ldots,\zeta_n):\exp\left(i\sqrt{\beta}\sum\nolimits_{j=1}^{n}\sigma_j\phi(x_j)\right): \tag{1.14}$$

where the Ursell functions $\psi_n^c(t,\zeta_1,\ldots,\zeta_n)$ are translational invariant and invariant under the action of the symmetric group $\mathbb{S}_n$ of permutations of the index set $\{1,\ldots,n\}$. The present work will take the system of equations satisfied by the $\psi_n^c(t,\zeta_1,\ldots,\zeta_n)$ (see (3.5) and (3.6) below), together with the scale decomposition (1.10) for the Yukawa potential, as the starting point for our analysis.

Stability condition and minimal specific energy. Stability of the interaction v is a condition under which there exist the thermodynamic functions describing an infinitely large statistical system. Let U_n be the total energy potential of the classical charged system of n point particles at positions $x_1,\ldots,x_n$ of $\mathbb{R}^2$, with respective charges $\sigma_1,\ldots,\sigma_n\in\{-1,1\}$, interacting through a pair Yukawa potential:

$$U_n(\zeta_1,\ldots,\zeta_n;v)=\sum_{1\leq i<j\leq n}\sigma_i v\left(x_i-x_j\right)\sigma_j\ . \tag{1.15}$$

An interacting potential v satisfies the *stability condition* if there exists $B>0$ such that

$$U_n(\zeta_1,\ldots,\zeta_n;v)\geq -nB \tag{1.16}$$

holds for all $(\zeta_1,\ldots,\zeta_n)$ on the configurations space $\bigcup_n\left(\{-1,1\}\times\mathbb{R}^2\right)^n$ (otherwise the specific energy U_n/n would not be bounded from below).

The standard stability theorem for charged system due to Fisher and Ruelle [14] (see Theorem I and Eq. (III.7) therein) assures that: if $\hat{v}(\xi)=\int_{\mathbb{R}^2}v(x)e^{-i\xi\cdot x}dx\geq 0$ and $v(0)=(1/2\pi)^2\int_{\mathbb{R}^2}\hat{v}(\xi)d\xi<\infty$, then

$$U_n(\zeta_1,\ldots,\zeta_n;v)\geq -\frac{1}{2}v(0)\sum_{j=1}^{n}\sigma_j^2 \tag{1.17}$$

and, since $\sigma_j^2=1$, (1.16) is satisfied with $B=v(0)/2$. The proof of (1.17) follows from the "if" direction of Bochner's theorem. For this, note that adding 1/2 of each $i=j$ diagonal terms to (1.15) (i.e., (1.17) with the right hand side passed to the left), the quadratic form has to be positive as v is positive definite. It follows from (1.3) that $\hat{v}(\xi)=\left(\xi^2+1\right)^{-1}\geq 0$ is a positive density but $v(x)$, which yields the stability constant B, grows unboundedly at $x=0$. As the self-energy $v(0)$ diverges logarithmically, the decomposition (1.7) or (1.4) has to be used instead.

Let v be the scale mixtures of Euclid's hat (1.10), cutoff on the short scales. We introduce the minimal specific energy $e = e(h)$ of h at the scale s

$$e = \inf_{n\geq 2} e_n$$
$$e_n = \inf_{(\zeta_1,\ldots,\zeta_n)} \frac{1}{n} U_n(\zeta_1, \ldots, \zeta_n; h(\cdot/s)) \tag{1.18}$$

and, analogously, the modified minimal specific energy $\bar{e} = \bar{e}(h)$,

$$\bar{e} = \inf_{n\geq 2} \inf_{\substack{(\zeta_1,\ldots,\zeta_n)\\ \text{non-neutral}}} \frac{1}{n-1} U_n(\zeta_1, \ldots, \zeta_n; h(\cdot/s)) \tag{1.19}$$

where the infimum is now taken over all non-neutral configurations $(\zeta_1, \ldots, \zeta_n)$: $(x_1, \ldots, x_n) \in \mathbb{R}^{2n}$ and $(\sigma_1, \ldots, \sigma_n) \in \{-1, 1\}^n$ such that $\sum_{j=1}^n \sigma_j \neq 0$. As will become clear, the minimal specific energy (modified or not) of the Euclid's hat $h(\cdot/s)$ does not depend on the scale s. So, replacing (1.10) into (1.15) yields

$$\frac{1}{n} U_n(\zeta_1, \ldots, \zeta_n; v) \geq \int_{t_0}^{t} e\left(h(\cdot/s)\right) g(s) ds = e\left(h\right) \cdot \int_{t_0}^{t} g(s) ds$$

from which, taking the infimum on the left side, we have

$$e\left(v\right) \geq e\left(h\right) \cdot \int_{t_0}^{t} g(s) ds\ . \tag{1.20}$$

In the present paper we determine both specific energies e and $\bar{e}$ and characterize the configurations they are attained for $h(\cdot/s)$. Since they are independent of s, equation (1.20) holds as an equality. By (1.17) and (1.18), we have $-e(h) \leq h(0)/2 = 1/2$. We show that this is in fact an equality and, moreover, $e = \bar{e} = -1/2$. More precisely, we have proven in Sec. 2 an improvement of (1.17)

$$U_n(\zeta_1, \ldots, \zeta_n; h(\cdot/s)) \geq \frac{1}{2}\left(\left|\sum\nolimits_{j=1}^{n} \sigma_j\right| - \sum\nolimits_{j=1}^{n} \sigma_j^2\right) \tag{1.21}$$

from which the equality of specific energies e and $\bar{e}$ follows at once. Observe that the inequality (1.21) turns out to be an equality for a configuration of n charges collapsed to a single point, with the total charge $\sum\nolimits_{j=1}^{n} \sigma_j$ being 0 or ± 1. As a consequence, equation (1.20) is as a matter of fact an equality.

One versus iterated Mayer expansion. The Ursell functions can be written by the well known formula introduced by Mayer (see [33]):

$$\psi_n^c(\zeta_1, \ldots, \zeta_n; v) = \sum_{G \text{ connected}} \prod_{\langle ij\rangle \in E(G)} \left(\exp\left(-\beta\sigma_i\sigma_j v\left(\left|x_i - x_j\right|\right)\right) - 1\right)\ , \tag{1.22}$$

where the sum runs over all connected linear graphs G with vertices in the index set $\{1, \ldots, n\}$ and $E(G)$ denotes the set of edges of G. As far as the estimation of pressure and correlation functions are concerned, equation (1.22) is not useful due the cardinality of its sum. To reduce the sum over connected Mayer graphs to labeled trees, Penrose [27,28] has exploited cancellations occurring on the formula under proper re-summation and proved that the Mayer series converge provided the potential v is stable, integrable at large distances and has, in addition, a hard core condition which recently has shown in [29] to be unnecessary (see also [9] for an overview and extensions). The cardinality of labeled trees of order n is n^{n-2} by the famous Cayley theorem, which makes the tree graph identities suitable for the estimation of thermodynamical functions. Among the proposed tree graph formulas now available, we indicate that in Theorem 3.1 of [8] as the most adequate to our purposes of representing the Ursell functions $\psi_n^c(t, \zeta_1, \ldots, \zeta_n)$ defined by (1.22) with the scale-dependent-interaction (1.10) in the place of v. Such Ursell functions satisfy the system of ordinary differential equations (3.5).

However, one particular tree graph identity due to Basuev [2] is worth mentioning in the context of the present work. Basuev's representation works for a radial potential $\phi(|x|)$ in $\mathbb{R}^d$ of the form $\phi = \phi^a + \delta$, where $\phi^a(r) = \phi(r)$ for $r = |x| > a$, $\phi^a(r) = \phi(a)$ for $r \le a$, is stable and $\delta(r) = \phi(r) - \phi^a(r) > 0$ for $r \le a$ and $\delta(r) = 0$ for $r > a$, which may include hard-core: $\delta(r) = \infty$ for $r \le a$. To estimate the Ursell functions efficiently, Basuev uses the modified stability condition (see references [1,30])

$$U_n(\zeta_1, \ldots, \zeta_n; v) \ge -(n-1)\bar{B} \tag{1.23}$$

instead of (1.16), where in the majority of cases important for applications $\bar{B}$ is equal or close to B. It might appear that a slight improvement on the stability bound would not affect the radius of convergence of Mayer series. It turns out, however, that the estimate of the Ursell functions through the Basuev tree graph identity works so well when (1.23) is applied (see particularly equations (15) and (16) of [2]) that impressive improvements on the convergence are reported (at low temperatures) in Basuev paper, as well as in [26].

Let us now explain how the estimate on the Ursell functions gets improved by (1.23) in our case. It is known that the Mayer series for the pressure of a two-dimensional Yukawa gas [3,8,22]

$$\beta p(\beta, z) = \sum_{k \ge 1} b_k z^k \,,$$
$$b_k = \frac{1}{k!} \int d^{k-1}\varrho \; \psi_k^c(\zeta_1, \ldots, \zeta_k; v) \,, \tag{1.24}$$

converges if $|z| < (4\pi - \beta)/(4\pi e \beta)$, the radius of convergence being positive provided $\beta < 4\pi$. Since the Yukawa potential (1.3) diverges logarithmically as $|x| \to 0$, the proof of such statement requires the use of Brydges–Kennedy approach or iterated Mayer expansion (no one-scale tree expansion formula would be able to deal with this issue). The problem at our hand is to extend the stability of 2-dimensional Yukawa gas to the inverse temperature in the range $4\pi \le \beta < 8\pi$, passing through the sequence of thresholds $\beta_{2r} = 8\pi\,(1 - 1/2r)$, $r \in \mathbb{N}$. Here, β_{2r} is the inverse temperature in which a clusters with r positive and r negative charges collapse altogether at once, heuristically given by an argument of entropy–energy (there are r^2 and $r(r-1)$ distinct pairings of opposite and, respectively, same charges):

$$W_{2r}(\delta) = \int_{|x_2| \le \delta} dx_2 \cdots \int_{|x_{2r}| \le \delta} dx_{2r} \exp\left(\beta\left(r^2 - r(r-1)\right) \int_\delta^1 g(s)ds\right) . \tag{1.25}$$

Since $g(s) \asymp 1/(2\pi s)$ as $s \to 0$, we have

$$\lim_{\delta \to 0} W_{2r}(\delta) = c \lim_{\delta \to 0} \delta^{2(2r-1) - \beta r/(2\pi)} = \begin{cases} 0 & \text{if } \beta < \beta_{2r} \\ \infty & \text{if } \beta > \beta_{2r} \end{cases}$$

for some constant $c > 0$. The balance favors the entropy $S(\delta) = \delta^{2(2r-1)}$ if $\beta < \beta_{2r}$ while the energy contribution $e^{-\beta E(\delta)} \simeq \delta^{-\beta r/(2\pi)}$ dominates if $\beta > \beta_{2r}$. The same entropy–energy argument applied to odd clusters, let us say $2r+1$ particles, where $r+1$ particles are charged with one sign and the remaining r with the other sign, shows that (1.25) tends to 0 as δ goes to 0:

$$\lim_{\delta\to 0} W_{2r+1}(\delta) = c \lim_{\delta\to 0} \delta^{4r-\beta(r(r+1)-r^2)/(2\pi)} = c \lim_{\delta\to 0} \delta^{(4-\beta/(2\pi))r} = 0 \,,$$

at any inverse temperature $\beta < 8\pi$.

This heuristic picture suggests us that the collapse of odd clusters should be disregarded when we pass through the sequence of thresholds $\beta_2 = 4\pi$, $\beta_4 = 6\pi$, ..., up to the accumulation point $\beta_\infty = 8\pi$. However, if the Mayer series (1.24) is majorized by using the stability bound (1.16), the majorant series will diverge at the inverse temperatures $\beta_{2r+1} = 8\pi\,(1 - 1/(2r+1))$ when t_0 tends to 0 and will be finite up to 8π whether the stability bound (1.23) is used instead. The confirmation of the heuristic picture by our calculations according to the principles of the statistical physics is, in our opinion, the most important contribution of the present paper. The improvement on the stability bound from $-Bn$ to $-\bar{B}(n-1)$ with $B = \bar{B}$ and n odd, although might seem of little importance, is exactly what is needed for (1.24) to be majorized by a convergent series for β inside any interval between two successive thresholds $[\beta_{2r}, \beta_{2r+2})$, $\forall r \in \mathbb{N}$.

Avoiding the collapse of neutral clusters: a conjecture As a consequence of the alluded collapses, the leading even coefficients b_{2j}, $j = 1, \ldots, n$, of the Mayer series (1.24) diverges for $\beta_{2n} \le \beta < \beta_{2(n+1)}$ when the short scale cutoff t_0, introduced in (1.10) (or in (1.7)) to make the system conditionally stable, is removed. A conjecture presented in [3] as an open problem may be formulated as follows:

Conjecture 1.1 *If the leading n even coefficients b_{2j}'s are removed from the Mayer series (1.24), the radius of convergence of the corresponding series remains positive as t_0 goes to 0 for any $\beta \in [\beta_{2n}, \beta_{2(n+1)})$ and, consequently, for any $\beta < \beta_{2(n+1)}$.*

Brydges-Kennedy [8] have proved convergence of (1.24) with $O\left(z^2\right)$ term omitted for $4\pi \le \beta < 16\pi/3 = \beta_3$ and have explained how it would be extended up to the second threshold 6π. It turns out that the claimed improvement on the estimate of the three-particle energy from $U_3(\xi_1, \xi_2, \xi_3; \dot{v}) \ge -3\dot{v}(t, 0)/2$ to $U_3(\xi_1, \xi_2, \xi_3; \dot{v}) \ge -\dot{v}(t, 0)$ does not hold uniformly on $\left(\{-1, 1\} \times \mathbb{R}^2\right)^3$ at each scale for the decomposition (1.7) used by the authors. It has been shown by numerical calculation in [22] that the factor 3 (the number n of charged particles involved) in the lower bound of U_3 may be improved to 2.14..., which is enough to extend the convergence of Mayer series to any $\beta \in [4\pi, 6\pi)$ but insufficient to go beyond a certain threshold (about $\beta_{15} = 112\pi/15$) up to 8π. Both statements are proven in the present work. We have proved in addition that, if the alternative decomposition (1.4) for the Yukawa potential is used, then the factor 3 can be replaced by 2 in the stability bound for U_3. More generally, by (1.21), (1.16) can be substituted by (1.23) with $B = \bar{B} = \dot{v}(t, 0)/2$ for any odd number $n \ge 3$.

We intend in the present paper to provide a majorant candidate for the pressure of the Yukawa gas at each interval $\beta_{2n} \le \beta < \beta_{2(n+1)}$, uniformly in the cutoff t_0. We use the majorant construction proposed in [22], with (1.7) replaced by (1.4), which leaves the leading even Mayer coefficients b_{2j}, $j = 1, \ldots, n$, bounded by free of divergence coefficients. We shall assume that the neutral parts of a cluster of order larger of $2n$ do not collapse. This assumption has been verified in the present paper only for the neutral pairs for $\beta < 6\pi$.

