

# Poisson Structures on the Phase Space of Mechanical Systems with Constraint

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*To the memory of Taro Yoshisawa*

**Abstract:** The existence of a Poisson structure on the phase space of mechanical systems with a fixed constraint satisfying the geometrical property that any conservative mechanical system with this constraint is itself hamiltonian (with respect to the Poisson structure) implies the integrability of the constraint. Two others equivalent geometrical properties are also presented.

**Key words:** Poisson Structure, Constrained Mechanical System, Integrability of Constrained.

## 1 Introduction

Let  $M^m$  be a connected  $C^\infty$  riemannian manifold of dimension  $m$ , with  $C^\infty$  metric  $\langle \cdot, \cdot \rangle_M$ .  $M$  is the configuration space of a mechanical system, the tangent bundle  $(TM, \tau, M)$  is the velocity phase space and the cotangent bundle  $(T^*M, \tau^*, M)$  is the momentum phase space. The metric defined on the manifold  $M$  induces a canonical  $C^\infty$  vector bundle isomorphism  $\mu : TM \rightarrow T^*M$ ,  $\tau^* \circ \mu = \tau$ , called the mass operator or Legendre transformation, defined by  $\mu(v)(w) = \langle v, w \rangle_M$ ,  $\forall v, w \in TM$ . The function  $K : TM \rightarrow \mathbb{R}$ , given by  $K(v) = \frac{1}{2} \langle v, v \rangle_M$  is the kinetic energy of the system. A constraint on the mechanical system is a  $C^\infty$  distribution  $\Sigma$  on  $M$ , that is, a  $C^\infty$  function that assigns to each  $q \in M$  an  $n$ -dimensional subspace  $\Sigma_q \subset T_q M$ . The vector sub-bundle  $\Sigma M = \bigcup_{q \in M} \Sigma_q$  is a  $(n+m)$ -dimension imbedded submanifold of  $TM$ . Using the metric, there is at each  $q \in M$  a well defined subspace  $\Sigma_q^\perp$ , which gives rise to another  $C^\infty$  vector sub-bundle  $\Sigma^\perp M$ , that is a  $(2m-n)$ -dimension imbedded submanifold of  $TM$ .

There are canonically defined  $C^\infty$  vector bundle maps  $P : TM \rightarrow \Sigma M$ ,  $Q : TM \rightarrow \Sigma^\perp M$  given, at each point  $q \in M$ , by the orthogonal projections on  $\Sigma$  and  $\Sigma^\perp$  respectively ( $P + Q = I$ ). By means of the mass operator, we can define  $\Sigma^* M = \mu(\Sigma M)$  and  $\Sigma^{\perp*} M = \mu(\Sigma^\perp M)$ , vector sub-bundles of  $T^*M$ . On  $T^*M$  we will consider the metric defined by  $\langle p_1, p_2 \rangle = \langle \mu^{-1}(p_1), \mu^{-1}(p_2) \rangle_M$ , for any  $p_1, p_2 \in T^*M$ , and the orthogonal projections  $P^* = \mu P \mu^{-1}$ ,  $Q^* = \mu Q \mu^{-1}$  on  $\Sigma^* M$  and  $\Sigma^{\perp*} M$ , respectively ( $P^* + Q^* = I$ ). Note that with these definitions,  $\Sigma^{\perp*} M = \Sigma^{*\perp} M$  (the orthogonal of  $\Sigma^* M$ ).

To completely define the mechanical system it is still necessary to choose a field of forces, that is, a  $C^\infty$  vector bundle homomorphism  $F : TM \rightarrow T^*M$ ,

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$\tau^* \circ F = \tau$ . In the case without constraints ( $\Sigma M = TM$ ), a physical curve  $q = q(t) \in M$  will be a solution of the Newton's law<sup>[A],[AM]</sup>:

$$\mu(\nabla_{\dot{q}}\dot{q}) = F(\dot{q}) , \quad (1)$$

where  $\nabla$  is the Levi-Civita connection associated to  $\langle , \rangle$  and  $\nabla_{\dot{q}}\dot{q}$  is the acceleration.,

If we are in the case of  $n < m$  ( $\Sigma$  involutive or not), to guarantee that the physical curves are compatible with the constraint ,that is,  $\dot{q} \in \Sigma_q$ , it is necessary to introduce a reaction field of forces,  $R$ , such that these physical curves will be solutions of the generalized Newton's law:

$$\mu(\nabla_{\dot{q}}\dot{q}) = (F + R)(\dot{q}), \quad \dot{q} \in \Sigma_q . \quad (2)$$

One assumes the *d'Alembert Principle*, that is,  $R(\dot{q}) = Q(R(\dot{q}))$ , and this implies that  $R$  is uniquely determined.

