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RESEARCH ARTICLE

Exact solution of the Einstein field equations for a spherical shell of fluid matter

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Abstract

We determine the exact solution of the Einstein field equations for the case of a spherically symmetric shell of liquid matter, characterized by an energy density which is constant with the Schwarzschild radial coordinate r between two values r_1 and r_2. The solution is given in three regions, one being the well-known analytical Schwarzschild solution in the outer vacuum region, one being determined analytically in the inner vacuum region, and one being determined mostly analytically but partially numerically, within the matter region. The solutions for the temporal coefficient of the metric and for the pressure within this region are given in terms of a non-elementary but fairly straightforward real integral. For some values of the parameters this integral can be written in terms of elementary functions. We show that in this solution there is a singularity at the origin, and give the parameters of that singularity in terms of the geometrical and physical parameters of the shell. This does not correspond to an infinite concentration of matter, but in fact to zero energy density at the center. It does, however, imply that the spacetime within the spherical cavity is not flat, so that there is a non-trivial gravitational field there, in contrast with Newtonian gravitation. This gravitational field is repulsive with respect to the origin, and thus has the effect of stabilizing the geometrical configuration of the matter, since any particle of the matter that wanders out into either one of the vacuum regions tends to be brought back to the bulk of the matter by the gravitational field.

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Contents

1 Introduction

The exterior Schwarzschild solution [1, 2] of the Einstein field equations has played a major role in General Relativity. It describes the effects of gravitation in the vacuum *outside* a time-independent spherically symmetric distribution of matter. One of the reasons for its importance is its generality—it only depends on the spherical symmetry and on the total energy of the matter distribution. Jebsen and Birkhoff [3, 4] have shown that this solution is still valid even in time-dependent situations, provided that the spherical symmetry is preserved. Another reason for its popularity is the association of the coordinate singularity of this solution, which occurs for a certain value of the radial coordinate, with the presence of an event horizon, thus leading to the concept of black holes.

Less known—even absent in many standard textbooks on General Relativity—is the interior Schwarzschild solution [2, 5]. It gives the metric of the space *inside* a spherically symmetric matter distribution with an energy density which is constant with the radial coordinate. This other solution can be continuously joined with the Schwarzschild vacuum solution that is valid outside the matter distribution. It is less general in that it only describes matter distributions with energy densities that do not depend on the radial coordinate r. In addition, it does not contain any singularities. This point is emphasized in many texts, for example in [2, 6]. Basically, in order to avoid singularities at the center of the matter distribution a certain integration constant is set equal to zero.

For a spherical matter shell characterized by an inner radius r_1, an outer radius r_2 and an energy density constant with r the situation is more involved. In the inner vacuum region, where $r < r_1$, the solution of the Einstein equations leads to an integration constant, heretofore denoted by r_μ, which determines the singularities in the entire inner vacuum region. There are no singularities only if $r_\mu = 0$. In analogy with what is done for the interior Schwarzschild solution one may feel tempted to set $r_\mu = 0$ by hand and eliminate all singularities. However, as we are going to show in this paper, the correct approach is to start in the outer vacuum region ($r > r_2$), where the exterior Schwarzschild solution holds, and use the continuity of the solution in the two boundaries of the three regions to determine the constant r_μ. The rather surprising result is that the imposition of the surface boundary conditions implies that $r_\mu > 0$, so that the solutions do contain a singularity at the origin. In addition, one can prove that this condition has to be satisfied in order to produce solutions with non-negative pressure inside the matter shell.

It is remarkable that the boundary conditions on matter interfaces for the Einstein field equations seem to play a smaller than expected role in the literature. A rare example in which the role of these boundary conditions is emphasized can be found in [7], although the author of that paper only obtained solutions containing a negative pressure region inside the matter shell. By analyzing these negative pressure solutions the author concluded that matter cannot collapse towards the center of black holes in general relativity. We are going to show in this paper that it is possible to obtain physically reasonable matter shell solutions of the Einstein equations with non-negative and finite pressure inside the shell. It is important to emphasize that the singularity at the origin in the inner vacuum region does not lead to any divergence of the matter quantities, and in fact stabilizes the matter shell structure. This is so because the gravitational field within the inner vacuum region turns out to be repulsive with respect to the origin. Our solutions for matter shells are expressed in terms of a single integral which for some values of the physical parameters can be written in terms of elementary functions and constitute a new class of exact solutions of the Einstein field equations.