The majorant construction is based on the idea already presented in early works (see e.g. [6,25]), according to which the Mayer series (1.24) is dominated by an expansion in powers of $ez\Gamma e^B$, where $\Gamma = \|\beta v\|_1$ is the L^1-norm of $\beta v(x)$ and $B = \beta v(0)/2$ is the particle self-energy. A one-step Mayer expansion is not suitable for potentials in which B is large in the range that Γ contributes little, as typically occurs for the two-dimensional Yukawa potential v at low scales (see Eqs. (3.8) and (3.9) in Proposition 3.4 and [8,20,25] for other applications). When v is decomposed into a continuum range of scales, the Mayer series becomes, roughly speaking, an expansion in powers of $ez\tau(t_0, t)$, where

$$\tau(t_0, t) = \int_{t_0}^{t} \Gamma(s) e^{2\int_s^t B(\tau)d\tau} ds \tag{1.26}$$

solves a linear equation $\dot{C}_2 = 2B(t)C_2 + \Gamma(t)$ satisfied by the majorant C_2 of two times the second Mayer coefficient: $2|b_2| \leq C_2$ (see (3.16) and (3.18)). It has been shown that the Mayer expansion (1.24) converges provided $\beta \in [0, 4\pi)$ and $e|z|\,\tau(t_0, t) < 1$, uniformly in t_0 (see [8, Theorem 4.1 together with pp. 41, 42] and Proposition 3.4, Remarks 3.5 and 3.6 below, for the decomposition (1.4)). The Mayer series past the first threshold converges, if the first divergent term is omitted, provided $\beta \in [4\pi, 16\pi/3)$ and $e|z| \displaystyle\int_{t_0}^{t} \Gamma(s) e^{3/2\int_s^t B(\tau)d\tau} ds < 1$ holds uniformly in t_0. We show in the Sect. 3.4 that, for both decompositions (1.7) and (1.4), the exponent factor $3/2$ may be replaced by $4/3$ and the result can be extended for $\beta \in [4\pi, 6\pi)$.

Let us explain further how the proposed majorant series converges provided $\beta \in [\beta_{k-1}, \beta_{k+1})$ and the exponent factor $4/3$ ($k = 3$) is in general replaced by $(k+1)/k$ for any $k > 1$ odd. Let C_n be the n-th majorant coefficient defined in (3.17) et seq. For fixed $t_0 > 0$, we have (see (3.16) and (3.18))

$$n\,|b_n| \leq C_n \tag{1.27}$$

where, by the hypothesis of Conjecture 1.1, the first $(k-1)/2$ even Mayer coefficients are set to 0: $b_{2j} = 0$ for $1 \leq j \leq (k-1)/2$. Equation (1.27) continues to holds if the corresponding even majorant coefficients have their divergent part (as t_0 tends to 0) extracted through a Lagrange multiplier L_k. The equation satisfied by the C_n after the extraction becomes $\dot{C}_n = (n-1)\,((k+1)/k)\,B(t)C_n +$ nonlinear terms for $n > 2$, and for $n = 2$, the C_2 satisfies a linear equation $\dot{C}_2 = ((k+1)/k)\,B(t)C_2 + \Gamma(t)$, whose solution $\tau_k(t_0, t) = \displaystyle\int_{t_0}^{t} \Gamma(s) e^{(k+1)/k\int_s^t B(\tau)d\tau} ds$, generalizes (1.26). We see that C_2 is finite as t_0 goes to 0 up to β_{k+1}, by (3.29), and may be used to build, according to the general principle for Mayer series, a convergent majorant expansion in powers of $ez\tau_k(t_0, t)$. Since the modified stability condition (1.23) applies for every $n > 1$ odd, the linear part of the equation satisfied by C_n in (1.27), given by $(n-1)BC_n < (n-1)\,((k+1)/k)\,BC_n$, implies that the same equation satisfied by C_n with $n \leq k$ even, improved "by hand", also holds for n odd. To understand why the modified stability bound (1.23) is so crucial, we observe that anything large than $(n-1)B$ would prevent the convergence of the majorant series in the whole interval of collapse $[4\pi, 8\pi)$.

Accomplishments and unresolved issues Regarding the Conjecture 1.1, the present paper establishes that the Mayer expansion (1.24) of the two-dimensional Yukawa gas, omitting $b_2 z^2$, is majorized by a convergent series for β in the first collapse interval $[4\pi, 6\pi)$, uniformly in t_0, for standard and mixtures of Euclid's hat decompositions of Yukawa potential v . The challenging unresolved question is summarized by the following

Claim 1.2 *Neutral subclusters of size smaller than or equal* $2n$ *inside a cluster of size larger than* $2n$ *are prevented to collapse for* $\beta \in [\beta_{2n}, \beta_{2(n+1)})$ *due to the interactions of their constituents with the remainder particles.*

Two distinct facts are implicitly assumed in Claim 1.2. The contributions of the neutral subclusters to the Mayer coefficients are supposed to be finite when the cutoff t_0 tends to 0 while the relations (1.27) hold uniformly in t_0, i.e., they are bounded by the corresponding majorant coefficients. The claim has been introduced in order to replace Conjecture 1.1 by Claim 1.2, together with the improved stability and its implication to the majorant coefficient C_n for n odd. We illustrate the usefulness of our strategy for a neutral pair of particles at Sect. 3.4, in which the estimate of "tripoles" by C_3 has played an important role. Until now, we have not succeeded in proving Claim 1.2 , except in the case of a pair of opposite charges. Although we do not intend to address Conjecture 1.1 beyond 6π, we shall argue in Sect. 3.4 that some technics available nowadays could be used to do so.

We now list what has been accomplished in the present paper:

1. An improved stability bound for U_3 for the standard decomposition (1.7) of v (see Proposition 2.1 and Remark 2.3). As a consequence, the convergence of the Mayer expansion with singular part of $n = 2$ term omitted (see [22, Sect. 6.3]) is proven to be extended from the interval $[4\pi, 16\pi/3)$ to $[4\pi, 6\pi)$, uniformly in t_0. The same extension applies to Theorem 4.3 in [8].
2. If the Yukawa potential v is represented by scale mixtures of Euclid's hat (1.4) and $n \in \mathbb{N}$ is odd, Theorem 2.5 establishes that $U_n(\xi_1, \ldots, \xi_k; v(t, \cdot))$ is bounded from below by $-(n-1)/2 \displaystyle\int_{t_0}^{t} g(s)ds$. This implies that clusters with an odd number of particles are not thresholds of the two-dimensional Yukawa gas in the sense that, assuming that Claim 1.2 holds true, the j-th Mayer coefficient b_j is majorized by a finite constant C_j for $j \geq n$ and $\beta \in [\beta_{n-1}, \beta_{n+1})$ and $\sum_{j \geq n} C_j z^j$ has a finite radius of convergence, uniformly in t_0.
3. If v is represented by a scale mixtures of Euclid's hat (1.10), then the Mayer series (1.24) for the two-dimensional Yukawa gas, with the singular part of $n = 2$ term omitted, converges uniformly in t_0 for β in the first threshold interval $[4\pi, 6\pi)$.

The proof of these statements are given in Sects. 2 and 3.

Outlines of the present work The present paper is organized as follows. Section 2 is dedicated to the proof of improved stability bonds Proposition 2.1, Theorem 2.5 (main) and Corollaries 2.7 and 2.8, together with results involved in the standard and scale mixtures of Euclid's hat (Proposition 2.9) representations of two-dimensional Yukawa potential v. Details of the calculations on the standard decomposition of v, the Euclid's hat function and the scale density mixtures are presented, respectively, in Appendices A, B and C

The main Theorem is then used to show that, assuming that Claim 1.2 holds in the two-dimensional Yukawa gas, then, as the heuristic picture suggests, odd number of particles do not collapse. We dedicate Sect. 3 to the Cauchy majorant method applied to the density function of Yukawa gas on the whole interval $[4\pi, 8\pi)$ of collapses. In Sect. 3.4, it is proven that the Mayer series converge for β in the first threshold interval $[4\pi, 6\pi)$ for the standard and scale mixtures of Euclid's hat representation of v.

2 Minimal Specific Energies: Main Theorem and Estimates Involving Modified Bessel Functions

We prove in this section our main theorem (1.21) and the implications of it on the minimal specific energies $e(v)$ and $\bar{e}(v)$ for v a mixture of the Euclid's hat h in $\mathbb{R}^2$.

2.1 Three Particles Minimal Specific Energy

To begin with, let $U_n = U_n(\zeta_1, \ldots, \zeta_n; v)$ be the n-particle total energy (1.15) for a given potential v where $\zeta_i = (\sigma_i, x_i)$, $i = 1, \ldots, n$, unite the i-th particle charge σ_i and position x_i. Let

$$e_n(v) = \frac{1}{n} \inf_{\substack{(\zeta_1,\ldots,\zeta_n),\\ \zeta_i \in \{-1,1\}\times\mathbb{R}^2}} U_n(\zeta_1, \ldots, \zeta_n; v) \tag{2.1}$$

and

$$\bar{e}_n(v) = \frac{1}{n-1} \inf_{\substack{(\zeta_1,\ldots,\zeta_n), \zeta_i \in \{-1,1\}\times\mathbb{R}^2: \\ \sigma_1+\cdots+\sigma_n \neq 0}} U_n(\zeta_1, \ldots, \zeta_n; v) \tag{2.2}$$

be, respectively, the n-particles minimal and constrained minimal specific energies. As the particles of our system have either $+1$ or -1 charges, these two quantities are related to each other when n is an odd number as $e_n(v) = \bar{e}_n(v)(n-1)/n$. Let us first consider the case $n = 3$ and let $v(x)$ be given by the two-dimensional Yukawa potential (1.3) under the standard decomposition (1.7), cut-off at short distances $s \le t_0$: $v(x) = \int_{t_0}^{1} \tilde{h}(|x|/s)/(2\pi s)\, ds$. Assuming $\bar{e}_n\left(\tilde{h}(\cdot/s)\right)$ is independent of s, we have

$$\bar{e}_n(v) \ge \int_{t_0}^{1} \frac{1}{2\pi s}\, \bar{e}_n\left(\tilde{h}(\cdot/s)\right) ds = \frac{1}{2\pi} \log\frac{1}{t_0}\, \bar{e}_n\left(\tilde{h}\right) ,$$

So, it is enough to consider the minimal specific energy of 3-particles $\bar{e}_3(\tilde{h})$ for $\tilde{h}(w) = wK_1(w)$, where K_1 is the modified Bessel function of second kind of order 1.

We shall need among other properties some general features of $\tilde{h}(w)$, which has been stated and proved in the Appendix A (see Fig. 1).

Because the particles interact via a pair potential, it is easy to see that the minimum potential energy is attained to a configuration in which two of the three particles have equal signs and the third has charge with the opposite sign. The potential energy (1.15) with $n = 3$ and $\sigma_1 = \sigma_3 = -\sigma_2$ is then given by

$$U_3(\zeta_1, \zeta_2, \zeta_3; \tilde{h}(\cdot/s)) = -\tilde{h}\left(|x_1 - x_2|/s\right) - \tilde{h}\left(|x_2 - x_3|/s\right) + \tilde{h}\left(|x_1 - x_3|/s\right) .$$

To simplify the expression, we write $r_1 = |x_1 - x_2|/s$, $r_2 = |x_2 - x_3|/s$ and $r_3 = |x_1 - x_3|/s$ can be written, as the particles are located at the vertices of a triangle, by the law of cosine:

$$r_3(r_1, r_2, \theta) = \sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \sin^2\theta/2} .$$

Since $\tilde{h}(w)$ is a strictly decreasing positive function, the minimal specific energy of 3-particles (2.1) thus reads

$$\begin{aligned} \bar{e}_3(\tilde{h}(\cdot/s)) &= \frac{1}{2} \min_{r_1,r_2\ge 0, 0\le\theta\le\pi} \left(\tilde{h}(r_3(r_1, r_2, \theta)) - \tilde{h}(r_1) - \tilde{h}(r_2)\right) \\ &= \frac{1}{2} \min_{r_1,r_2\ge 0} \left(\tilde{h}(r_1 + r_2) - \tilde{h}(r_1) - \tilde{h}(r_2)\right) \end{aligned} \tag{2.3}$$

and is independent on the scale s. The next proposition shows that this quantity does not attain to the value $-1/2 = \left(-\sum_{i=1}^{3}\sigma_i^2 + \left|\sum_{i=1}^{3}\sigma_i\right|\right)/(2\cdot(3-1))$ that one would expected

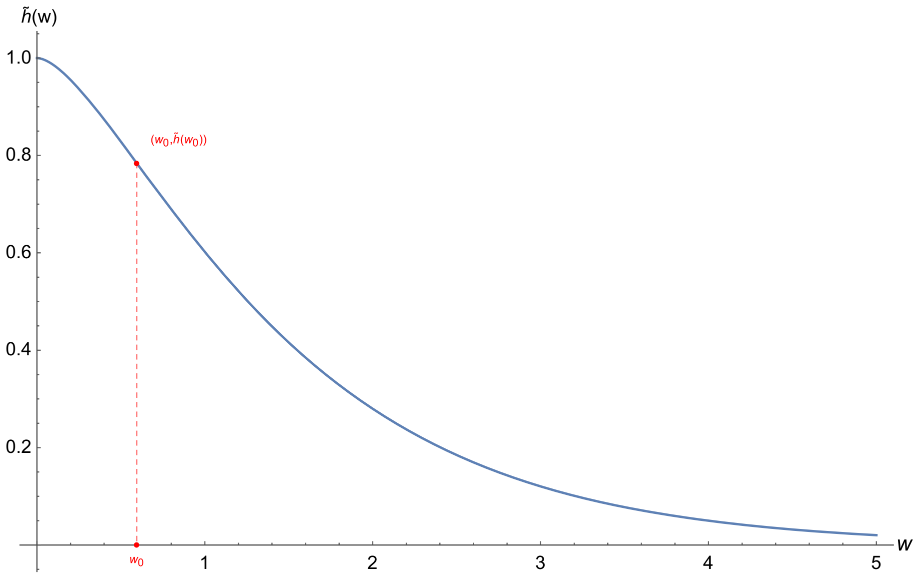

Fig. 1 Plot of $\tilde{h}(w)$

for a convex function h. Despite of that, its bound from bellow guarantee that "tripoles" do not collapse for $\beta \in [4\pi, 6\pi)$.

Proposition 2.1

$$-\frac{1}{2} > \bar{e}_3(\tilde{h}) > -0.535 . \tag{2.4}$$

Remark 2.2 As the numerical evaluations used in the proofs are sharp up to high decimal order, we may claim that $\bar{e}_3 = -0.530(\ldots)$.

Proof The function to minimize in (2.3) reaches the value $(K_1(1) - K_1(1/2))/2 = -0.527(...)$ at $r_1 = r_2 = 1/2$, proving the upper bound. To prove the lower bound, it is enough by (2.3) to show that

$$\tilde{h}(x + y) - \tilde{h}(x) - \tilde{h}(y) + 1.07 > 0 \tag{2.5}$$

holds for all $x, y \geq 0$. Defining $f(x) = \tilde{h}(x) - 1.07$, equation (2.5) is equivalent to show superadditivity of $f(x)$:

$$f(x + y) > f(x) + f(y) . \tag{2.6}$$

But this is implied by the following argument. Let $q(x) = f(x)/x$ be defined for $x > 0$ and suppose that $q(x)$ is monotone increasing. Then $q(x + y) \geq q(x)$, $q(x + y) > q(y)$ and it follows that

$$f(x + y) = xq(x + y) + yq(x + y) > xq(x) + yq(y) = f(x) + f(y) .$$

The proof of (2.5) is thus reduced to prove that $q(x) = (\tilde{h}(x) - 1.07)/x = K_1(x) - 1.07/x$ is monotone increasing. From (A.1) with $n = 1$, we deduce

$$K_1(x) + xK_1'(x) = (xK_1(x))' = -xK_0(x)$$

which implies that

$$q'(x) = K_1'(x) + \frac{1.07}{x^2} = \frac{-1}{x^2}\left(xK_1(x) + x^2K_0(x) - 1.07\right) > 0$$

for $x > 0$ provided

$$xK_1(x) + x^2K_0(x) < 1.07 .$$

This inequality, however, holds in view of Lemma A.2 in Appendix A (see Fig. 6).

The numerical estimate for the specific energy $\bar{e}_3(\tilde{h})$ stated in Remark 2.2 is obtained when 1.07 is replaced by the maximum values $p(x_0) = 1.061(\ldots)$ since, at this point, $f(x_0) = \tilde{h}(x_0) - 1.061(\ldots)$ satisfies (2.6) as an equality and consequently, by (2.5), $\tilde{h}(2x_0) - \tilde{h}(x_0) - \tilde{h}(x_0) = -1.061(\ldots)$.