Remark that the derivatives of the solutions of (1) are the trajectories of a vector field  $X^{(1)}$  on  $TM$  and the derivatives of the solutions of (2) are the trajectories of a vector field  $X^{(2)}$  on  $\Sigma M$ ;  $X^{(2)}$  is called the GMA (Gibbs, Maggi, Appell) vector field. Remark also that the manifold  $\Sigma^* M = \mu(\Sigma M)$  depends on the data  $(M, \Sigma, \langle , \rangle)$  only, and can be considered as the phase space of any mechanical system with constraint  $\Sigma$ .

In [FO], Th.2.3, Fusco and Oliva have shown that even in the case of a noninvolutive distribution  $\Sigma$ , and under suitable conditions, the GMA vector field  $X^{(2)}$  is obtained from  $X^{(1)}$  through the derivative  $DP$  of the projection  $P$ , that is,  $DP(v)X_v^{(1)} = X_v^{(2)}$ , for all  $v \in \Sigma M$ . This theorem shows how the properties of the constrained system are determined by the properties of the unconstrained one, the geometrical properties of the distribution, and the metric. It is usual to work on  $T^*M$  and  $\Sigma^*M$  with the vector fields  $\mu \circ X^{(1)}$  and  $\mu \circ X^{(2)}$ , respectively, the last one also called GMA vector field.

An interesting special case is the one of conservative systems, where the field of forces  $F$  comes from a potential, that is,  $F = -dg$ ,  $g : M \rightarrow R$ , a  $C^\infty$  function. In this case, the system  $\mu \circ X^{(1)}$  is hamiltonian, that is, if we consider the canonical symplectic form on  $T^*M$ , and the associated Poisson bracket, the physical curves are obtained from the solutions of the Hamilton's equations,  $\dot{q} = \{q, H\}$  and  $\dot{p} = \{p, H\}$ ,  $H = K \circ \mu^{-1} + g \circ \tau^*$ . A natural question that arises is to ask if there exists (and under which conditions) a Poisson structure<sup>[W]</sup> on  $\Sigma^*M$  that turns the GMA vector field  $\mu \circ X^{(2)}$  into a hamiltonian system relatively to this Poisson structure, that is, such that the derivative of  $P^*$  takes hamiltonian vector fields on  $T^*M$  into hamiltonian vector fields on  $\Sigma^*M$ . We will give an answer to this and some other related questions in theorems 1 and 2 of section 3. In section 2 we present three natural geometrical properties of a Poisson structure and for each one of them, we study its consequences in local coordinates of  $\Sigma^*M$ . It turns out that these consequences are the same in the three cases. Theorem 1 of section 3 shows that those three natural properties are

indeed equivalent to the involutiveness of  $\Sigma$  and, by theorem 2, this involutiveness of  $\Sigma$  or any one of the three equivalent conditions assures the existence of a global Poisson structure on  $\Sigma^*M$  satisfying those conditions.

## 2 Basic definitions. Three natural properties for a Poisson structure on $\Sigma^*M$ .

Let  $(q_1, q_2, \dots, q_m)$  be a local coordinate system in a neighborhood  $U \subset M$ . Then, we have in  $T^*U \subset T^*M$  the local coordinate system  $(q_1, \dots, q_m, p_1, \dots, p_m)$ , defined by  $p_q = \sum_{i=1}^m p_i dq_i$  for all  $q \in U$  and  $p_q \in \Sigma^*U$ . In this coordinate system the canonical symplectic form is given by  $\omega = dp \wedge dq$ , and the Poisson bracket is given by:

$$\{f, g\} = \sum_{i=1}^m \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} = X_g(f) = -X_f(g) \quad (3)$$

for any  $f, g \in C^\infty(T^*M)$ . Here  $X_f$  denotes the hamiltonian vector field associated to  $f$ .