Results similar to the ones we present here were obtained numerically for the case of neutron stars, with a Chandrasekhar-style equation of state [8], by Ni [9], including the presence of inner and outer matter-vacuum interfaces. However, the crucial consideration of the interface boundary conditions was missing from that analysis, thus leading to incomplete results. The discussion of the interface boundary conditions was subsequently introduced by Neslušan [10], thus completing the analysis of the case of the neutron stars. Just as in the present work, the discussion of the interface boundary conditions led, also in that case, to an inner vacuum region containing a singularity at the origin and a gravitational field pointing away from the origin, that is, repulsive with respect to the origin. The present work can be considered as an exactly solvable laboratory model that illustrates some of the properties of that numerical solution. It also shows that the properties of the inner vacuum region are not artifacts of that particular problem or of that particular type of equation of state.

This paper is organized as follows. In Sect. 2 we state and solve the problem; in Sect. 3 we derive the main physical properties of the solution; in Sect. 4 we present a two-parameter family of explicit solutions and a few numerical examples; and in Sect. 5 we present our conclusions.

2 The problem and its solution

We will present, in the case of a spherically symmetric shell of liquid fluid with constant energy density, the exact solution of the Einstein field equations of General Relativity [11],

$$R_{\mu}^{\ \nu} - \frac{1}{2} R\, g_{\mu}^{\ \nu} = -\kappa\, T_{\mu}^{\ \nu}, \tag{1}$$

where $\kappa = 8\pi G/c^4$, G is the universal gravitational constant and c is the speed of light. Under the conditions of time independence and of spherical symmetry around the origin of a spherical system of coordinates (t, r, θ, ϕ), the Schwarzschild system of coordinates, the most general possible metric is given by the invariant interval, written in terms of this spherical system of coordinates,

$$ds^2 = \mathrm{e}^{2\nu(r)} c^2 dt^2 - \mathrm{e}^{2\lambda(r)} dr^2 - r^2 \left[d\theta^2 + \sin^2(\theta) d\phi^2 \right], \tag{2}$$

where $\exp[\nu(r)]$ and $\exp[\lambda(r)]$ are two positive functions of only r. As one can see, in this work we will use the time-like signature $(+, -, -, -)$, following [11]. Under these conditions the matter stress-energy tensor density $T_{\mu}^{\ \nu}$ on the right-hand side of the equation is diagonal, and given by the four diagonal components $T_0^{\ 0}(r) = \rho(r)$, where $\rho(r)$ is the energy density of the matter, and $T_1^{\ 1}(r) = T_2^{\ 2}(r) = T_3^{\ 3}(r) = -P(r)$, where $P(r)$ is the pressure, which is isotropic, thus characterizing a fluid.

Since under these conditions $R_{\mu}^{\ \nu}$ and $T_{\mu}^{\ \nu}$ are both diagonal, there are just four non-trivial field equations contained in Eq. (1). In addition to these four field equations we have the consistency condition

$$D_{\nu} T_{\mu}^{\ \nu} = 0, \tag{3}$$

which is due to the fact that the combination of tensors that constitutes the left-hand side of the Einstein field equation satisfies the Bianchi identity of the Ricci curvature tensor. Writing these equations explicitly in the chosen coordinate system, one finds that the component equations involving $T_2^{\ 2}(r)$ and $T_3^{\ 3}(r)$ turn out to be identical, so that we are left with the set of four equations, including the consistency condition,

$$\left\{1 - 2\left[r\lambda'(r)\right]\right\} \mathrm{e}^{-2\lambda(r)} = 1 - \kappa r^2 \rho(r), \tag{4}$$

$$\left\{1 + 2\left[r\nu'(r)\right]\right\} \mathrm{e}^{-2\lambda(r)} = 1 + \kappa r^2 P(r), \tag{5}$$

$$\Big\{r^2 \nu''(r) - \left[r\lambda'(r)\right]\left[r\nu'(r)\right] + \left[r\nu'(r)\right]^2 + \left[r\nu'(r)\right] - \left[r\lambda'(r)\right]\Big\} \mathrm{e}^{-2\lambda(r)} = \kappa r^2 P(r), \tag{6}$$

$$[\rho(r) + P(r)]\, \nu'(r) = -P'(r), \tag{7}$$

where the primes indicate differentiation with respect to r. Next, it can be shown that Eq. (6) can be obtained from the others, being in fact a linear combination of the derivative of Eq. (5) and of Eqs. (4), (5) and (7). If we denote Eqs. (4) through (7) respectively by E_t, E_r, E_θ and E_c, we have that

$$E_\theta = \frac{1}{2}\left[-r\nu'(r)\,(E_t - E_r) + rE_r' + \kappa r^2 E_c\right]. \tag{8}$$

This leaves us with a set of just three differential equations to solve. In addition to this, we will assume that we have an energy density $\rho(r) = \rho_0$ which is constant as a function of r within the shell of fluid matter, thus characterizing a liquid fluid. The equations that we propose to solve are therefore those given in Eqs. (4), (5) and (7). It is important to note that, in this way, we are left with a system of just three *first-order* differential equations. Therefore, the discussion of boundary conditions can be limited to the discussion of the behavior of the functions involved, thus eliminating the need for any discussion of the behavior of their derivatives.