The proof of the lower bound and Proposition 2.1 is now completed. □

Remark 2.3 It does not seem easy to extend the superadditivity method used to estimated the (restricted) minimum specific energy of 3-particles to $(2k+1)$-particles with $k > 1$. As we shall see in the next section, the result on the minimal specific energy $\bar{e}_3\left(\tilde{h}\right)$ prevents the third Mayer coefficient to be defined uniformly in the cutoff t_0 in the entire collapse interval $[4\pi, 8\pi]$, although it is enough for concluding convergence of the Mayer series up to the second threshold $[4\pi, 6\pi)$. Numerical calculations performed in [22] indicate that $\bar{e}_{2k+1}\left(\tilde{h}\right)$ remains for $k > 1$ strictly smaller than $-1/2$. We should mention that if $\tilde{h}(w)$ were convex, the minimal of (2.3) would be attained at $r_1 = r_2 = 0$, obtaining the expected value $\bar{e}_3 = -1/2$ as it is exactly the case when decomposition (1.4) is used. Since the method based on superadditivity cannot be easily extended to $k > 1$, a different method will be employed to obtain $\bar{e}_{2k+1}(h) = -1/2$ with h the Euclid's hat function (1.5).

2.2 The Main Theorem

We shall now turn to the representation of Yukawa potential (1.3) given by $v(x) = v_{(0,\infty)}(x) = K_0(|x|)/(2\pi)$ where (see (1.4)):

$$v_{(t_0,t)}(x) = \int_{t_0}^{t} h(|x|/s)g(s)ds , \tag{2.7}$$

is a scale mixtures of Euclid's hat (see Fig. 2). Here, for $x \in \mathbb{R}^2$ and $s \in \mathbb{R}_+$,

$$h(|x|/s) = \frac{4}{\pi s^2}\chi_{[0,s/2]} * \chi_{[0,s/2]}(x) \tag{2.8}$$

is the self-convolution of indicator function $\chi_{[0,s/2]}(x) := \theta(s/2 - |x|)$ of the 2-dimensional disc $B_r \equiv B_r(0)$ of radius $r = s/2$ centered at origin and $g(s)$ is the scale mixtures density given by Hainzl–Seiringer's formula: [23]

$$g(s) = \frac{-s}{4\pi}\int_s^{\infty} K_0'''(r)\frac{r}{\sqrt{r^2 - s^2}}dr . \tag{2.9}$$

Remark 2.4 (2.9) differs from the $g(s)$ in equation (11) of [23] by a pre-factor $\pi\ (s/2)^2$ that we have used in (2.8) in order to normalize h at origin: $h(0) = 1$. This normalization is suitable when the radial function $\varphi(|x|) = v(x)$ is the characteristic function of a spherically symmetric probability distribution in $\mathbb{R}^d$ or the covariance of a stationary and isotropic random field on d-dimensional Euclidean space. The latter is the point of view of the present paper, while

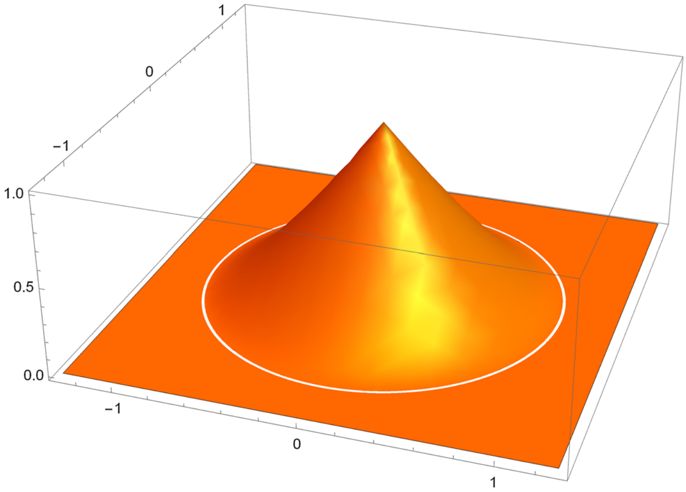

Fig. 2 Euclid's hat function

the former were the focus of Gneiting paper [19], for which the classes H_d of radial positive definite functions generated by scale mixtures of d-dimensional Euclid's hat $h_d(|x|)$ played an important role in the proof of an analogue of Pólya's criterion for $d > 1$. We observe however that the scale mixture used in [19] is of the form $\varphi(|x|) = \int_0^\infty h_d(r\,|x|)dG(r)$, where $G(r)$ is a probability distribution function in $(0, \infty)$ with $G(0+) = c \in [0, 1]$. Since $\int_{t_0}^t g(s)ds$ diverges logarithmically as t_0 tends to 0, our measure $g(s)ds$, written in terms of $r = 1/s$ (see (2.7)), does not satisfy the properties of $dG(r)$.

Equation (2.9) can be written in terms of a Meijer G-functions that is regular at $s = 0$ as

$$2\pi s g(s) = \sqrt{\pi} G_{13}^{30}\left(s^2/4 \,\middle|\, {1/2 \atop 0,1,2}\right) \tag{2.10}$$

as one can check using Mathematica program together with the shift property: $t^2 G_{13}^{30}\left(t \,\middle|\, {-3/2 \atop -2,-1,0}\right) = G_{13}^{30}\left(t \,\middle|\, {1/2 \atop 0,1,2}\right)$.

The general features of $h(w)$ are described in Appendix B for the reader's convenience and further use (see Fig. 3 for a comparison of $h(w)$ with $\tilde{h}(w)$). We shall now state and prove our main theorem and return afterwards to the asymptotic properties of (2.10) required for the next section.

Theorem 2.5 *For any integer $n \geq 2$, any configuration of n-particle $(\zeta_1, \ldots, \zeta_n)$, $\zeta_j = (\sigma_j, x_j) \in \{-1, 1\} \times \mathbb{R}^2$ and any $s \in \mathbb{R}_+$, the total energy with interacting potential h satisfies*

$$U_n(\zeta_1, \ldots, \zeta_n; h(\cdot/s)) = \sum_{1 \leq i < j \leq n} \sigma_i \sigma_j h\left(\left|x_i - x_j\right|/s\right) \geq -\frac{1}{2}\left(n - \left|\sum_{j=1}^n \sigma_j\right|\right). \tag{2.11}$$

Proof Since $h(0) = 1$, we add $n/2$ to the total energy in order to include the $i = j$ terms into the sum in (2.11):

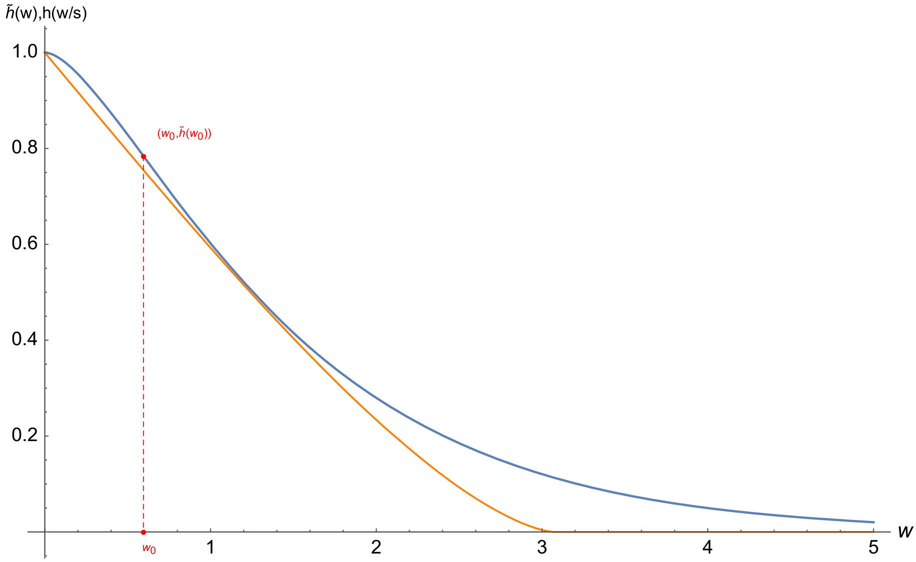

Fig. 3 Plot of $h(w/s)$ scaled by $s = 3.07$ and $\tilde{h}(w)$ together

$$\sum_{1\le i<j\le n} \sigma_i\sigma_j h\left(\left|x_i - x_j\right|/s\right) = -\frac{1}{2}\sum_{j=1}^{n}\sigma_j^2 h(0) + \frac{1}{2}\sum_{1\le i,j\le n}\sigma_i\sigma_j h\left(\left|x_i - x_j\right|/s\right) .$$

So, the result is proven if we show that

$$\sum_{1\le i,j\le n}\sigma_i\sigma_j h\left(\left|x_i - x_j\right|/s\right) \ge \left|\sum_{j=1}^{n}\sigma_j\right| .$$

Now, we use (2.8) to write

$$\sum_{1\le i,j\le n}\sigma_i\sigma_j h\left(\left|x_i - x_j\right|/s\right) = \frac{4}{\pi s^2}\int_{\mathbb{R}^2}\sum_{1\le i,j\le n}\sigma_i\sigma_j\chi_{[0,s/2]}(x_i - x_j - y)\chi_{[0,s/2]}(y)dy .$$

Changing the integration variables for each term of the sum to $z = y + x_j$ yields

$$\begin{aligned}\sum_{1\le i,j\le n}\sigma_i\sigma_j h\left(\left|x_i - x_j\right|/s\right) &= \frac{4}{\pi s^2}\int_{\mathbb{R}^2}\sum_{1\le i,j\le n}\sigma_i\sigma_j\chi_{[0,s/2]}(x_i - z)\chi_{[0,s/2]}(z - x_j)dz \\ &= \frac{4}{\pi s^2}\int_{\mathbb{R}^2}\left(\sum_{j=1}^{n}\sigma_j\chi_{[0,s/2]}(z - x_j)\right)^2 dz \end{aligned} \tag{2.12}$$

in view of the fact that $\chi_{[0,s/2]}(x)$ is even. Since $\sum_{j=1}^{n}\sigma_j\chi_{[0,s/2]}(z - x_j)$ is always an integer number, we have

$$\left(\sum_{j=1}^{n}\sigma_j\chi_{[0,s/2]}(z - x_j)\right)^2 \ge \left|\sum_{j=1}^{n}\sigma_j\chi_{[0,s/2]}(z - x_j)\right|$$

and this together with (2.12) implies that

$$\begin{aligned}\sum_{1\le i,j\le n}\sigma_i\sigma_j h\left(\left|x_i-x_j\right|/s\right) &\ge \frac{4}{\pi s^2}\int_{\mathbb{R}^2}\left|\sum\nolimits_{j=1}^n \sigma_j\chi_{[0,s/2]}(z-x_j)\right|dz\\ &\ge \left|\frac{4}{\pi s^2}\int_{\mathbb{R}^2}\sum\nolimits_{j=1}^n \sigma_j\chi_{[0,s/2]}(z-x_j)dz\right|\\ &= \left|\sum\nolimits_{j=1}^n \sigma_j\frac{4}{\pi s^2}\int_{\mathbb{R}^2}\chi_{[0,s/2]}(z-x_j)dz\right|\\ &= \left|\sum\nolimits_{j=1}^n \sigma_j\right| ,\end{aligned}$$

concluding the proof. □

Remark 2.6 The proof of Theorem 2.5 holds for any $d \ge 2$ provided $h(w)$ is replaced by the Euclid's hat $h_d(w)$ (see Sect. 2 of [19] for the proof of Proposition B.1 for d-dimensional Euclid's hat).

Theorem 2.5 implies the following

Corollary 2.7 *The minimal specific energy $e(h(\cdot/s))$ and the minimal constrained specific energy $\bar{e}(h(\cdot/s))$, defined by (1.18) and (1.19), are both $-1/2$.*

Proof This result follows from the definitions (2.1) and (2.2) and the inequality (2.11). The minimal specific energy $e(h) = \inf_{n\ge2} e_n(h)$ of h is attained for even number of particles n satisfying $\sum_{j=1}^n \sigma_j = 0$ and $\sum_{j=1}^n \sigma_j^2 = n$ when they collapse to a single point since, in this case, the inequality (2.11) becomes an equality. Likewise, the constrained minimal specific energy $\bar{e}(h) = \inf_{n\ge2} \bar{e}_n(h)$ of h is attained for odd number of particles n satisfying $\left|\sum_{j=1}^n \sigma_j\right| = 1$ and $\sum_{j=1}^n \sigma_j^2 = n$ when they collapse to a single point. Note that, for a calculation similar to the energy in (1.25), the potential energy (1.15) with $n = 2r + 1$, $\sigma_1 = \cdots = \sigma_r = -\sigma_{r+1} = \cdots = -\sigma_{2r+1}$ and $x_1 = \cdots = x_{2r+1} = x_0 \in \mathbb{R}^2$, is given by $(h(0) = 1)$

$$U_n(\zeta_1,\ldots,\zeta_n;h) = -(r+1)r + \frac{r(r-1)}{2} + \frac{(r+1)r}{2} = -r = \frac{-1}{2}(n-1) .$$

□

As $e(h(\cdot/s))$ and $\bar{e}(h(\cdot/s))$ do not depend on the scale s , Theorem 2.5 immediately implies

Corollary 2.8 *Let the Yukawa potential v be represented as scale mixtures of Euclid's hat (2.7) regularized at short distances and let the potential energy of n-particles be defined by (1.15). If n is odd, then the stability bound (1.16) can be replaced by (1.23) with*

$$B = \bar{B} = \frac{1}{2}\int_{t_0}^{t} g(s)ds .$$

2.3 Properties of the Mixture Function

Regarding the mixture function, we have the following (see Figs. 4 and 5)

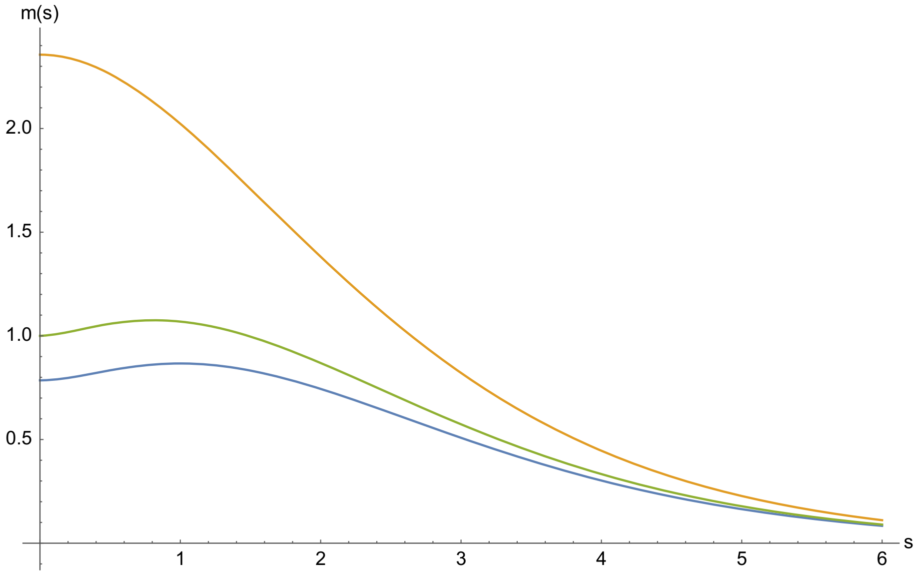

Fig. 4 Plot of $m(s)$ together with its upper and lower functions

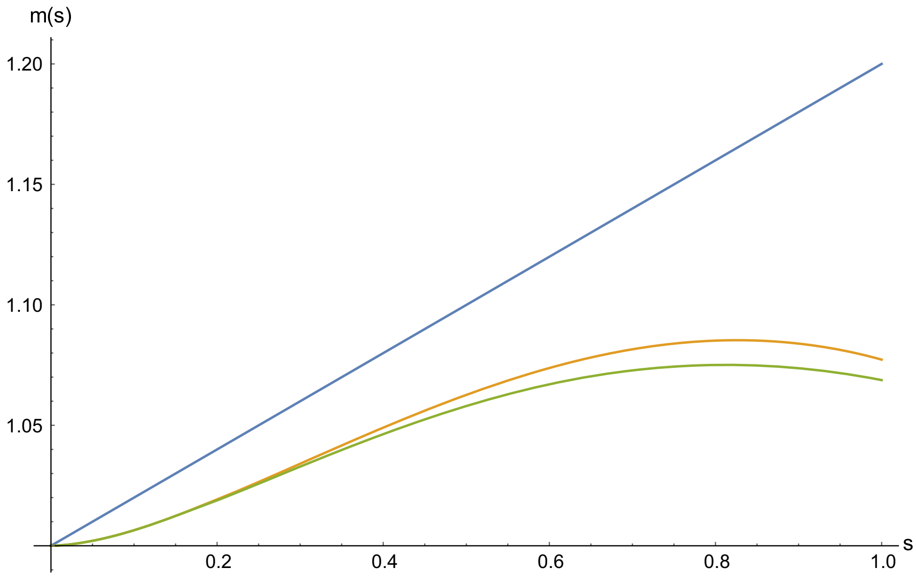

Fig. 5 Plot of $m(s)$ together with its best and linear (upper) asymptotes

Proposition 2.9 *The function* $g : (0, \infty) \longrightarrow (0, \infty)$ *given by (2.9) can be written as*

$$g(s) = \frac{1}{2\pi s} m(s)$$

where

$$m(s)=\frac{1}{2}\int_s^\infty y^2K_1(y)\frac{y}{\sqrt{y^2-s^2}}dy \tag{2.13}$$

is a regular function such that $m(0)=1$*, increases monotonously in* $(0,s_0)$*, where* $s_0=0.812(\ldots)$ *and* $m(s_0)=m_0=1.075(\ldots)$*, then decreases monotonously in* (s_0,∞) *to* 0*, exponentially fast. Globally, it is bounded from above and from below as*

$$\frac{\pi}{4}e^{-s}(1+s+s^2)<m(s)<\frac{\pi}{4}e^{-s}(3+3s+s^2)\,,\qquad \forall\, s\in[0,\infty)\,. \tag{2.14}$$