Let us consider in  $\Sigma^*U \subset \Sigma^*M$  an orthonormal basis (upon restriction of the neighborhood, if necessary)  $v_1, \dots, v_n$ , that is  $\langle v_i(q), v_j(q) \rangle = \delta_{i,j}$ ,  $\forall q \in U$ . We have in  $\Sigma^*U \subset \Sigma^*M$  the local coordinate system  $(q_1, \dots, q_m, V_1, \dots, V_n)$ , defined by  $p_q = \sum_{i=1}^n V_i(q) v_i(q)$  for all  $q \in U$  and  $p_q \in \Sigma^*U$ . Since  $v_i(q) \in T_q^*M$ ,  $i = 1, \dots, n$ , there are, locally determined, functions  $b_{ij} \in C^\infty(U)$ , for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , given by:

$$v_j(q) = \sum_{i=1}^m b_{ij}(q) dq_i(q). \quad (4)$$

A Poisson manifold<sup>[W]</sup>  $N$ , of dimension  $r$ , is a manifold with a Lie algebra structure over  $C^\infty(N)$ , that satisfies the Liebnitz identity

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \quad \text{for all } f, g, h \in C^\infty(N). \quad (5)$$

In a local coordinate system  $(x_1, \dots, x_r)$  the Poisson structure is determined by the component functions  $w_{ij}(x) = \{x_i, x_j\}$ , defined by the relation

$$\{f, g\} = \sum_{i,j=1}^r w_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \quad (6)$$

where  $w_{ij} = -w_{ji}$  (skew-symmetry) satisfy the Jacobi Identity :

$$\sum_{l=1}^r \left( w_{lj} \frac{\partial w_{ik}}{\partial x_l} + w_{li} \frac{\partial w_{kj}}{\partial x_l} + w_{lk} \frac{\partial w_{ji}}{\partial x_l} \right) = 0. \quad (7)$$

Conversely, any set of functions satisfying the skew-symmetry property and equations (7) determine, locally, by means of (6), a Poisson structure on  $N$ .

In our case, as  $N$  we consider the manifold  $\Sigma^*M$ . In the local coordinate system described above, every Poisson bracket has the form :

$$\begin{aligned} \{f, g\}_\Sigma = & \sum_{i=1}^m \sum_{j=1}^n W_{ij} \left( \frac{\partial f}{\partial q_i^\Sigma} \frac{\partial g}{\partial V_j} - \frac{\partial f}{\partial V_j} \frac{\partial g}{\partial q_i^\Sigma} \right) + \sum_{i,j=1}^m L_{ij} \frac{\partial f}{\partial q_i^\Sigma} \frac{\partial g}{\partial q_j^\Sigma} + \\ & + \sum_{i,j=1}^n K_{ij} \frac{\partial f}{\partial V_i} \frac{\partial g}{\partial V_j}, \end{aligned} \quad (8)$$

where the component functions satisfy equations (7) and the skew-symmetric relations  $L_{ij} = -L_{ji}$  and  $K_{ij} = -K_{ji}$  ( $(\frac{\partial}{\partial q_i^\Sigma}, \frac{\partial}{\partial V_j})$   $i = 1, \dots, m$ ;  $j = 1, \dots, n$  is the natural basis corresponding to the local coordinates  $(q_i, V_j)$  on  $\Sigma^*M$ ). The hamiltonian vector field  $X_f^\Sigma \in \mathcal{X}(\Sigma^*M)$  is given by:

$$\begin{aligned} X_f^\Sigma = & \sum_{i=1}^m \left( \sum_{j=1}^n W_{ij} \frac{\partial f}{\partial V_j} + \sum_{j=1}^m L_{ij} \frac{\partial f}{\partial q_j^\Sigma} \right) \frac{\partial}{\partial q_i^\Sigma} - \\ & - \sum_{j=1}^n \left( \sum_{i=1}^m W_{ij} \frac{\partial f}{\partial q_i^\Sigma} + \sum_{i=1}^m K_{ij} \frac{\partial f}{\partial V_i} \right) \frac{\partial}{\partial V_j}. \end{aligned} \quad (9)$$

