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THE RELATIONSHIP BETWEEN THE COHERENT SYSTEM SIGNATURE AND THE BARLOW AND PROSCHAN COMPONENT RELIABILITY IMPORTANCE

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The relationship between the coherent system signature and the Barlow and Proschan component reliability importance

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Abstract. In this note we analyse the relationship between the signature influence measure and the Barlow and Proschan component reliability importance measure for a coherent system.

Keywords: Coherent system, signature, Barlow-Proschan reliability importance, quality functions.

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1.Introduction.

Studying a complex coherent system, it is important to the analyst determine which events, such as the occurrence of a system and \ or component failure, merit additional research to improve overall system reliability. Generally they use several indexes to measure the influence or the importance of such event to system reliability. Two of these indexes, the coherent system signature and the Barlow-Proschan reliability importance of a component have similar concepts, but different interpretations.

If $T_1, ..., T_n$ are independent, identically and continuously distributed component lifetimes of a coherent system with lifetime T the system signature vector, as defined by Samaniego (1985), is the vector $\mathrm{s} = (s_1, ..., s_n)$ where $s_k = P(T = T_{(k:n)})$ and $T_{(1:n)}) < ... < T_{(n:n)}$ are the ordered $T_1, ..., T_n$.

Under the definition assumption s_k is distribution free and s is a structural measure for the system whose distribution function can be written as a mixture of the ordered lifetimes

$$P(T \leq t) = \sum_{k=1}^{n} s_k P(T_{(k:n)}) \leq t).$$

The signature s_k is interpreted as a measure of the influence on the structure function of adding a k-th component to the set of failed ones.

Otherwise, if $T_1, ..., T_n$ are independent, identically and continuously distributed component lifetimes of a coherent system with lifetime T the Barlow-Proschan reliability importance of the component k, for the system reliability, as defined by Barlow and Proschan (1975), is $I^k_{BP} = P(T = T_k)$ given that the system failed

$$I^k_{BP} = \int_0^{\infty} [P(T > s | T_k > s) - P(T > s | T_k \leq s)] dF_k(s),$$

where $F_k(s) = P(T_k \leq s)$. Also

$$P(T \leq t) = \sum_{k=1}^{n} \int_0^t [P(T > s|T_k > s) - P(T > s|T_k \leq s)] dF_k(s)$$

The component reliability importance I^k_{BP} is interpreted as the, accumulated, system reliability perturbation caused by failure of the component k.

The similarity between the two definitions motivate to analyse its relationship. Anyway we can ask about the importance of a component k for the system to fail in the set $\{T = T_{(i:n)}\}$.

As a coherent structure function is a pseudo-Boolean function, recently, see [2], [3], [4] and [5] in the references, an approach through Boolean algebra has been used to study these indexes measures for complex systems which allows to generalize those measures for the case of dependent lifetimes without ties, that is, $P(T_i = T_j) = 0, \quad 1 \leq i \leq j \leq n$, and to analyse, with more detail, the relationship between the influence index and the pertubation one. In this note we prove that the signature element s_k is equal to the sum of the Barlow-Proschan components reliability importance of the survival components on the $(k-1)$ failure. In Section 2, first we give a mathematical details and then, prove the main result.

2.Mathematical details.

The approach using the pseudo boolean functions, as in Marichal and Mathonet (2011a), (2011a), (2013) give an important contribution to the development of classical results extension in the context of dependent lifetimes without simultaneous failures. To follow we introduce the necessary details found in Marichal and Mathonet (2011a), (2013):

Through the usual identification of the elements of $[0,1]^n$ with the subsets of $[n] = \{1, ..., n\}$, a pseudo-Boolean function $f : [0,1]^n \to \Re$ can be equivalently described by a set function $v_f : 2^n \to \Re$. We simply write $v_f(A) = f(\mathbf{1}_A)$, where $\mathbf{1}_A$ denotes the n-tuple whose i-th coordenate is 1, if $i \in A$, and 0, otherwise. To avoid cumbersome notation, we hencefort use the same symbol to denote both a given pseudo-Boolean function and its underlying set function, thus writing $f : [0,1]^n \to \Re$ as $f : 2^n \to \Re$ interchangeably.