In the vicinity of the origin, it satisfies

$$m(s)\le 1+\left(a-\frac{1}{4}\log s\right)s^2,\qquad s\in[0,1] \tag{2.15}$$

where $a=(1-3\gamma+\log 4-\psi(-1/2))/8=0.07726(\ldots)$*, being the r.h.s. of (2.15) asymptotic to* $m(s)$ *at* $s=0$*.*

Proof We begin by showing that (2.9) multiplied by $2\pi s$ can be written as (2.13). For this, we use $K_1(w)=-K_0'(w)$ and the representation (see [18, Sect. 7.2])

$$K_0(w)=\int_0^\infty e^{-w\sqrt{k^2+1}}\frac{dk}{\sqrt{k^2+1}}\,, \tag{2.16}$$

from which we infer that K_0 is regular in $(0,\infty)$. We may thus differentiate (2.16) three times, replace it into (2.9), switch the integration order and, after multiplying by $2\pi s$ it can be written as

$$m(s)=\int_0^\infty\left(k^2+1\right)F(s,k)dk \tag{2.17}$$

where

$$\begin{aligned}F(s,k)&=\frac{s^2}{2}\int_s^\infty e^{-r\sqrt{k^2+1}}\frac{rdr}{\sqrt{r^2-s^2}}\\&=\frac{s^3}{2}\int_0^\infty e^{-s\sqrt{k^2+1}\sqrt{z^2+1}}dz\\&=-\frac{s^3}{2}K_0'\left(s\sqrt{k^2+1}\right)=\frac{s^3}{2}K_1\left(s\sqrt{k^2+1}\right)\,.\end{aligned} \tag{2.18}$$

We have changed variable $sz=\sqrt{r^2-s^2}$, so $r=s\sqrt{z^2+1}$ and $rdr/\sqrt{r^2-s^2}=sdz$. Replacing (2.18) back into (2.17), making one more change of variable: $s\sqrt{k^2+1}=y$, so that $sk=\sqrt{y^2-s^2}$ and $sdk=y/\sqrt{y^2-s^2}$, yields (2.13).

The sequence of operations bringing (2.9) into the form (2.13) will be applied some more times. Let us start by finding a lower bound for (2.13). By monotonicity of the modified Bessel functions with respect to their order (see [11]) together with partial integration, we have

$$\begin{aligned}m(s)&>\frac{1}{2}\int_s^\infty y^2K_0(y)\frac{y}{\sqrt{y^2-s^2}}dy\\&=\frac{-1}{2}\int_s^\infty\left(y^2K_0(y)\right)'\sqrt{y^2-s^2}dy\equiv L(s)-J(s)\end{aligned} \tag{2.19}$$

where

$$L(s)=\frac{1}{2}\int_s^\infty y^2K_1(y)\sqrt{y^2-s^2}dy=\frac{\pi}{4}\left(3+3s+s^2\right)e^{-s} \tag{2.20}$$

$$J(s) = \int_s^\infty y K_0(y)\sqrt{y^2 - s^2}dy = \frac{\pi}{2}(1+s)\, e^{-s} \,. \tag{2.21}$$

as we shall prove in Appendix C. Observe that the boundary term in the partial integration, $y^2 K_0(y)\sqrt{y^2-s^2}/2\Big|_{y=s}^{\infty}$ vanishes for all $s \in (0,\infty)$ because the exponential decay of $K_0(y)$ and boundedness of $y^2 K_0(y)$.

Equations (2.20) and (2.21) replaced into (2.19) gives the lower bound (2.14). An upper bound is obtained similarly. By monotonicity of the modified Bessel functions with respect to their order (see [11]) and integration by parts, we have

$$\begin{aligned} m(s) &< \frac{1}{2}\int_s^\infty y^2 K_2(y)\frac{y}{\sqrt{y^2-s^2}}dy \\ &= \frac{-1}{2}\int_s^\infty \left(y^2 K_2\right)'(y)\sqrt{y^2-s^2}dy = L(s) \end{aligned} \tag{2.22}$$

by (A.1), where $L(s)$ is given as in (2.20). Equation (2.22) together with (2.20) gives the upper bound (2.14).

The asymptotic behavior (2.15) of $m(s)$ follows from the mean value theorem

$$m(s) - m(0) = \int_0^s m'(t)dt = m'(\tilde{s})s \tag{2.23}$$

for some $\tilde{s} = \tilde{s}(s) \in [0,s]$ depending on s. The value $m(0)$ may be calculated using the integral representation (2.16) for $K_1(y) = -K_0'(y)$ and Fubini's theorem:

$$m(0) = \int_0^\infty y^2 K_1(y)dy = \int_0^\infty \left(\int_0^\infty y^2 e^{-y\sqrt{k^2+1}}dy\right)dk = -\int_0^\infty \frac{1}{\left(k^2+1\right)^{3/2}}dk = 1\,.$$

To calculate the derivative of $m(s)$ we apply partial integration twice, before and after the derivative with respect to s:

$$\begin{aligned} m(s) &= \frac{-1}{2}\int_s^\infty \left(y^2 K_1(y)\right)'\sqrt{y^2-s^2}dy \\ &= \frac{-1}{2}\int_s^\infty \left(yK_1(y) - y^2 K_0(y)\right)\sqrt{y^2-s^2}dy\,, \end{aligned}$$

by $(y\cdot(yK_1))' = yK_1 + y\,(yK_1)'$ together with (A.1). We continue

$$\begin{aligned} m'(s) &= \frac{s}{2}\int_s^\infty \left(K_1(y) - yK_0(y)\right)\frac{y}{\sqrt{y^2-s^2}}dy \\ &= \frac{-s}{2}\int_s^\infty \left(K_1(y) - yK_0(y)\right)'\sqrt{y^2-s^2}dy \equiv \frac{s}{2}\left(M(s)+N(s)\right) \end{aligned} \tag{2.24}$$

where, by $K_1 + yK_1' = (yK_1)' = -yK_0$ we have $-K_1' = K_1/y + K_0$ and

$$M(s) = \int_s^\infty K_1(y)\frac{\sqrt{y^2-s^2}}{y}dy \le \int_s^\infty K_1(y)dy = \frac{s}{2}K_0(s) \tag{2.25}$$

in view of the inequality $\sqrt{y^2-s^2}/y \le 1$ for $s \le y < \infty$, $-K_0'(y) = K_1(y) > 0$ and the fundamental theorem of calculus;

$$N(s) = \int_s^\infty \left(2yK_0(y) - y^2 K_1(y)\right)\frac{\sqrt{y^2-s^2}}{y}dy$$

$$\leq \int_s^\infty \left(2yK_0(y) - y^2K_1(y)\right) dy = -s^2K_0(s) \leq 0 \,, \tag{2.26}$$

the first inequality holds provided $s \in [0, 1]$ and the second equality follows from $2yK_0(y) - y^2K_1(y) = \left(y^2K_0(y)\right)'$. The boundary terms in both partial integrations vanish. We observe that

$$K_0(s) = -\left.\frac{\partial I_\nu(s)}{\partial \nu}\right|_{\nu=0} = -\log(s/2)\sum_{n=0}^\infty \frac{(s/2)^{2n}}{(n!)^2} + \sum_{n=0}^\infty \frac{(s/2)^{2n}}{(n!)^2}\psi(1+n)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and so, $K_0(s) = -\log(s/2) - \gamma + O(s^2)$ where $\gamma = -\psi(1)$ is the Euler-Mascheroni constant.

To obtain (2.15) and conclude the proof of Proposition 2.9 , we need to optimize the choice of $\tilde{s}(s)$ in (2.23). So far, by (2.23), (2.24), (2.25) and (2.26) we have

$$\begin{aligned} m(s) &\leq 1 + \frac{1}{2}\int_0^s tK_0(t)ds \\ &= 1 + \frac{1}{2}(1 - sK_1(s)) \end{aligned} \tag{2.27}$$

by Proposition A.1 and this upper bound is asymptotic as s tends to 0: $m(s) = 1 + O\left(s^2\right)$ the s^2 order term in the upper bound is $(1 - 2\gamma - 2\log(s/2))/8 = 0.1539(\dots) - (\log s)/4$. The best upper bound up to $O\left(s^2\right)$ term is, however, stated in Proposition 2.9, given by the asymptotic expansion of (2.10), calculated algebraically by the software Mathematica. □

3 Majorant of the Density Function

3.1 Set Up and Ingredients

Let $(\Omega, \mathscr{B}, \varrho)$ denote the (translational invariant) σ-finite measure space on $\{-1, 1\} \times \mathbb{R}^2$; the set Ω corresponds to the possible configurations of a single particle (we united σ and x into $\zeta = (\sigma, x)$) and $\int d\varrho(\zeta)\,\cdot\; = 1/2 \sum_{\sigma\in\{-1,1\}} \int_{\mathbb{R}^2} d^2x\;\cdot$ denotes the integration with respect to ρ. Let

$$\beta p(\beta, z) = \sum_{n\geq 1} \frac{z^n}{n!} \int d\varrho(\zeta_2)\cdots d\varrho(\zeta_n)\psi_n^c(\zeta_1, \dots, \zeta_n; \beta v) \tag{3.1}$$

be the pressure of the Yukawa gas in the infinite volume limit, where v is the Yukawa potential regularized at short distances $s \leq t_0$, given by the scale decomposition (1.10). We observe that, as v decays exponentially fast at infinity and has its singularity at origin removed, the finite volume pressure p_{Λ_i}, defined for any increasing and absorbing sequence $(\Lambda_i)_{i\geq 1}$ of squares with $\lim_i \Lambda_i = \mathbb{R}^2$, converges by standard methods (see e.g. [31]) to the expression (3.1).

The density function $\rho(\beta, z) = z\partial p/\partial z(\beta, z)$ as a series in power of the activity z reads

$$\frac{\beta}{z}\rho(\beta, z) = \sum_{n\geq 1} nb_n z^{n-1} \tag{3.2}$$

where $b_1 = 1$ and, for $n > 1$,

$$b_n = \frac{1}{n!}\int d\varrho(\zeta_2)\cdots d\varrho(\zeta_n)\psi_n^c(\zeta_1, \dots, \zeta_n; \beta v)$$

is the so called n-th Mayer coefficient in the infinite volume limit. Note that $\beta\rho(\beta, z)/z = 1$ is the equation of state of an ideal gas and the series (3.2) provides corrections about it to all orders in terms of the Ursell (cluster) functions ψ_n^c.

Definition 3.1 A formal power series in z

$$\Theta^*(\beta, z) = \sum_{n\geq 1} C_n^* z^{n-1}$$

is said to be a majorant of $\beta\rho(\beta, z)/z$ if the $C_n^* = C_n^*(\beta)$ are nonnegative and

$$n\,|b_n| \leq C_n^*$$

holds for all $n \in \mathbb{N}$. We write

$$\beta\rho/z \ll \Theta^*$$

for the majorant relation.

It follows that, if the series $\Theta^*(z)$ converges on the open disc $D(r) := \{z \in \mathbb{C} : |z| < r\}$ for some $r > 0$, then $\rho(\beta, z)$ is holomorphic function of z on the same disc. The largest r provides a lower bound on the radius of convergence of the Mayer series (3.2) and (1.24).

Our construction of majorants combines (multi)scale decomposition of v together with some basic ingredients. Beginning with the following

Lemma 3.2 *Let $a = (a_n)_{n\geq 1}$, $b = (b_n)_{n\geq 1}$, $\tilde{a} = (\tilde{a}_n)_{n\geq 1}$ and $\tilde{b} = \left(\tilde{b}_n\right)_{n\geq 1}$ be positive numerical sequences (a, b, $\tilde{a}$ and $\tilde{b} > 0$) such that $\tilde{a} - a$ and $\tilde{b} - b$ are both positive sequences (i.e., $\tilde{a}_n - a_n > 0$ and $\tilde{b}_n - b_n > 0$ hold for all $n \geq 1$). Let the convolution product $e = c * d$ and the pointwise product $f = c \cdot d$ of two sequences $c = (c_n)_{n\geq 1}$ and $d = (d_n)_{n\geq 1}$ be defined by the sequences $e = (e_n)_{n\geq 1}$ and $f = (f_n)_{n\geq 1}$ where $e_1 = 0$ and*

$$e_n = \sum_{k=1}^{n-1} c_k d_{n-k} \ , \qquad n \geq 2$$

and

$$f_n = c_n d_n \ , \qquad n \geq 1 \ .$$

*Then, **(i)** $\tilde{a} \cdot \tilde{b} - a \cdot b > 0$; **(ii)** $\tilde{a} * \tilde{b} - a * b > 0$; in particular **(iii)** $\tilde{a} * \tilde{a} - a * a > 0$ and $\tilde{b} * \tilde{b} - b * b > 0$ hold.*

Proof The conclusions **(i)**, **(ii)** and **(iii)** follow immediately from the following elementary inequality: If a, b, c and d are positive numbers such that $a - c$ and $b - d$ are positive, then

$$\begin{aligned} ab - cd &= ab - \frac{1}{2}(ad + bc) - \left(cd - \frac{1}{2}(ad + bc)\right) \\ &= \frac{1}{2}\left((a + c)(b - d) + (a - c)(b + d)\right) > 0 \ . \end{aligned} \tag{3.3}$$

For **(i)** each element of the sequence is of the form (3.3); for **(ii)** and **(iii)** each element of the sequence is a sum of terms of the form (3.3). □

Remark 3.3 The statements of Lemma 3.2 hold true if the assumption of positivity is replaced by nonnegativity.