Various conditions can be imposed on this Poisson structure on  $\Sigma^*M$  in order to reflect the theorem 2.3 in [FO] mentioned in the introduction. As a first try one could impose that every hamiltonian vector field of  $T^*M$  should be projected on a hamiltonian vector field of  $\Sigma^*M$ . More precisely, let  $i : \Sigma^*M \rightarrow T^*M$  be the inclusion map and let  $f \in C^\infty(T^*M)$  be an arbitrary function. Then  $\tilde{f} = f \circ i \in C^\infty(\Sigma^*M)$  and we can impose that:

$$DP_{p_q}^*(X_f) = X_{\tilde{f}}^\Sigma(p_q) \quad , \quad \forall p_q \in \Sigma^*M. \quad (10)$$

However, it is not difficult to see that the problem of existence of a Poisson structure on  $\Sigma^*M$  satisfying this property has no solution for  $m > n$ . Fortunately, if we try to solve the same problem under more relaxed conditions, we will obtain a much more interesting answer. We will consider below three such conditions.

**Condition I-** As a first condition, let us impose that the hamiltonian vector fields (of  $T^*M$ ) that arise from a function which is constant on the fibers of  $\Sigma^{\perp*}M$  project onto hamiltonian vector fields of  $\Sigma^*M$ . More precisely, let  $f \in C^\infty(T^*M)$  be a function such that  $f = \tilde{f} \circ P^*$  for some  $\tilde{f} \in C^\infty(\Sigma^*M)$ . We shall impose that the hamiltonian vector fields of such  $f$  and  $\tilde{f}$  are related by the last condition (10). Let us study what are the consequences of this condition in the local coordinate system defined above.

Note that , relatively to the basis  $\left( \frac{\partial}{\partial q_1}(p_q), \dots, \frac{\partial}{\partial q_m}(p_q), \frac{\partial}{\partial p_1}(p_q), \dots, \frac{\partial}{\partial p_m}(p_q) \right)$  of  $T_{p_q}(T^*M)$  and  $\left( \frac{\partial}{\partial q_1^\Sigma}(p_q), \dots, \frac{\partial}{\partial q_m^\Sigma}(p_q), \frac{\partial}{\partial V_1}(p_q), \dots, \frac{\partial}{\partial V_n}(p_q) \right)$  of  $T_{p_q}\Sigma^*M$  ,  $q \in U$  , the matrix of  $DP_{p_q}^*$  is given by:

$$[DP^*] = \begin{pmatrix} Id_{m \times m} & 0_{m \times m} \\ \sum_{l,k=1}^m p_l \frac{\partial}{\partial q_j}(b_{ki}g_{lk}) & \sum_{k=1}^m b_{ki}g_{jk} \end{pmatrix}_{m+n, m+m}, \quad (11)$$

where  $g_{ij}(p_q) = g_{ij}(q) = \langle dq_i(q), dq_j(q) \rangle$  . For the sake of simplicity we will omite , from now on, the point dependence of the functions, wherever the meaning is clear. Note that ,with this notation, in view of definition (4) we have:

$$\langle dq_i, v_j \rangle = \sum_{k=1}^m b_{kj}g_{ki}. \quad (12)$$

Since  $Df_{p_q} = D\tilde{f}_{p_q} \circ DP_{p_q}^*$  , for any  $p_q \in \Sigma^*M$  , the relations between the derivatives of  $f$  and  $\tilde{f}$  in a point  $p_q \in \Sigma^*U$  are given by:

$$\frac{\partial f}{\partial q_i} = \frac{\partial \tilde{f}}{\partial q_i^\Sigma} + \sum_{j=1}^m \sum_{k=1}^n p_j \frac{\partial}{\partial q_i} \langle dq_j, v_k \rangle \frac{\partial \tilde{f}}{\partial V_k}, \quad (13)$$

$$\frac{\partial f}{\partial p_i} = \sum_{k=1}^n \langle dq_i, v_k \rangle \frac{\partial \tilde{f}}{\partial V_k}. \quad (14)$$