If $T_1, ...T_n$ denote the component lifetimes with jointly absolutely continuous distribution function, we define the associated relative quality function $q : 2^n \to [0,1]$ as

$$q(A) = P(\max_{i \in [n] \setminus A} T_i < \min_{j \in A} T_j)$$

with the convention that $q(\emptyset) = q([n]) = 1$. $q(A)$ is the probability that the lifetime of every component in A is greater than the lifetime of every component in $[n] \setminus A$, that is, $q(A)$ is a measure of the overall quality of the components in A when compared with the components in $[n] \setminus A$.

Since that the random variables $T_1, ...T_n$ are continuous the function q can be written as

$$q(A) = \sum_{\sigma \in \wp : \{\sigma(n-|A|+1), ..., \sigma(n)\} = A} P(T_{\sigma(1)} < ... < T_{\sigma(n)})$$

where $\wp$ denote the class of permutations on $[n]$.

Also, for $1 \leq k \leq n$ we have

$$\sum_{A \subseteq [n], |A|=k} q(A) = \sum_{\sigma \in \wp} P(T_{\sigma(1)} < ... < T_{\sigma(n)}) = 1$$

in the way the, if $T_1, ...T_n$ are exchangeable $q(A) = \frac{1}{\binom{n}{|A|}}$.

With these notations Marichal and Mathonet (2011a) proves

Lemma 2.1 Under the above assumptions, for every $k \in [n]$, we have

$$P(T \geq T_{(k:n)}) = \sum_{|A|=n-k+1} q(\mathrm{x})\phi(\mathrm{x}).$$

Continuing, under the assumption of no ties between the component lifetimes, Marichal and Mathonet (2013) define, for every $j \in [n]$, the function $q_j : 2^n \to [0,1]$ as

$$q_j(A) == P(\max_{i \notin A \cup \{j\}} T_i < T_j < \min_{i \in A} T_i).$$

Similarly, as $q(A)$, we can write

$$q_j(A) = \sum_{\sigma \in \wp : \{\sigma(n-|A|+1),...,\sigma(n)\} = A, \sigma(nn-|A|)=j} P(T_{\sigma(1)} < ... < T_{\sigma(n)}).$$

$q_j(A)$ is the probability that the components that are better than component j are precisely those in A. Follows that

$$\sum_{A \subseteq [n] \setminus \{j\}} q_j(A) = 1, \quad j \in [n].$$

We also observe that

$$q(A) = \sum_{j \notin A} q_j(A), \quad A \neq [n]$$

and

$$q(A) = \sum_{j \in A} q_j(A \setminus \{j\}), \quad A \neq \emptyset.$$

Moreover, $q_j(\emptyset) = q(\{j\})$ is the probability that component j is the best component, while $q_j([n]\setminus\{j\}) = q([n]\setminus\{j\})$ is the probability that component j is the worse component.

If the variables $T_1, ...T_n$ are exchangeable

$$q_j(A) = \frac{1}{n\binom{n-1}{|A|}} = \frac{1}{(n-|A|)\binom{n}{|A|}}.$$

Marichal and Mathonet (2013) shows that

Theorem 2.2 For every $j \in [n]$, the Barlow-Proschan reliability importance of component j is

$$I_{BP}^{(j)} = \sum_{A \subseteq [n] \setminus \{j\}} q_j(A) \triangle_j \phi(A).$$

where $\triangle_j \varphi(A) = \varphi(A \cup \{j\}) - \phi(A)$.

Example 2.3 For the If ϕ is the k-out-of-n structure, a perfect symmetric syatem with lifetime $T = T_{(k:n)})$, we have $\phi(A) = 1$ if and only if $|A| \geq n - k + 1$and hence, for every $j \in [n]$, we have $\triangle_j(\phi(A)) = 1$ if and only if $|A| = n - k$. Then, by Theorem 2.2,

$$I_{BP}^{j} = \sum_{A \subseteq [n] \setminus \{j\}, |A| = n-k} q_j(A),$$

Where A is related with the survival components.