Now, using the scale decomposition (1.10), the Ursell function in (3.1) is defined by a scaling limit

$$\psi_n^c(\zeta_1, \ldots, \zeta_n; \beta v) = \lim_{t\to\infty} \psi_n^c(t, \zeta_1, \ldots, \zeta_n; \beta v(t, \cdot)) \tag{3.4}$$

where $\psi_n^c(t, \zeta_1, \ldots, \zeta_n; \beta v(t, \cdot)) \equiv \psi^c(t, \zeta_{\{1,\ldots,n\}})$ is the unique solution of the infinite system of ordinary differential equations for $f_I = f_I(t) \equiv f(t, \zeta_I)$, where $\zeta_I = \left(\zeta_{i_1}, \ldots, \zeta_{i_k}\right)$ is the set of variables indexed by $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ and $n \in \mathbb{N}$: (see Lemma 3.3 of [8])

$$\dot{f}_I = -\sum_{i,j\in I,\ i<j} \beta\dot{v}_{ij}(t) f_I - \frac{1}{2}\sum_{J\subset I}\sum_{i\in J,\ j\in I\setminus J} \beta\dot{v}_{ij}(t) f_J f_{I\setminus J} \tag{3.5}$$

with (ideal gas) initial condition[1]

$$f_I(t_0) = \begin{cases} 1 \text{ if } & |I| = 1 \\ 0 \text{ otherwise} & \end{cases} .$$

Here $\dot{v}_{ij}(t) \equiv \dot{v}(t, \zeta_i, \zeta_j) = \sigma_i\sigma_j g(t) h(\left|x_i - x_j\right|/t)$ so, as $\dot{v}(t, \zeta_i, \zeta_j)$ is a measurable and translational invariant function on the 2-particle configuration space, $\psi^c(t, \zeta_I)$ is a measurable and translational invariant function on the k-particle configuration space $\left(\{-1, 1\} \times \mathbb{R}^2\right)^k$.

By the variation of constants formula the system of equations (3.5) is equivalent to a system of integrable equations: $f_I(t) = 1$ if $|I| = 1$ and

$$f_I(t) = \frac{-1}{2}\int_{t_0}^{t} \exp\left(-\sum_{i,j\in I,\ i<j}\int_s^t \beta\dot{v}_{ij}(\tau)\, d\tau\right)\sum_{J\subset I}\sum_{i\in J,\ j\in I\setminus J} \beta\dot{v}_{ij}(s) f_J(s) f_{I\setminus J}(s)\, ds\ , \tag{3.6}$$

if $|I| > 1$, which will be useful to our application.

3.2 Majorant Construction for $\beta < 4\pi$

Using (3.6), Guidi one of the authors have proven in [22] (see Theorem 2.2 and Eqs. (4.10)–(4.12) therein) the following

Proposition 3.4 *Let $\Theta = \Theta(s, z)$ be the classical solution of*

$$\Theta_t = \Gamma(z^2\Theta^2)_z + B\left((z\Theta)_z - 1\right)\ , \qquad (s, z) \in (t_0, \infty) \times \mathbb{R}_+ \tag{3.7}$$

with $\Theta(t_0, z) = 1$ for all $z \geq 0$, where by (1.10), (1.5) and explicit calculation, $\Gamma = \Gamma(s) = \|\beta\dot{v}(s, \cdot)\|_1$ and $B = B(s) = |\beta\dot{v}(s, 0)|/2$ are given by (see Proposition B.1)

$$\Gamma = \beta g(s)\int_{\mathbb{R}^2} h(|x|/s) d^2x = \frac{\beta\pi}{4} s^2 g(s) \tag{3.8}$$

and

$$B = \frac{\beta}{2} g(s)\ . \tag{3.9}$$

Then, the following majorant relations

$$\frac{\beta}{z}\rho(\beta, z) \ll \Theta(\infty, z) \ll \frac{-1}{\tau(t_0, \infty)z} W\left(-\tau(t_0, \infty)z\right) \tag{3.10}$$

[1] By (1.10), the interaction $v(t, x)$ between particles is turned off at $t = t_0$.

hold for all (β, z) *satisfying*

$$ez\tau(t_0, \infty) < 1 \tag{3.11}$$

where

$$\tau(t_0, t) = \int_{t_0}^{t} \Gamma(s) \exp\left(2\int_s^t B(\tau)d\tau\right) ds \tag{3.12}$$

and $W(x)$ *denotes the Lambert* W*-function.[12]*

Remark 3.5 The analogue of Proposition 3.4 in Subsec. 6.1 of [22] uses the scale decomposition (1.7) of v, for which $B = \beta/(4\pi s)$ and $\Gamma = 2\beta s$ can be exactly calculated (for comparison, we have set therein $\kappa(s) = 1/s^2$ for $s \in (0, 1]$). Here v is given by (1.10) whose $g(s)$ agree with the scaling function $1/(2\pi s)$ of (1.7) only asymptotically as $s \to 0$. Writing $\tau(t_0, \infty) = \tau(t_0, 1) \exp\left(\beta \int_1^\infty g(\tau)d\tau\right) + \tau(1, \infty)$ together with $0 < g(s) \le (1+s/5)/(2\pi s)$ if $0 \le s \le 1$ and $g(s) \le (3+3s+s^2)e^{-s}/(2\pi s)$ if $1 \le s < \infty$, by Proposition 2.9, for any $0 < \beta < 4\pi$ the limit

$$\begin{aligned} \lim_{t_0\to 0} \tau(t_0, 1) &= \frac{\beta\pi}{4}\int_0^1 s^2 g(s) \exp\left(\beta\int_s^1 g(\tau)d\tau\right) ds \\ &\le \frac{\beta\pi}{4} e^{\beta/10\pi} \int_0^1 \frac{1}{2\pi}\left(s^{1-\beta/2\pi} + \frac{1}{5}s^{2-\beta/2\pi}\right) ds \\ &= \frac{\beta\pi}{4} e^{\beta/10\pi}\left(\frac{1}{4\pi-\beta} + \frac{1}{5}\frac{1}{6\pi-\beta}\right) \end{aligned} \tag{3.13}$$

exists and $\exp\left(\beta \int_1^\infty g(\tau)d\tau\right)$ and $\tau(1, \infty)$ are finite since $g(s)$ decays exponentially fast as $s \to \infty$.

Remark 3.6 The existence of $\tau = \lim_{t_0\to 0} \tau(t_0, \infty)$ implies by (3.10) and (3.11) that the radius of convergence $r = \sup\{|z| : e\,|z|\,\tau < 1,\ z \in \mathbb{C}\}$ of the Mayer series (3.2) remains strictly positive. This fact is already remarkable considering that v, given by (1.10) with $t_0 = 0$, does not satisfies the stability condition (1.16) (see also (1.17)).

We include a neat short version of the original proof in [22] for the reader's convenience.

Proof of Proposition 3.4 By (3.4), (3.6) and stability (1.17), the sequence $(A_n)_{n\ge 1}$ of positive functions $A_n : [t_0, \infty) \longrightarrow \mathbb{R}$, defined by

$$A_n(t) = \frac{1}{n!}\int d\varrho(\zeta_2)\cdots d\varrho(\zeta_n)\left|\psi_n^c(\zeta_1, \ldots, \zeta_n; \beta v(t, \cdot))\right| \tag{3.14}$$

satisfies a system of integral inequality equations

$$nA_n(t) \le \frac{n}{2}\int_{t_0}^t ds \exp\left(n\int_s^t B(s')ds'\right)\Gamma(s)\sum_{k=1}^{n-1} kA_k(s)(n-k)A_{n-k}(s)\,, \quad n > 1 \tag{3.15}$$

with $A_1(t) \equiv 1$. Hence, the Mayer coefficients of the series (3.2) are majorized by

$$n\,|b_n| \le nA_n(\infty)\,. \tag{3.16}$$

If $\Theta(t, z)$ is defined by the series

$$\Theta(t, z) = 1 + \sum_{n\ge 2} C_n(t)z^{n-1} \tag{3.17}$$

where the sequence $(C_n)_{n\geq 1}$ of positive functions $[t_0,\infty)\ni t\longmapsto C_n(t)\in\mathbb{R}_+$ satisfies equations (3.15) for $(nA_n)_{n\geq 1}$ as **equalities**, then $A_1(t)=C_1(t)\equiv 1$ and

$$nA_n(t)\leq C_n(t)\ ,\qquad n\geq 2 \text{ and } t\geq t_0\ . \tag{3.18}$$

It can be shown (see [22, Sect. 4]) that (3.17) satisfies the quasi-linear first order PDE (3.7). So, the first majorant relation of (3.10) holds and all one needs to determine is a domain in $(t_0,\infty)\times\mathbb{R}_+$ for which the classical solution of (3.7) exists. Observe that (3.7) can be written as a system of first order differential equations for the coefficients $(C_n)_{n\geq 1}$. For this, by (3.17), we have

$$\begin{aligned}\Theta_t &= \sum_{n\geq 1} C_n' z^{n-1}\\ (z\Theta)_z &= \sum_{n\geq 1} nC_n z^{n-1}\\ (z^2\Theta^2)_z &= \sum_{n\geq 2} n\left(\sum_{k=1}^{n-1} C_k C_{n-k}\right) z^{n-1}\ .\end{aligned} \tag{3.19}$$

Substituting these series back into the equation, yields

$$C_n' = nBC_n + \frac{n\Gamma}{2}\sum_{k=1}^{n-1} C_k C_{n-k}\ ,\qquad n>1 \tag{3.20}$$

with $C_1(t)\equiv 1$, $t\in[t_0,\infty)$, and initial data $C_n(t_0)=0$ for all $n\geq 2$.

The first non-trivial equation for $n=2$,

$$C_2' = 2BC_2 + \Gamma \tag{3.21}$$

with $C_2(t_0)=0$, has a unique solution, $\tau(t_0,t)$ given by (3.12), which can be written as

$$C_2(t) = f_1(t)\int_{t_0}^t \Gamma_1(s)ds$$

where $\Gamma_1(s)=\Gamma(s)/f_1(s)$ and

$$f_1(t) = \exp\left(2\int_{t_0}^t B(\tau)d\tau\right)$$

is an integrating factor of (3.21). As we shall see $C_2(t)=\tau(t_0,t)$ determines the radius of convergence of the series (3.17) for Θ:

$$e\,|z|\,\tau(t_0,t) < 1, \tag{3.22}$$

uniformly in t_0, for $\beta<\beta_2$, $0<t_0<t<\infty$, where $\beta_2=4\pi$ is the first threshold (see Remark 3.5). For this, let $\left(C_n^{(1)}(t)\right)_{n\geq 1}$ be a sequence of positive functions defined by

$$\Psi(t,w) = \Theta\left(t, w/f_1(t)\right) = 1 + \sum_{n\geq 2} C_n^{(1)} w^{n-1}\ . \tag{3.23}$$

Since $C_n^{(1)} = C_n/f_1^{n-1}$ and

$$C_n^{(1)\prime} = \frac{C_n'}{f_1^{n-1}} - (n-1)\frac{f_1'}{f_1}\,\frac{C_n}{f_1^{n-1}}$$

$$= \frac{C_n'}{f_1^{n-1}} - 2(n-1)B\,\frac{C_n}{f_1^{n-1}}\,,$$

equation (3.20), in terms of the new $C_n^{(1)}$'s, reads[2]

$$C_n^{(1)\prime} = -(n-2)BC_n^{(1)} + \frac{n\Gamma_1}{2}\sum_{k=1}^{n-1} C_k^{(1)} C_{n-k}^{(1)}\,, \qquad n > 1 \tag{3.24}$$

with $C_1^{(1)}(t) \equiv 1$ and initial data $C_n^{(1)}(t_0) = 0$ for all $n \geq 2$. Since the coefficient $-(n-2)B$ of the linear term is nonpositive for all $n \geq 2$, the solution of the above initial value problem (IVP) can, in turn, be majorized by another sequence $\left(\tilde{C}_n^{(1)}\right)_{n\geq 1}$:

$$C_n^{(1)}(t) \leq \tilde{C}_n^{(1)}(t) \tag{3.25}$$

which solves the IVP

$$\tilde{C}_n^{(1)\prime} = \frac{n\Gamma_1}{2}\sum_{k=1}^{n-1} \tilde{C}_k^{(1)} \tilde{C}_{n-k}^{(1)}\,, \qquad n > 1$$

with $\tilde{C}_1^{(1)}(t) \equiv 1$ and initial data $\tilde{C}_n^{(1)}(t_0) = 0$ for all $n \geq 2$.

Proof of (3.25) Using the notation introduced in Lemma 3.2, we write $a = (a_n)_{n\geq 1}$ and $b = (b_n)_{n\geq 1}$ with $a_1 = b_1 \equiv 0$, $a_n(t) = (n-2)B(t)$ and $b_n(t) = n\Gamma_1(t)/2$ for $n > 1$. The difference sequence $\Delta = (\Delta_n)_{n\geq 1}$, given by $\Delta_1 \equiv 0$ and $\Delta_n(t) = \tilde{C}_n^{(1)}(t) - C_n^{(1)}(t)$ for $n > 1$, thus satisfies

$$\Delta' = a \cdot \Delta + b \cdot \left(\tilde{C}^{(1)} * \tilde{C}^{(1)} - C^{(1)} * C^{(1)}\right)\,.$$

Let us assume that (3.25) holds for some $t \geq t_0$. Then, by Lemma 3.2 we have $\Delta'(t) \geq 0$ which, together with $\Delta(t_0) \equiv 0$, implies that $\Delta(t) \geq 0$. Consequently, (3.25) holds for all $t \geq t_0$. □

The proof of Proposition 3.4 is now resumed. It is shown in Sec. 5 of [22] that the power series analogous to (3.23): $\tilde{\Psi}(t, w) = 1 + \sum_{n\geq 2} \tilde{C}_n^{(1)} w^{n-1}$ satisfies an equation given by (3.7) setting $B = 0$, $\Gamma = \Gamma_1$ and together with $\tilde{\psi}(t_0, z) \equiv 1$ has by the method of characteristics the classical solution

$$\tilde{\Psi}(t, w) = \frac{-1}{\tilde{\tau}_1(t_0, t)w} W\left(-\tilde{\tau}_1(t_0, t)w\right)$$

provided $e\,|w|\,\tilde{\tau}_1(t_0, t) < 1$ holds, where $\tilde{\tau}_1(t_0, t) = \int_{t_0}^{t} \Gamma_1(s)ds$. Here, $W(x)$ denotes the Lambert W-function, implicitly defined by $We^W = x$, whose Taylor series about $x = 0$ (see Lagrange–Bürmann theorem [13]): $W(x) = \sum_{n\geq 1} (-n)^{n-1} x^n/n!$ converges for $|x| < 1/e$, including the branching point at $x = -1/e$ (see e.g. [12]).

[2] The idea alluded in the introduction (see paragraph of Eq. (1.26)), of balancing large self-energies $B(s) = \dot{v}(s)/2$ against small norms $\Gamma(s) = \|\beta\dot{v}(s)\|_1$ is implemented in the majorant series through the new coefficients. The choice that optimizes the interval $[0, \beta_2)$ for which the series converge redefines the particle activity by the corresponding integrating factor of equation satisfied by C_2.

Joining equations (3.16), (3.18) and (3.25) together, we conclude

$$\Theta(t,z)=\Psi(t,w)\ll\tilde{\Psi}(t,w)=\frac{-1}{\tau_1(t_0,t)z}W\left(-\tau(t_0,t)z\right)$$

with $\tau_1(t_0,t)$ given by (3.12), establishing the second majorant relation of (3.10). □

Note that, since $f_1(t)>1$ for any t_0 and t fixed, the radius of convergence of the majorant series $\Theta(t,z)=\Psi(t,w)$ is smaller in z than in w variable. However, in view of Remarks 3.5 and 3.6, it remains strictly positive when the cutoff t_0 is removed provided $\beta<\beta_2$ where $\beta_2=4\pi$ is the first threshold.

3.3 Majorant Construction for β Inside the Threshold Intervals I_n

The procedure of finding a majorant series for the density function can be extended for the inverse temperature β in the threshold interval $I_n=[\beta_{2n},\beta_{2(n+1)})$, $n\in\mathbb{N}$ where, for convenience, we write $\beta_{k+1}=8\pi\left(1-1/(k+1)\right)=8\pi k/(k+1)$. We shall present an scheme of three stages which holds for any threshold intervals.