Then, condition (10) takes the form:

$$\sum_{k=1}^n \langle dq_i, v_k \rangle \frac{\partial \tilde{f}}{\partial V_k} = \sum_{j=1}^n W_{ij} \frac{\partial \tilde{f}}{\partial V_j} + \sum_{j=1}^m L_{ij} \frac{\partial \tilde{f}}{\partial q_j^\Sigma}, \quad (15)$$

$$\begin{aligned} & \sum_{i=1}^m \left( -\frac{\partial \tilde{f}}{\partial q_i^\Sigma} - \sum_{s=1}^m \sum_{k=1}^n p_s \frac{\partial}{\partial q_i} \langle dq_j, v_k \rangle \frac{\partial \tilde{f}}{\partial V_k} \right) \langle dq_i, v_j \rangle + \\ & + \sum_{i,l=1}^m \sum_{s=1}^n p_l \frac{\partial}{\partial q_i} \langle dq_l, v_j \rangle \frac{\partial \tilde{f}}{\partial V_s} \langle dq_i, v_s \rangle = \\ & = -\sum_{i=1}^m W_{ij} \frac{\partial \tilde{f}}{\partial q_i^\Sigma} - \sum_{i=1}^n K_{ij} \frac{\partial \tilde{f}}{\partial V_i}. \end{aligned} \quad (16)$$

It is not difficult to see that equations (15) and (16) are equivalent to:

$$L_{ij} = 0 \quad , \quad i, j = 1, \dots, m, \quad (17)$$

$$W_{ij} = \langle dq_i, v_j \rangle, \quad i = 1, \dots, m \text{ and } j = 1, \dots, n, \quad (18)$$

$$K_{ij} = \sum_{k,l=1}^m p_l \left( \langle dq_k, v_j \rangle \frac{\partial}{\partial q_k} \langle dq_l, v_i \rangle - \langle dq_k, v_i \rangle \frac{\partial}{\partial q_k} \langle dq_l, v_j \rangle \right),$$

$$i, j = 1, \dots, n. \quad (19)$$

**Condition II-** A more natural condition in the theory of Poisson manifolds<sup>[W]</sup>, is that  $P^* : T^*M \rightarrow \Sigma^*M$  is a Poisson map. More precisely,

$$\{\tilde{f}, \tilde{g}\}_{\Sigma} \circ P^* = \{\tilde{f} \circ P^*, \tilde{g} \circ P^*\}, \quad \forall \tilde{f}, \tilde{g} \in C^\infty(\Sigma^*M). \quad (20)$$

If we try to study what are the consequences of this condition in the above local coordinate system, we find out, after tedious computations, the same expressions (17), (18) and (19) of condition I.

**Condition III-** Another condition, more directly related to classical mechanics, is that equation (10) above is satisfied only for functions  $f \in C^\infty(T^*M)$  that are classical hamiltonians (conservative mechanical system), that is,

$f(p_q) = \frac{1}{2} \langle p, p \rangle + g(q)$ , where  $g \in C^\infty(M)$ . The consequences of this condition in local coordinates, are again the same expressions (17), (18) and (19) of condition I.

It is rather unexpected that the three conditions above leads to the same local expression for a Poisson structure on  $\Sigma^*M$ .

### 3 Main Results

In this section we will present the main results of this paper.

**Theorem 1** *If there exists a Poisson structure on  $\Sigma^*M$ , such that, either condition I, condition II or condition III is satisfied, then the distribution  $\Sigma$  is involutive (integrable). In this case, the Poisson structure on  $\Sigma^*M$  satisfies all the three conditions and is locally given by:*

$$\{f, g\}_{\Sigma}(q_1, \dots, q_m, V_1, \dots, V_n) =$$

$$= \sum_{i=1}^m \sum_{j=1}^n \langle dq_i, v_j \rangle (q) \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial V_j} - \frac{\partial f}{\partial V_j} \frac{\partial g}{\partial q_i} \right) (q_1, \dots, q_m, V_1, \dots, V_n) -$$

$$- \sum_{i,j=1}^n \sum_{k=1}^n \langle [w_i, w_j], w_k \rangle (q) V_k \frac{\partial f}{\partial V_i} \frac{\partial g}{\partial V_j} (q_1, \dots, q_m, V_1, \dots, V_n), \quad (21)$$

where  $w_1, \dots, w_n$  is any orthonormal (local) basis of  $\Sigma M$  and  $(v_i) = (\mu w_i)$ ,  $i = 1, \dots, n$  is the orthonormal basis of  $\Sigma^*M$  considered in the beginning of section 2.