We will assume that the matter consists of a spherical shell of liquid, located between the radial positions r_1 and r_2, meaning that we will have an inner vacuum region within $(0, r_1)$, a matter region within (r_1, r_2), and an outer vacuum region within (r_2, ∞). This means that we will have for $\rho(r)$ and $P(r)$

$$\rho(r) = \begin{cases} 0 & \text{for} \quad 0 \leq r < r_1, \\ \rho_0 & \text{for} \quad r_1 < r < r_2, \\ 0 & \text{for} \quad r_2 < r < \infty, \end{cases} \tag{9}$$

$$P(r) = \begin{cases} 0 & \text{for} \quad 0 \leq r \leq r_1, \\ 0 & \text{for} \quad r_2 \leq r < \infty. \end{cases} \tag{10}$$

The function $P(r)$ within the matter region is, of course, one of the unknowns of our problem. In addition to this, we have the boundary conditions for $P(r)$ at the two interfaces, in the limits coming from within the liquid,

$$P(r_1) = 0, \tag{11}$$

$$P(r_2) = 0, \tag{12}$$

since these constitute a requirement in any interface between fluid matter and a vacuum. The remaining boundary conditions are those requiring the continuity of $\lambda(r)$ and $\nu(r)$ across the interfaces, and the asymptotic conditions leading to the Newtonian limit at radial infinity.

2.1 Solutions in the vacuum regions

Within either vacuum region the consistency condition in Eq. (7) becomes a mere identity, so that we are left with only two equations, in which we replace both $\rho(r)$ and $P(r)$ by zero,

$$1 - 2\left[r\lambda'(r)\right] = \mathrm{e}^{2\lambda(r)}, \tag{13}$$

$$1 + 2\left[r\nu'(r)\right] = \mathrm{e}^{2\lambda(r)}. \tag{14}$$

This immediately implies that $\lambda'(r) + \nu'(r) = 0$, and hence that $\lambda(r) + \nu(r) = A$, where A is a dimensionless integration constant. The first of these two equations involves only $\lambda(r)$, and can also be written as

$$\left[r\,\mathrm{e}^{-2\lambda(r)}\right]' = 1, \tag{15}$$

which can be immediately integrated to

$$\mathrm{e}^{-2\lambda(r)} = 1 - \frac{R}{r}, \tag{16}$$

where R is an integration constant with dimensions of length.

We must now discriminate between the inner and outer vacuum regions. In the outer vacuum region we must get flat space at radial infinity, which requires that both $\lambda(r)$ and $\nu(r)$ go to zero for $r \to \infty$. This in turn implies that $A = 0$ in the outer vacuum region, thus leading to $\nu(r) = -\lambda(r)$. As is well known, the condition that the Newtonian limit be realized at radial infinity requires that $R = r_M$, the Schwarzschild radius $r_M = 2MG/c^2$ associated to the asymptotic gravitational mass M of the system. Thus we arrive at the time-honored Schwarzschild solution [1, 2] in the outer vacuum region,

$$\lambda_s(r) = -\frac{1}{2}\ln\left(\frac{r - r_M}{r}\right), \tag{17}$$

$$\nu_s(r) = \frac{1}{2}\ln\left(\frac{r - r_M}{r}\right), \tag{18}$$

where the subscript s denotes the outer vacuum region. Note that there is a limitation on the values of the parameters r_2 and r_M describing the distribution of matter, because these expressions have a singular behavior at $r = r_M$. We must have $r_M < r_2$ to ensure that there is no event horizon formed outside the distribution of matter.

In the inner vacuum region there are no asymptotic conditions to be applied, and thus the integration constants A and R will have to be left undetermined, to be determined later on via the boundary conditions at the interfaces between the vacuum and the matter, as we come in from radial infinity towards the origin. For convenience we will put $R = -r_\mu$, and write the solution in the inner vacuum region as

$$\lambda_i(r) = -\frac{1}{2}\ln\left(\frac{r + r_\mu}{r}\right), \tag{19}$$

$$\nu_i(r) = A + \frac{1}{2}\ln\left(\frac{r + r_\mu}{r}\right), \tag{20}$$

where the subscript i denotes the inner vacuum region. Note that the value of r_μ determines the singularity structure of this solution within the inner vacuum region. If $r_\mu < 0$ then there is a singularity at the strictly positive radial position $r = -r_\mu$, corresponding to the formation of an event horizon at that position. If $r_\mu = 0$ then there are no singularities at all within this region. If $r_\mu > 0$ then there is only one point of singularity, located at the origin $r = 0$. We will show later on that we do indeed have that $r_\mu > 0$.

We therefore have the complete analytical solutions in the inner and outer vacuum regions, which contain one input parameter of the problem, the mass M associated to the Schwarzschild radius r_M, and two integration constants still to be determined, A and r_μ.