1st stage. For $\beta\in I_{(k-1)/2}$ where $k>1$ is an odd number, assuming that neutral subclusters of size less than k do not collapse (see Claim 1.2), we remove from the system any (odd or even) clusters of size $<k$. The removal of the clusters, which prevents the corresponding terms in the Mayer expansion to increase as t_0 goes to 0, is implemented in the majorant series through Lagrange multipliers added to equation (3.7). The Lagrange multipliers $(L_k)_{k\geq 0}$ are given by the Cesàro mean of the Taylor series of Θ around $z=0$, i.e., L_k is the arithmetic mean of its first k Taylor polynomials. Our choice is optimal in the sense that it subtracts from the linear term of (3.20) an exact amount, allowing the solution of (3.7) be majorized by a series with positive radius of convergence, uniformly on t_0 for $\beta<\beta_{k+1}$.

The following proposition extends the second majorant relation of (3.10). Its original proof in Sec. 7 of [22]) is included for the reader's benefit.

Proposition 3.7 *For any $k\in\mathbb{N}$, let $\Theta=\Theta(t,z)$ be the classical solution of*

$$\Theta_t=\Gamma(z^2\Theta^2)_z+B\left((z\Theta)_z-L_k\right)\ ,\qquad (t,z)\in(t_0,\infty)\times\mathbb{R}_+ \tag{3.26}$$

with $\Theta(t_0,z)=1$ for all $z\geq 0$, where $\Gamma=\Gamma(t)=\|\beta\dot{v}(t,\cdot)\|_1$ and $B=B(t)=|\beta\dot{v}(t,0)|/2$ are given in Proposition 3.4 and

$$L_k=L_k(t)=1+\sum_{j=1}^{k-1}\left(1-\frac{j}{k}\right)\frac{1}{j!}z^j\,\Theta_{\underbrace{z\cdots z}_{j-\text{times}}}(t,0)$$

is a Lagrange multiplier. Then, the following majorant relation

$$\Theta(t,z)\ll\frac{-1}{\tau_k(t_0,t)z}W\left(-\tau_k(t_0,t)z\right) \tag{3.27}$$

holds for all (β,z) satisfying

$$ez\tau_k(t_0,\infty)<1$$

where

$$\tau_k(t_0,t)=\int_{t_0}^{t}\Gamma(s)\exp\left(\frac{k+1}{k}\int_s^t B(\tau)d\tau\right)ds \tag{3.28}$$

and $W(x)$ denotes the Lambert W-function.[12]

Remark 3.8 By a calculation analogous to (3.13)

$$\lim_{t_0\to 0}\tau_k(t_0,1)=\frac{\beta\pi}{4}\int_0^1 s^2 g(s)\exp\left(\frac{k+1}{k}\frac{\beta}{2}\int_s^1 g(\tau)d\tau\right)ds$$
$$\leq \frac{\beta}{16}e^{2\beta/5\beta_{k+1}}\left(\frac{1}{1-\beta/\beta_{k+1}}+\frac{1}{5}\frac{1}{2-\beta/\beta_{k+1}}\right), \tag{3.29}$$

exists for $\beta < \beta_{k+1}$ and the radius of convergence of the majorant series (3.27) is strictly positive.

Proof of Proposition 3.7 Let $\Theta(t,z)$ be defined by the series (3.17). Observe that, by

$$(z\Theta)_z - L_k=\sum_{n=2}^{k}\left(n-\frac{k-n+1}{k}\right)C_n z^{n-1}+\sum_{n\geq k+1} nC_n z^{n-1}$$

and the remaining series of (3.19), (3.26) can be written as a system of first order differential equations for $(C_n)_{n\geq 1}$:

$$C_n'=\frac{k+1}{k}(n-1)BC_n+\frac{n\Gamma}{2}\sum_{k=1}^{n-1}C_kC_{n-k}\,,\qquad 1<n\leq k \tag{3.30}$$

and (3.20) for $n>k$, with $C_1(t)\equiv 1$ and initial data $C_n(t_0)=0$ for $n\geq 2$. The equation for $n=2$

$$C_2'=\frac{k+1}{k}BC_2+\Gamma \tag{3.31}$$

with $C_2(0)=0$ has a unique solution $C_2(t)=f_k(t)\int_{t_0}^t\Gamma_k(s)ds$ where $\Gamma_k(s)=\Gamma(s)/f_k(s)$ and

$$f_k(t)=\exp\left(\frac{k+1}{k}\int_{t_0}^t B(\tau)d\tau\right) \tag{3.32}$$

is an integrating factor of (3.31), given by (3.28)

As in the proof of Proposition 3.4, we introduce a sequence $\left(C_n^{(k)}(t)\right)_{n\geq 1}$ of positive functions defined by

$$\Psi(t,w)=\Theta\left(t,w/f_k(t)\right)=1+\sum_{n\geq 2}C_n^{(k)}(t)w^{n-1}\,. \tag{3.33}$$

Since $C_n^{(k)}=C_n/f_k^{n-1}$ and

$$C_n^{(k)\prime}=\frac{C_n'}{f_k^{n-1}}-\frac{k+1}{k}(n-1)B\,\frac{C_n}{f_k^{n-1}}\,,$$

the equations (3.30) for $1<n\leq k$ and (3.20) for $n>k$ in terms of the new $C_n^{(k)}$'s read

$$C_n^{(k)\prime}=\frac{n\Gamma_k}{2}\sum_{j=1}^{n-1}C_j^{(k)}C_{n-j}^{(k)}\,,\qquad 1<n\leq k$$
$$C_n^{(k)\prime}=-\left(\frac{n-k-1}{k}\right)BC_n^{(k)}+\frac{n\Gamma_k}{2}\sum_{j=1}^{n-1}C_j^{(k)}C_{n-j}^{(k)}\,,\qquad n>k \tag{3.34}$$

with $C_1^{(k)}(t) \equiv 1$ and initial data $C_n^{(k)}(t_0) = 0$ for $n \geq 2$. Since the coefficient $-(n-k-1)B/k$ of the linear term of (3.34) is nonpositive for all $n \geq k+1$, the solution of the above IVP can be majorized by another sequence $\left(\tilde{C}_n^{(k)}\right)_{n\geq 1}$:

$$C_n^{(k)}(t) \leq \tilde{C}_n^{(k)}(t) \tag{3.35}$$

which solves the IVP

$$\tilde{C}_n^{(k)\prime} = \frac{n\Gamma_k}{2} \sum_{j=1}^{n-1} \tilde{C}_j^{(k)} \tilde{C}_{n-j}^{(k)} , \qquad n > 1 \tag{3.36}$$

with $\tilde{C}_1^{(k)}(t) \equiv 1$ and initial data $\tilde{C}_n^{(k)}(t_0) = 0$ for $n \geq 2$. For (3.35), one may apply the same proof of (3.25) based in Lemma 3.2. It follows that (see [22, Sect. 7]) the power series $\tilde{\Psi}(t, w) = 1 + \sum_{n\geq 2} \tilde{C}_n^{(k)} w^{n-1}$ satisfies (3.7) setting $B = 0$, $\Gamma = \Gamma_k$ and together with $\tilde{\psi}(t_0, z) \equiv 1$ has the classical solution

$$\tilde{\Psi}(t, w) = \frac{-1}{\tilde{\tau}_k(t_0, t)w} W\left(-\tilde{\tau}_k(t_0, t)w\right)$$

provided $e\,|w|\,\tilde{\tau}_k(t_0, t) < 1$ holds, where $\tilde{\tau}_k(t_0, t) = \int_{t_0}^{t} \Gamma_k(s)ds$ and $W(x) = \sum_{n\geq 1} (-n)^{n-1} x^n/n!$ denotes the Lambert W-function.

We thus have

$$\Theta(t, z) = \Psi(t, w) \ll \tilde{\Psi}(t, w) = \frac{-1}{\tau_k(t_0, t)z} W\left(-\tau_k(t_0, t)z\right) ,$$

concluding the proof of Proposition 3.7. □

2^{nd} stage. Non neutral clusters of size smaller or equal to k wasn't properly estimated in the majorant equation (3.7) because the stability bound (1.16) has indistinctly used in (3.15) for n odd or even. The consequences of applying the stability bound (1.23) in equation (3.15) are now exploited. The apparently insignificant improvement replaces equation (3.7) by

$$\Theta_t = \Gamma(z^2\Theta^2)_z + Bz\Theta_z \tag{3.37}$$

and the coefficient nB of the linear term of (3.20) by $(n-1)B$. To verify these statements, let the argument n of the exponential in (3.15) be replaced by $n-1$. The modified coefficients $(C_n)_{n\geq 1}$ of the power series (3.17) thus satisfy a system of integral equations

$$C_n(t) = \frac{n}{2} \int_{t_0}^{t} dse^{(n-1)\gamma(s,t)} \Gamma(s) \sum_{k=1}^{n-1} C_k(s) C_{n-k}(s) , \quad n > 1 \tag{3.38}$$

with $C_1(t) \equiv 1$ where $\gamma(s, t) = \int_s^t B(\tau)d\tau$. Summing equation (3.38) multiplied by z^{n-1} over n yields an integral equation for Θ:

$$\Theta(t, z) = 1 + \frac{1}{2} \int_{t_0}^{t} dse^{-\gamma(s,t)} \Gamma(s) \left(z^2 e^{2\gamma(s,t)} \Theta^2(s, ze^{\gamma(s,t)})\right)_z \tag{3.39}$$

and from this we deduce (3.37). Observe that an extra factor $e^{-\gamma(s,t)}$ inside the integration results from the stability improvement (1.23) and the derivative with respect to t applied to this factor produces an additional term $B(\Theta - 1)$ which has to be subtracted (due to the minus sign of the exponent) from the last term on the right hand side of (3.7): $B((z\Theta)_z - 1) - B(\Theta - 1) = Bz\Theta_z$.

Equation (3.37) yields a significant improved regarding the convergence of the Mayer series (3.2) for β inside the threshold interval $I_{(k-1)/2} = [\beta_{k-1}, \beta_{k+1})$, for all $k > 1$ odd. The proof of the following result shows that, as far as majorant coefficients C_n with $n < k$ are concerned, if neutral clusters of size smaller than k are all discarded then (3.20) may be replaced by (3.30) for any threshold interval $I_{(k-1)/2}$.

Proposition 3.9 *Let $\Theta = \Theta(t, z)$ be the classical solution of (3.37) with B and Γ as in Proposition 3.4. Then the following majorant relation*

$$\Theta(t, z) \leq \frac{-1}{\tau_k(t_0, t)z} W(-\tau_k(t_0, t)z)$$

holds for all $k \in \mathbb{N}$ and (β, z) satisfying $ez\tau_k(t_0, t) < 1$, where τ_k is given by (3.28).

Proof Let $\Theta(t, z)$ be defined by the series (3.17) and observe that, by

$$z\Theta_z = \sum_{n=2}^{\infty} (n-1) C_n z^{n-1} ,$$

(3.37) can be written as a system of first order differential equations for $(C_n)_{n\geq 1}$:

$$C_n' = (n-1) BC_n + \frac{n\Gamma}{2} \sum_{k=1}^{n-1} C_k C_{n-k} , \qquad 1 < n \leq k \tag{3.40}$$

with $C_1(t) \equiv 1$ and initial data $C_n(t_0) = 0$ for $n \geq 2$. Since the coefficient $(n-1)B$ of the linear term of (3.40) is smaller than $(n-1)(k+1)B/k$ for $2 \leq n \leq k$ and smaller than nB for all $n \geq k+1$,[3] for any $k \in \mathbb{N}$, the solution of the above IVP can be majorized, in view of Lemma 3.2, by the solution of the IVP in (3.30), which by Proposition 3.7 satisfies (3.27). The proof of Proposition 3.9 is concluded. □

3.4 Stability of a Neutral Pair in the Presence of Other Particles

3rd stage. The last stage of our scheme deals with neutral subclusters of size smaller than k that are part of a cluster of size larger or equal to k. So far, we have proved the Conjecture 1.1 assuming that Claim 1.2 holds true. Let $k > 1$ be an odd number and suppose in addition to Claim 1.2 that all neutral clusters of size smaller than k have their singularities been removed. Then, the density function (3.2) satisfies

$$\frac{\beta}{z} |\rho(\beta, z)| \ll \frac{-1}{\tau_k(t_0, \infty)z} W(-\tau_k(t_0, \infty)z) , \tag{3.41}$$

with the radius of convergence of the majorant series strictly positive, uniformly in the cutoff t_0 for $\beta < \beta_{k+1}$. From the point of view of the Mayer coefficients b_n with $n \geq k$ however, the contribution coming from their neutral subclusters of size smaller than k has to be reevaluated from an equation analogous to (3.15) and its limit as $t_0 \to 0$ has to be shown to exist.

[3] This part would not be necessary for keeping the radius of convergence positive, uniformly in t_0, at $\beta < \beta_{k+1}$.

To deal with the new situation, we introduce a sequence $\left(\tilde{A}_m\right)_{m\geq 1}$ of appended at ζ_0 functions analogous to the sequence $(A_n)_{n\geq 1}$ defined by (3.14). Recalling the role of $\dot{v}_{ij}(s) = \sigma_i g(s)h(\left|x_j - x_i\right|/s)\sigma_j$ played in (3.6), $\tilde{A}_1 \equiv 1$ and for $m \geq 2$

$$\tilde{A}_m(s, \sigma_1, \ldots, \sigma_m) = \frac{1}{m!}\int_{\mathbb{R}^2\times\cdots\times\mathbb{R}^2} dx_1\cdots dx_m \left|\sum_{j=1}^{m}\sigma_0 g(s)h(\left|x_j - x_0\right|/s)\sigma_j\psi_m^c(s, \zeta_1, \ldots, \zeta_m)\right| \tag{3.42}$$

is choosing to be dependent on the charges of the m-particle subcluster but independent on $\zeta_0 = (x_0, \sigma_0)$ by translational invariance and $|\sigma_0| = 1$. The $\tilde{A}_m$ functions might satisfy a system of integral inequality equations similar to (3.15) but we shouldn't write it down here since we restrict ourselves in the present paper to the simplest case of smallest size $m = 2$.

Referring to (3.6), let I be an index set of a cluster of size $|I| = n$ and let J be the index set of a pair $|J| = 2$ of particles with opposite charges: $\sigma_1\sigma_2 = -1$ located at x_1 and x_2. The second Ursell function at scale s is given by

$$\psi_2^c(s, \zeta_1, \zeta_2) = \beta\int_{t_0}^{s} g(\tilde{s})h(r/\tilde{s})\exp\left(\beta\int_{\tilde{s}}^{s} g(\tau)h(r/\tau)d\tau\right)d\tilde{s}\,. \tag{3.43}$$

where $r = |x_2 - x_1|$. Let $\Delta = \Delta(s, \tilde{s}, x_0, x_1, x_2)$ be the function that gather the h part of (3.42) with $m = 2$ together with the h in (3.43):

$$\Delta = (h(|x_0 - x_1|/s) - h(|x_0 - x_2|/s))\, h(|x_1 - x_2|/\tilde{s})\,, \tag{3.44}$$

for $s > \tilde{s} \geq t_0$. An estimate for the integral of (3.44) with respect to x_1 and x_2 is proven in Appendix B (see (B.5) in Proposition B.2). We now apply (B.5) in order to show that (3.42) with $m = 2$ and $\sigma_1\sigma_2 = -1$ is bounded uniformly with respect to the cutoff t_0, provided $\beta \leq 6\pi$.