**Proof:** Let us choose a local coordinate system, as in Section (2), and an orthonormal basis  $w_1, \dots, w_n$  of  $\Sigma U$ , that is,  $w_i(q) \in \Sigma_q$  and  $\langle w_i(q), w_j(q) \rangle = \delta_{ij}$  for all  $q \in U$ ,  $i, j = 1, \dots, n$ . Then,  $v_i = \mu w_i$ , for  $i = 1, \dots, n$ , is an orthonormal basis of  $\Sigma^* U$ . By the hypotheses that one of our three conditions is satisfied, the matrix of the Poisson structure (8) is locally determined by equations (17), (18), (19). But, in order to have a Poisson manifold, we also need that equations (7) be satisfied. These equations reduce to some others that are automatically satisfied if equations (17), (18), (19) are, and to the following equations:

$$\Gamma_{jk}^i - \sum_{l=1}^n W_{il} \frac{\partial K_{kj}}{\partial v_l} = 0, \quad (22)$$

$$\begin{aligned} \sum_{l,t=1}^m \left\{ W_{lj} \frac{\partial (\Gamma_{ki}^t b_{ts})}{\partial q_l} + W_{li} \frac{\partial (\Gamma_{jk}^t b_{ts})}{\partial q_l} + W_{lk} \frac{\partial (\Gamma_{ij}^t b_{ts})}{\partial q_l} + \right. \\ \left. \sum_{\tau=1}^m (\Gamma_{jl}^t b_{ts} \Gamma_{ki}^\tau b_{\tau l} + \Gamma_{il}^t b_{ts} \Gamma_{jk}^\tau b_{\tau l} + \Gamma_{kl}^t b_{ts} \Gamma_{ij}^\tau b_{\tau l}) \right\} = 0, \end{aligned} \quad (23)$$

where  $b_{ij}$  are defined by (4) and  $\Gamma_{ij}^k$ ,  $i, j = 1, \dots, n$ ,  $k = 1, \dots, m$ , are defined by:

$$[w_i, w_j] = \sum_{k=1}^m \Gamma_{ij}^k \frac{\partial}{\partial q_k}. \quad (24)$$

With this notation, it is not difficult to show that  $b_{ji} = \langle w_i, \frac{\partial}{\partial q_j} \rangle$ , which implies that:

$$P[w_i, w_j] = \sum_{l=1}^m \sum_{s=1}^n \Gamma_{ij}^l b_{ls} w_s. \quad (25)$$

It follows from (25) that equations (22) and (23) are equivalent to :

$$[w_j, w_k] = P[w_j, w_k], \quad (26)$$

$$P[P[w_k, w_i], w_j] + P[P[w_j, w_k], w_i] + P[P[w_i, w_j], w_k] = 0, \quad (27)$$

which are satisfied if and only if the distribution is involutive. To conclude the proof of the theorem, it is enough to note that equation (21) is obtained from equation (8), using conditions (19) rewritten in terms of the  $w_i$ ,  $i = 1, \dots, n$ . 2

As a converse of theorem 1, we have the following:

**Theorem 2** *Let  $M^m$  be a riemaniann  $C^\infty$  manifold and  $\Sigma$  be a  $C^\infty$   $n$ -dimensional involutive constraint in  $M$ . Then*

$$\{f, g\}_\Sigma(p_q) = \{f \circ P^*, g \circ P^*\}(p_q), \forall f, g \in C^\infty(\Sigma^*M), \forall p_q \in \Sigma^*M \quad (28)$$

*defines a unique global Poisson structure on  $\Sigma^*M$  which satisfies conditions I, II and III above. Moreover, the local expression for the Poisson bracket is given by equality (21).*

**Proof:** Note that expression (28) defines, locally, a bracket that satisfies condition II above, and in the involutive case this bracket defines a local Poisson structure on  $\Sigma^*M$ . Conversely, it is clear that any local Poisson structure on  $\Sigma^*M$ , satisfying condition II, also satisfies (28). So, by the hypotheses of theorem 2, any point in  $M$  has a neighborhood where there exists a unique local Poisson structure which agrees with the one given by definition (28). Since the expression (28) is globally defined, we have a unique global Poisson struture on  $\Sigma^*M$ . 2

**Remark :** A classical example of a constraint is obtained by considering in a Lie group  $G$  with a left invariant metric, a distribution generated by left translations of a subspace  $\Sigma_e$  of the Lie algebra  $\mathcal{G}$  of  $G^{[K]}$ . The distribution is involutive if, and only if,  $\Sigma_e$  is a Lie sub-algebra of  $\mathcal{G}$ . For this example, with  $\Sigma$  involutive, theorems 1 and 2 can be applied.

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