Now we can prove our main result

Theorem 2.4 If T is a coherent system with exchangeable components and signature vector s $= (s_1, ..., s_n)$, then

$$s_k = P(T = T_{(k:n)}) = \sum_{j \in A} I_{BP}^{j}$$

where A is the set of components which survive $T_{(k-1:n)}$.

Proof First we prove that

$$P(T \geq T_{(k:n)}) = \sum_{|A| = n-k+1} \frac{1}{\binom{n}{|A|}} \phi(A) = \sum_{j \in A} \sum_{|B| = n-k} q_j(B) \phi(B).$$

To proof we are going to write the set A, of cardinality $|A| = n - k + 1$ in the form $A = B \cup \{j\}$ where j is the value of a random variable X uniformly in the set A.Therefore

$$P(T \geq T_{(k:n)}) = E[P(T \geq T_{(k:n)} | X)] = \sum_{j \in A} P(T \geq T_{(k:n)} | X = j) \frac{1}{|A|} =$$

$$\sum_{j \in A} \sum_{|A| = n-k+1, A = B \cup \{j\}} \frac{1}{\binom{n}{|B|+1}} \phi(B \cup \{j\}) \frac{1}{|B| + 1} =$$

$$\sum_{j \in A} \sum_{|A| = n-k+1, A = B \cup \{j\}} \frac{1}{\binom{n}{|B|} \frac{n-|B|}{|B|+1}} \phi(B \cup \{j\}) \frac{1}{|B| + 1} =$$

$$\sum_{j \in A} \sum_{|B| = n-k} q_j(B) \phi(B \cup \{j\}).$$

Also

$$P(T \geq T_{(k+1:n)}) = \sum_{|B|=n-k} \frac{1}{\binom{n}{|B|}} \phi(B) =$$

$$(n-|B|) \sum_{|B|=n-k} \frac{1}{\binom{n}{|B|}(n-|B|)} \phi(B) =$$

$$\sum_{j \in A} \sum_{|B|=n-k} \frac{1}{\binom{n}{|B|}(n-|B|)} \phi(B) =$$

$$\sum_{j \in A} \sum_{|B|=n-k} q_j(B)\phi(B).$$

Therefore

$$s_k = P(T = T_{(k:n)}) = P(T \geq T_{(k:n)}) - P(T \geq T_{(k+1:n)}) =$$

$$\sum_{j \in A} \sum_{|B|=n-k} q_j(B)\phi(B \cup \{j\}) -$$

$$\sum_{j \in A} \sum_{|B|=n-k} q_j(B)\phi(B) =$$

$$\sum_{j \in A} \sum_{|B|=n-k} q_j(B)[\phi(B \cup \{j\}) - \phi(B)] = \sum_{j \in A} I^{BP}(j).$$

The generalization of Theorem 2.2.4 follows the same argument:

Theorem 2.5 If T is a coherent system with component lifetimes $T_1, ..., T_n$, with no ties, and signature vector $\mathbf{s} = (s_1, ..., s_n)$, then

$$s_k = P(T = T_{(k:n)}) = \sum_{j \in A} I^j_{BP}$$

where A is the set of components which survive $T_{(k-1:n)}$.

Proof

Observing that

$$\sum_{|A|=n-k+1} q(A)\phi(A) = \sum_{|A|=n-k+1} [\sum_{j \notin A} q_j(A)]\phi(A) =$$

$$\sum_{|B|=n-k, A=B\cup\{j\}} [\sum_{j \notin B} q_j(B \cup \{j\})]\phi(B \cup \{j\}).$$

Therefore

$$s_k = \sum_{|A|=n-k+1} q(A)\phi(A) - \sum_{|B|=n-k} q(B)\phi(B) =$$

$$\sum_{|B|=n-k} [\sum_{j\notin B} q_j(B\cup\{j\})]\phi(B\cup\{j\}) - \sum_{|B|=n-k} [\sum_{j\notin B} q_j(B)]\phi(B) =$$

$$\sum_{|B|=n-k} \sum_{j\notin B} q_j(B\cup\{j\})[\phi(B\cup\{j\}) - \phi(B)].$$

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