By Propositions B.2 and 2.9,

$$\begin{aligned} 2\tilde{A}_2(s, \pm, \mp) &= \beta g(s)\int_{\mathbb{R}^2\times\mathbb{R}^2} dx_1 dx_2 \int_{t_0}^{s} g(\tilde{s})\,|\Delta|\exp\left(\beta\int_{\tilde{s}}^{s} g(\tau)h(|x_1 - x_2|/\tau)d\tau\right)d\tilde{s} \\ &\leq \frac{\beta}{8}sm(s)\cdot\frac{1}{4s}\int_{t_0}^{s}\tilde{s}^2 m(\tilde{s})\exp\left(\frac{\beta}{2\pi}\int_{\tilde{s}}^{s}\frac{1}{\tau}m(\tau)d\tau\right)d\tilde{s} \equiv \Gamma(s)\tilde{C}_2(s) \end{aligned} \tag{3.45}$$

where $\Gamma(s) = \beta sm(s)/8$ is the L^1-norm (3.8) of $\beta\dot{v}_{ij}(s)$ and $\tilde{C}_2(s)$, defined accordingly by (3.45), can be estimated for s small as (see (3.29))

$$\tilde{C}_2(s) < \frac{C}{2\pi}s^{\beta/2\pi - 1}\int_{t_0}^{s}\tilde{s}^{2-\beta/2\pi}d\tilde{s} < \frac{C}{6\pi - \beta}s^2 < \infty \tag{3.46}$$

if $\beta < 6\pi$, uniformly in t_0. Together with the two other components $2\tilde{A}_2(s, \pm, \pm)$, whose estimate is even better in view of the stability bound improvement, we have

$$1/4\sum_{\sigma,\sigma'} 2\tilde{A}_2(s, \sigma, \sigma') \leq \Gamma(s)\tilde{C}_2(s)\,.$$

Moreover, the function $\tilde{C}_2(s)$ that majorizes the contribution of a neutral pair inside a cluster of size $n \geq 3$ is bounded, for β in the first threshold interval $I_1 = [4\pi, 6\pi)$, by the corresponding term of the majorant series (recall (3.41) and Proposition 3.7): $f_3(s)\tilde{C}_2^{(3)}(s) = \tau_3(t_0, s)$, defined respectively by (3.32), (3.36) and (3.28) with $k = 3$:

$$\tilde{C}_2(s) \leq \tau_3(t_0, s)\,. \tag{3.47}$$

To prove the inequality we compare the integrand of (3.45) with the integrand of (3.28) with $k = 3$:

$$\frac{\tilde{s}e^{\frac{\beta}{2\pi}\int_{\tilde{s}}^{s} m(\tau)/\tau\ d\tau}}{se^{\frac{\beta}{3\pi}\int_{\tilde{s}}^{s} m(\tau)/\tau\ d\tau}} = \frac{\tilde{s}}{s}e^{\frac{-\beta}{6\pi}\int_{\tilde{s}}^{s} m(\tau)/\tau\ d\tau} \le \left(\frac{\tilde{s}}{s}\right)^{1-\frac{\beta}{6\pi}} \le 1 \tag{3.48}$$

holds for any $\tilde{s} \le s$ and $\beta \le 6\pi$, by the asymptotics of $m(\tau)$ given in Proposition 2.9 and the fact that $m(\tau)/\tau$ is a monotone decreasing function of $\tau > 0$. Observe that (3.47), together with (3.46), ensure that the two implicit assumptions of Claim 1.2 hold. We refer to Sec. 6.3 of [22] or the proof of Theorem 4.3, in [8] for alternative calculations by which the convergence of the Mayer series can also be established for $\beta \in [4\pi, 6\pi)$, in both treatments, provided the modified specific energy $\bar{e}_3(\tilde{h})$ of "tripole" exceeds $-(k+1)/(2k)$ for $k = 3$, i.e. $-2/3$. We have shown in Proposition 2.1 that $\bar{e}_3(\tilde{h})$, being smaller than $-1/2$ $(= \bar{e}_3(h))$ but larger than -0.535, exceeds $-2/3 = -0.666$ and equation (3.41) thus holds with $k = 3$ by Propositions 3.7 and 3.9. Observe that, by (3.30) with $n = k = 3$, $B(s) = \beta/(4\pi s)$ and $\Gamma(s) = 2\beta s$,

$$3b_3 \le f_3(1)^2 C_3^{(3)}(1) = \frac{3^2}{3!}\tau_3(t_0, 1)^2$$

and $\tau_3(t_0, s) = \int_{t_0}^{s} \Gamma(\tilde{s})e^{4/3\int_{\tilde{s}}^{s} B(\tau)d\tau} d\tilde{s}$ is finite uniformly in t_0 for the decomposition (1.7) of the Yukawa potential v (see Remark 3.5) and for the decomposition (1.4) it follows directly from Proposition 3.9.

The interval for which a neutral pair of charges yields finite contribution to the Mayer series has been extended from $[0, 4\pi)$, if the pair was an isolate cluster, to $[0, 6\pi)$, if they are in the presence of other charges, due to a redistribution of the powers of $\tilde{s}$ and s in a homogeneous of degree 4 estimate for $\iint |\Delta|\, dx_1 dx_2$ in favor of $\tilde{s}$ (see (B.5)). The rearrangement of powers wasn't enough to prevent the collapse of a neutral pair inside threshold intervals $I_k = [\beta_{k-1}, \beta_{k+1})$ of order k higher than 3, supposing of course the neutral clusters of size smaller than k have been discarded from the Mayer series. However, such an extension would be possible if the exponential term of (3.43), which also depends of h, were included in the definition of Δ. In this case we need to be more careful when $|x_1 - x_2|/\tau$ is small. Let us explain an argument used in [22]. By the first mean value theorem, there exist $\tau^* \in [\tilde{s}, s]$ such that ($r = |x_1 - x_2|$)

$$\frac{\beta}{2\pi}\int_{\tilde{s}}^{s} m(\tau)h(r/\tau)\frac{d\tau}{\tau} = m(\tau^*)h(r/\tau^*)\frac{\beta}{2\pi}\log\frac{s}{\tilde{s}} .$$

For s a small fixed number, let $\Lambda = \{(r, \tilde{s}) \in \mathbb{R}_+ \times [t_0, s] : r \le 0.2\tau^*\}$ and note that $m(\tau^*)h(r/\tau^*) < 3/4$ for $(r, \tilde{s})$ in the complementary set $(\mathbb{R}_+ \times [t_0, s])\backslash\Lambda$, by Propositions B.1 and 2.9. Under this condition $\tilde{A}_2(s, \pm, \mp)$ can be bounded by the last integral in (3.45) with the exponent $2 - \beta/2\pi$ of $\tilde{s}$ in (3.46) replaced by $2 - (3/4)\beta/2\pi = 2 - 3\beta/8\pi$, which is finite for $\beta < 8\pi$. On the other hand, we expect that the integral of Δ with respect to $(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ in (B.5), can be improved when restricted to Λ in such way that the exponent $2 - \beta/2\pi$ of $\tilde{s}$ in the last integral of (3.46), replaced by $3 - \beta/2\pi$, becomes integrable for $\beta < 8\pi$. We refer to Subsec. 6.3, pgs. 29-30 of [22] for detail of this calculation with $\tilde{h}$ in place of h for the standard decomposition and Lemmas 3 and 4 of [21]. We observe that, the analogue of both (3.46) and (3.48) hold under the improvement. So, the two implicit assumptions of Claim 1.2 are verified.

Let us comment on how to proceed in the next threshold interval $[6\pi, 20\pi/3)$. Given that the contribution to the Mayer series due to a neutral pair of charges satisfies the two assumptions of Claim 1.2 and "tripoles" and "quintupoles" do not collapse, the next contribution comes from neutral quadrupoles and superstability could be useful in this case (see [17, Sect. V.B.]).

4 Summary and Open Question

The main result of the present paper, Theorem 2.5, states that the energy $U_n(\xi; h)$ of a configuration $\xi = (x, \sigma) = (x_1, \ldots, x_n, \sigma_1, \ldots, \sigma_n)$ of n particles, with $(x_i, \sigma_i) \in \mathbb{R}^2 \times \{+1, -1\}$, interacting through the two-dimensional Euclid's hat pair potential $h\left(\cdot/s\right)$ at scale s satisfies (2.11). Since the inequality saturates when the n particles collapses all together to a single point with net charge 0 if n is even and ± 1 if n is odd, a corollary to this (see Corollary 2.7) is that the minimal specific energy $e(h)$ and the minimal constrained specific energy $\bar{e}(h)$, defined by (1.18) and (1.19), are both $-1/2$. The same statement holds to positive radial potentials of positive type in any dimension $d \geq 2$ provided it can be written as scale mixtures of Euclid's hat: $v(x) = \displaystyle\int g(s) h\left(|x|/s\right) ds$, $g(s) \geq 0$ and the right hand side of (2.11) is multiplied by $\displaystyle\int g(s) ds$. Consequently (see Corollary 2.8), if n is odd the stability bound (1.16) can be replaced by (1.23) for any potential of this class with $B = \bar{B} = \dfrac{1}{2} \displaystyle\int g(s) ds$.

We have applied the main result to the two-dimensional Yukawa gas with particles activity z at the inverse temperature β in the interval of collapse $[4\pi, 8\pi)$. A Cauchy majorant, proposed in [22] for the pressure and density function, can be written in terms of the principal branch of the W-Lambert function which is analytic provided its argument $-z\tau_k$, with $\tau_k = \tau_k(t_0, t)$ given by (3.28), satisfies $e\,|z|\,\tau_k < 1$, $\beta < \beta_{k+1} = 8\pi k/\left(k+1\right)$ when the divergent part of the leading even Mayer coefficients up to order $2n \leq k+1$, $k > 1$, are extracted. It has been assumed in addition that an improved stability condition (see [22, Conjecture 2.3]) holds for any odd number of particles $2n - 1 \leq k$. However, the numerical evaluation (see [22, Remark 7.5]) of the total energy $U_{2n-1}(\xi; \beta\dot{v})$ for the standard scaling decomposition (1.7) when $n = 2$ and 3 have indicated that it would fail for sufficient large k and Proposition 2.1 now proves it for $U_3(\xi; \beta\dot{v})$. We have in the present paper proved that when the Yukawa potential v is represented as scale mixtures of Euclid's hat it satisfies Conjecture 2.3 of [22] for any $k > 1$, provided Claim 1.2 holds true. Moreover, all the estimates necessary to establish convergence of the majorant series in [22] holds for this representation of v due to Proposition 2.9. We have restated Propositions 3.4, 3.7 and 3.9 accordingly for the reader convenience.

It is important to stress at this point that the classical solution $\Theta_k = \Theta_k(t, z)$ of (3.26) is actually a majorant for the density function (3.2) and the statement (3.10) holds in Proposition 3.7 as long as $t_0 > 0$. One open question is whether the majorant Θ_k remain faithful when the cutoff t_0 tends to 0. We answer the question affirmatively only for $k = 3$ and argue that this question might be dealt using nowadays available tools.

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Appendix A: Characteristic Features of $\tilde{h}$ and Related Function

Proposition A.1 *$w \longmapsto \tilde{h}(w) = wK_1(w)$ is a regular function at every point $w \in (0, \infty)$. The function $\tilde{h}(w)$ strictly decreases from its maximum $\tilde{h}(0) = 1$, decays to 0 at ∞ exponentially fast and changes its concavity: $\tilde{h}''(w) < 0$ for $w < w_0$ and $\tilde{h}''(w) > 0$ for $w > w_0$ at $1/2 < w_0 < \left(1 + \sqrt{17}\right)/8$, whose numerical value is $w_0 = 0.5950(\ldots)$.*

Proof Regularity and positivity of $K_\nu(x)$ for every $\nu \in \mathbb{R}$ and $x > 0$ are known facts (see e.g. [16, Appendix A]). It follows from the equation

$$\left(x^n K_n(x)\right)' = -x^n K_{n-1}(x) \tag{A.1}$$

with $n = 1$ together with $\lim_{w\to 0} wK_0(w) = 0$ and $\lim_{w\to 0} wK_1(w) = 1$ (see [16] and [34, Lemma 2.2]) that

$$\tilde{h}'(w) = (wK_1(w))' = -wK_0(w) < 0$$

for $w > 0$, proving the strictly decreasing property of $\tilde{h}$ and $\tilde{h}(0) = 1$. The inequalities for $x > 0$:

$$\begin{aligned} \frac{\sqrt{\pi}e^{-x}}{\sqrt{2x+1/2}} &< K_0(x) < \frac{\sqrt{\pi}e^{-x}}{\sqrt{2x}} \\ 1 + \frac{1}{2x+1/2} &< \frac{K_1(x)}{K_0(x)} < 1 + \frac{1}{2x} \ , \end{aligned} \tag{A.2}$$

find in ref. [34], imply the exponential decaying of $\tilde{h}(w)$.

$$\tilde{h}''(w) = -\left(wK_0(w)\right)' = K_0(w)\left(\frac{wK_1(w)}{K_0(w)} - 1\right)$$

together with (A.2) yield

$$\tilde{h}''(w) < K_0(w)\left(w - \frac{1}{2}\right) < 0$$

provided $w < 1/2$ and

$$\tilde{h}''(w) > K_0(w)\left(\frac{w}{2w+1/2} + w - 1\right) > 0$$

provided $w > (2w + 1/2)(1 - w) = 3w/2 + 1/2 - 2w^2$ or, equivalently, $w > \left(1 + \sqrt{17}\right)/8 = 0.640\,39$. The unique solution of $wK_1(w)/K_0(w) - 1 = 0$, whose numerical value is $w_0 = 0.5950(\ldots)$, satisfies $1/2 < w_0 < \left(1 + \sqrt{17}\right)/8$ (see proof of Lemma A.2 below). This concludes the proof. □

Lemma A.2 *The function $x \longmapsto p(x) = xK_1(x) + x^2K_0(x)$ defined in $\mathbb{R}_+$ has a global maximum at x_0, $1/2 < x_0 < \left(1 + \sqrt{17}\right)/8$. It strictly increases from $p(0) = 1$ as x varies from 0 to $1/2$ and strictly decreases to 0, exponentially fast, as x varies from $\left(1 + \sqrt{17}\right)/8$ to ∞. The second derivative $p''(x)$ of $p(x)$ is negative in the interval $1/2 \le x_0 \le \left(1 + \sqrt{17}\right)/8$. Numerically, $x_0 = 0.5950\,(\ldots)$ and its (global) maximum values $p(x_0) = 1.061(\ldots) < 1.07$.*

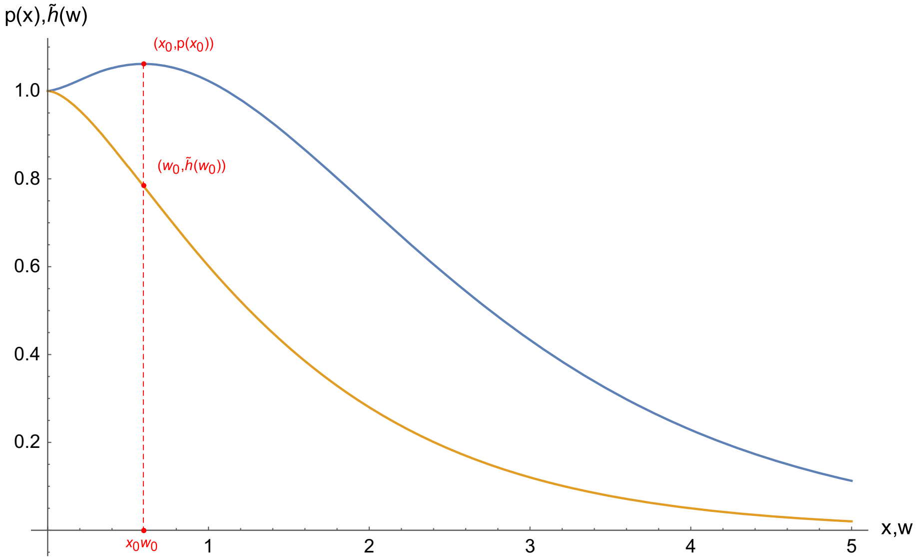

Fig. 6 Plot of $p(x)$ and $\tilde{h}(w)$ together

Proof Using (A.1) with $n = 1$ together with $K_0'(x) = -K_1(x)$, as in the proof of Proposition A.1, we have

$$p'(x) = xK_0(x)\left(1 - \frac{xK_1(x)}{K_0(x)}\right) < xK_0(x)\left(1 - \frac{x}{2x + 1/2} - x\right) < 0$$

provided $x > \left(1 + \sqrt{17}\right)/8$ and

$$p'(x) > xK_0(x)\left(x - \frac{1}{2}\right) > 0$$

provided $x < 1/2$. These prove that $p(x)$ increases in $(0, 1/2)$ and decreases in $\left(\left(1 + \sqrt{17}\right)/8, \infty\right)$, exponentially fast in view of (A.2).

$p(x)$ attains its maximum value at the same point at which $\tilde{h}(w)$ changes its concavity. The maximum x_0 of $p(x)$ solves $K_0(x) - xK_1(x) = 0$ and satisfies $1/2 < x_0 < \left(1 + \sqrt{17}\right)/8 \approx 0.64$, as stated above. To prove that x_0 is the global maximum, it suffices to show that the second derivative of $p(x)$, which may be calculated exactly as before,

$$\begin{aligned} p''(x) &= (x\,(K_0(x) - xK_1(x)))' \\ &= (1 + x^2)K_0(x) - 2xK_1(x) \\ &= -2K_0(x)\left(\frac{xK_1(x)}{K_0(x)} - \frac{1 + x^2}{2}\right), \end{aligned}$$

takes negative values for $x \in \left[1/2, \left(1 + \sqrt{17}\right)/8\right]$. By equation (A.2) and positivity of $K_0(x)$, this is implied by

$$\frac{xK_1(x)}{K_0(x)} - \frac{1 + x^2}{2} > x + \frac{x}{2x + 1/2} - \frac{1 + x^2}{2} > 0\,.$$

Denoting the function on the right hand side by $l(x) = x + x/(2x + 1/2) - (1 + x^2)/2$, we need to show that $l(x) > 0$ for $x \in \left[1/2, \left(1 + \sqrt{17}\right)/8\right]$. But $l(1/2) = 5/24 \approx 0.20$ and $l(\left(1 + \sqrt{17}\right)/8) = \left(23 + \sqrt{17}\right)/64 \approx 0.29$ are both positive and the second derivative of $l(x)$,

$$l''(x) = -\left(17 + 12x + 48x^2 + 64x^2\right)/(1 + 4x)^3 < 0$$

for all $x > 0$, proving therefore the statement.

We have thus proven that x_0 is a global maximum of $p(x)$, concluding the proof of Lemma A.2. □

Appendix B: General Features of *h* and a Certain Integral Estimate

Proposition B.1 $w \longmapsto h(w)$ *defined by (2.8) is regular at every point* $w \in (0, 1)$*, convex and non increasing function in* $(0, \infty)$*. Moreover, it can be written as*

$$h(w) = \frac{2}{\pi}\left(\arccos w - w\sqrt{1 - w^2}\right) , \qquad \text{if} \quad 0 \leq w \leq 1 \tag{B.1}$$

$h(w) = 0$ *if* $w > 1$ *so, writing* $\varphi(x) = h(|x|)$ *we have* $\varphi(0) = h(0) = 1$ *and* $\hat{\varphi}(0) = \pi/4$ *is its Fourier transform* $\hat{\varphi}(\xi) = \int_{\mathbb{R}^2} h(|x|)e^{-2\pi i \xi \cdot x} dx$ *at* $\xi = 0$.

Proof We shall deduce (B.1) from (2.8) by means of a geometric representation of the convolution integral

$$\frac{\pi s^2}{4} h(w) = \int_{\mathbb{R}^2} \chi_{[0,s/2]}(x - y)\chi_{[0,s/2]}(y)dy \tag{B.2}$$

(see e.g [19, Sect. 2]). Since Mittal's integral representation of the Euclid's hat is used instead (see [19, Fig. 4 and Eq. (49)]), we shall give details of its derivation for the reader convenience.

The product of indicator functions does not vanish if their support, the discs $B_{s/2}(x)$ and $B_{s/2}(0)$ centered at x and 0, intersect and this occurs if and only if the distance $|x|$ between their centers is less than their diameter s. Writing $w = |x|/s$, we have

$$h(w) \neq 0 \Longleftrightarrow 0 \leq w < 1 \ .$$

From this point of view, the convolution integral (B.2) is given by the area $A(\theta)$ of two "caps", of common bases, made of a sector of opening angle θ and radius $s/2$ with the triangular region inside removed (see Fig. 7):

$$\frac{\pi s^2}{4} h(w) = A(\theta) = 2 \times \left(\frac{1}{2}\left(\frac{s}{2}\right)^2 \theta - \frac{1}{2}\left(\frac{s}{2}\right)^2 \sin\theta\right) , \tag{B.3}$$

where, with b the length of the caps common bases,

$$\begin{aligned} s\cos\theta/2 &= |x| \\ s\sin\theta/2 &= b \ . \end{aligned} \tag{B.4}$$

By Pitagoras $|x|^2 + b^2 = s^2$, we deduce $b = s\sqrt{1 - w^2}$. Solving equations (B.4) for θ and $\sin\theta$: $\theta = 2\arccos w$ and $\sin\theta = 2\sin\theta/2\ \cos\theta/2 = 2w\sqrt{1 - w^2}$, together with (B.3), yields (B.1).

The regularity of $h(w)$ in $(0, 1)$ follows from this representation. Clearly, $h(0) = 1$ by (B.1). Since $h'(w) = -2\sqrt{1 - w^2} < 0$ and $h''(w) = 2w/\sqrt{1 - w^2} > 0$ for any $w \in$

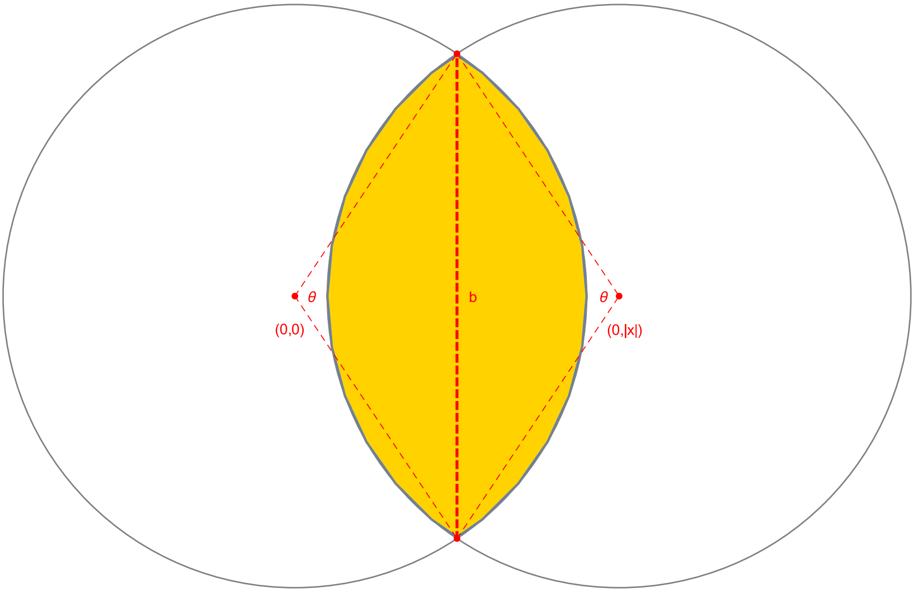

Fig. 7 Geometric interpretation of the Euclid's hat function

(0, 1), we conclude that $h(w)$ is strictly decreasing in (0, 1) monotone non increasing and convex in $(0,\infty)$. By the convolution theorem and (B.2) with $s = 1$, we have $\hat{h}(0) = (4/\pi)\,\widehat{\chi_{[0,1/2]}}(0)^2 = (4/\pi)\,(\pi/4)^2 = \pi/4$, concluding the proof. □

Using the self-convolution form (B.2) of the Euclid's hat $h(w)$ together with its geometric interpretation as the area of "caps" (see proof of Proposition B.1 above), the following estimate holds:

Proposition B.2 *The integral of* $\Delta := (h(|x_0 - x_1|/s) - h(|x_0 - x_2|/s))\,h(|x_1 - x_2|/\tilde{s})$ *in absolute value with respect to* x_1 *and* x_2 *satisfies*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} |\Delta(s,\tilde{s},x_0,x_1,x_2)|\,dx_1dx_2 \le \frac{1}{8}\pi^2\tilde{s}^3 s\ . \tag{B.5}$$

Proof Writing Δ in terms of indicator functions of two discs centered, let us say, in z and $\tilde{z}$ (analogous to Fig. 7, with different radius):

$$\Delta = \frac{4}{\pi s^2}\int_{\mathbb{R}^2} dz \cdot \frac{4}{\pi \tilde{s}^2}\int_{\mathbb{R}^2} d\tilde{z}\chi_{s/2}(x_0 - z)\left(\chi_{s/2}(z - x_1) - \chi_{s/2}(z - x_2)\right)\chi_{\tilde{s}/2}(x_1 - \tilde{z})\chi_{\tilde{s}/2}(\tilde{z} - x_2) \tag{B.6}$$

we observe that the integrand of Δ differs from 0 if, and only if, either x_1 is inside of the non null intersection $B_{s/2}(z) \cap B_{\tilde{s}/2}(\tilde{z})$ and x_2 is inside the complementary region $B_{\tilde{s}/2}(\tilde{z})\backslash(B_{s/2}(z) \cap B_{\tilde{s}/2}(\tilde{z})) \neq \emptyset$ or vice-versa. As a consequence, we have

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} dx_1dx_2\left|\chi_{s/2}(z - x_1) - \chi_{s/2}(z - x_2)\right|\chi_{\tilde{s}/2}(x_1 - \tilde{z})\chi_{\tilde{s}/2}(\tilde{z} - x_2) = 2A\cdot B \tag{B.7}$$

where $A = \left|B_{s/2}(z) \cap B_{\tilde{s}/2}(\tilde{z})\right|$, $B = \left|B_{\tilde{s}/2}(\tilde{z}) \backslash (B_{s/2}(z) \cap B_{\tilde{s}/2}(\tilde{z}))\right| = \left(\pi \tilde{s}^2/4\right) - A$ and $|D|$ denotes the area of a bounded domain $D \subset \mathbb{R}^2$. Using

$$2A \cdot B = \frac{1}{2}(A+B)^2 - \frac{1}{2}(A-B)^2 \le \frac{1}{2}(A+B)^2 = \frac{1}{2}\left(\frac{\pi \tilde{s}^2}{4}\right)^2$$

and the fact that A and B are different from 0 if and only if

$$\frac{s-\tilde{s}}{2} < |z - \tilde{z}| < \frac{s+\tilde{s}}{2}$$

we have

$$\frac{4}{\pi s^2}\int_{\mathbb{R}^2} dz\, \chi_{s/2}(x_0 - z) \cdot \frac{4}{\pi \tilde{s}^2}\int_{\mathbb{R}^2} d\tilde{z}\, 2A \cdot B \le \frac{1}{\tilde{s}^2}\left((s+\tilde{s})^2 - (s-\tilde{s})^2\right) \cdot \frac{1}{2}\left(\frac{\pi \tilde{s}^2}{4}\right)^2$$

which together with (B.6) and (B.7) yields (B.5). □

Appendix C: Some Auxiliary Equations Related to the Mixture Function *g*

Some results stated in Sect. 2 are proven below. We begin with an auxiliary Lemma.

Lemma C.1 *Let* $I : [0, \infty) \longrightarrow \mathbb{R}$ *be defined by*

$$I(s) = \int_s^\infty K_1(y)\sqrt{y^2 - s^2}\,dy\ . \tag{C.1}$$

The integral can be written as

$$I(s) = \int_s^\infty K_1(y)\frac{s}{\sqrt{y^2 - s^2}}dy \tag{C.2}$$

and from these we conclude that $I(s) = \pi e^{-s}/2$.

Proof The integral (C.1) converge uniformly. The integral (C.2) is obtained by taking (minus) the derivative of (C.1):

$$I'(s) = \int_s^\infty K_1(y)\frac{-s}{\sqrt{y^2 - s^2}}dy = -I(s)\ , \qquad s > 0\ . \tag{C.3}$$

Observe that $K_1(y)\sqrt{y^2 - s^2}\Big|_{y=s} = 0$ for the same reason as before. Since ae^{-s} solves (C.3) for any $a \in \mathbb{R}$, to complete the proof we to show that (C.1) implies (C.2) and $I(0) = \pi/2$.

Repeating the operations bringing (2.9) into the form (2.13), equation (C.1) can be written as

$$\begin{aligned}
\int_s^\infty K_1(y)\sqrt{y^2 - s^2}dy &= \int_s^\infty \left(\int_0^\infty e^{-y\sqrt{k^2+1}}dk\right)\sqrt{y^2 - s^2}dy \\
&= \int_0^\infty \left(\int_s^\infty e^{-y\sqrt{k^2+1}}\sqrt{y^2 - s^2}dy\right)dk \\
&= \int_0^\infty \left(\int_s^\infty e^{-y\sqrt{k^2+1}}\frac{y}{\sqrt{y^2 - s^2}}dy\right)\frac{dk}{\sqrt{k^2+1}}
\end{aligned}$$

$$= s\int_0^\infty \left(\int_0^\infty e^{-s\sqrt{k^2+1}\sqrt{r^2+1}}dr\right)\frac{dk}{\sqrt{k^2+1}}$$
$$= s\int_0^\infty K_1\left(s\sqrt{k^2+1}\right)\frac{dk}{\sqrt{k^2+1}}$$
$$= s\int_s^\infty K_1(y)\frac{dy}{\sqrt{y^2-s^2}}$$

where the integrations order has been switched in the second equality, following by partial integration; we have changed variable $y = s\sqrt{r^2+1}$ in the fourth equality, used $K_0'(w) = -K_1(w)$ together with (2.16) and changed to the variable $y = s\sqrt{k^2+1}$ in the last equality. Likewise, using $K_0'(y) = -K_1(y)$ and inserting the integral representation (2.16), we find

$$I(0) = \int_0^\infty yK_1(y)dy = \frac{\pi}{2}\ . \tag{C.4}$$

This concludes the proof of the lemma. □

Proof of the second equality of equations (2.21) and (2.20) We shall deduce an equation for $J(s)$ and $L(s)$ in terms of $I(s)$. Differentiating (2.21) with respect to s, gives

$$\begin{aligned} J'(s) &= -\int_s^\infty yK_0(y)\frac{s}{\sqrt{y^2-s^2}}dy \\ &= -s\int_s^\infty K_0(y)\left(\sqrt{y^2-s^2}\right)'dy \\ &= -s\int_s^\infty K_1(y)\sqrt{y^2-s^2}dy = -sI(s) = \frac{-\pi}{2}se^{-s}\ , \end{aligned} \tag{C.5}$$

by Lemma C.1. Analogously, differentiating (2.20) with respect to s, together with (A.1), gives

$$\begin{aligned} L'(s) &= \frac{-s}{2}\int_s^\infty yK_1(y)\frac{y}{\sqrt{y^2-s^2}}dy \\ &= \frac{s}{2}\int_s^\infty (yK_1)'(y)\sqrt{y^2-s^2}dy \\ &= \frac{-s}{2}\int_s^\infty yK_0(y)\sqrt{y^2-s^2}dy = -\frac{1}{2}sJ(s)\ . \end{aligned} \tag{C.6}$$

We also need initial condition to both equations. Using $K_0'(y) = -K_1(y)$ and inserting the integral representation (2.16), we find as in (C.4),

$$J(0) = \int_0^\infty y^2K_0(y)dy = \frac{\pi}{2} \tag{C.7}$$

and

$$L(0) = \frac{1}{2}\int_0^\infty y^3K_1(y)dy = \frac{3\pi}{4}\ . \tag{C.8}$$

Integrating (C.5) together with (C.7), yields

$$J(s) = \frac{\pi}{2}(1+s)e^{-s}\ . \tag{C.9}$$

Analogously, integrating (C.6) together with (C.9) and (C.8), yields

$$L(s) = \frac{\pi}{4}\left(3 + 3s + s^2\right) e^{-s} \tag{C.10}$$

concluding the proof. □

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