2.2 Solution in the matter region

In the matter region Eq. (4) for $\lambda(r)$ can be written as

$$\left[r\,\mathrm{e}^{-2\lambda(r)}\right]' = 1 - \kappa\rho_0 r^2, \tag{21}$$

which can be immediately integrated to

$$\mathrm{e}^{-2\lambda(r)} = 1 + \frac{B}{r} - \frac{\kappa\rho_0}{3}\,r^2, \tag{22}$$

where B is an integration constant with dimensions of length, thus leading to the general solution for $\lambda(r)$ in the matter region,

$$\lambda_m(r) = -\frac{1}{2}\ln\left(1 + \frac{B}{r} - \frac{\kappa\rho_0}{3}\,r^2\right), \tag{23}$$

where the subscript m denotes the matter region. This solution contains one integration constant, the constant B, and one parameter characterizing the system, namely ρ_0, which is not, however, a free input parameter of the problem, since it will depend on M and thus on r_M.

In order to deal with $\nu(r)$ in the matter region, we consider the consistency condition given in Eq. (7), which can be written in this region as

$$\nu'(r) = -\frac{P'(r)}{\rho_0 + P(r)}, \tag{24}$$

thus allowing us to separate variables and hence to write $\nu(r)$ in terms of $P(r)$,

$$\begin{aligned} d\nu &= -\frac{dP}{\rho_0 + P} \\ &= -d\ln(\rho_0 + P)\,. \end{aligned} \tag{25}$$

If we integrate from the left end r_1 of the matter interval to a generic point r within that interval, we get

$$\nu(r) - \nu(r_1) = -\ln\left[\frac{\rho_0 + P(r)}{\rho_0 + P(r_1)}\right]. \tag{26}$$

However, the boundary conditions for $P(r)$ at the interfaces tell us that we must have $P(r_1) = 0$, and hence we get the general solution for $\nu(r)$ within the matter region, written in terms of $P(r)$,

$$\nu_m(r) = \nu_1 - \ln\left[\frac{\rho_0 + P(r)}{\rho_0}\right], \tag{27}$$

where $\nu_1 = \nu(r_1)$. The solutions for $\lambda(r)$ and $\nu(r)$ within the matter region involve therefore two integration constants, B and ν_1. The solution for $\nu(r)$ is not yet completely determined, since it is given in terms of $P(r)$, which is also as yet undetermined. However, the information obtained so far already allows us to impose the boundary conditions at the interfaces, in order to determine the integration constants, which is what we turn to now.

2.3 Interface boundary conditions

The condition of the continuity of $\lambda(r)$ at the interface r_1 implies that we must have that $\lambda_i(r_1) = \lambda_m(r_1)$, which from Eqs. (19) and (23) gives us the following relation between the parameters

$$B - r_\mu = \frac{\kappa\rho_0}{3} r_1^3. \tag{28}$$

In addition to this, the condition of the continuity of $\lambda(r)$ at the interface r_2 implies that we must have $\lambda_m(r_2) = \lambda_s(r_2)$, which from Eqs. (17) and (23) gives us the following relation between B and the parameters

$$B + r_M = \frac{\kappa\rho_0}{3} r_2^3. \tag{29}$$

This last condition already determines the integration constant B in terms of the parameters of the problem,

$$B = -r_M + \frac{\kappa\rho_0}{3} r_2^3, \tag{30}$$

and the difference of the two conditions just obtained determines the integration parameter r_μ in terms of the parameters of the problem,

$$r_\mu = -r_M + \frac{\kappa\rho_0}{3}\left(r_2^3 - r_1^3\right). \tag{31}$$

We have therefore the solution for $\lambda(r)$ in the matter region, in terms of the parameters of the problem,

$$\lambda_m(r) = -\frac{1}{2}\ln\left[\frac{\kappa\rho_0\left(r_2^3 - r^3\right) + 3\left(r - r_M\right)}{3r}\right]. \tag{32}$$

Let us point out that there is a consistency condition to be applied to this result, since we must have that the cubic polynomial appearing in the argument of the logarithm be strictly positive for all values of r within the matter region, that is

$$\kappa\rho_0\left(r_2^3 - r^3\right) + 3\,(r - r_M) > 0, \tag{33}$$

for all $r \in [r_1, r_2]$. Note that the term with the cubes is necessarily non-negative, but that the other term may be negative, if r_M is not smaller than r_1. Therefore, so long as $r_M < r_1$, this strict positivity condition is automatically satisfied. If, however, we have that $r_1 < r_M < r_2$, then the condition must be actively verified for all $r \in [r_M, r_2]$. If it fails, then there is no solution for that particular set of input parameters.

Since we have $\nu_m(r)$ written in terms of $P(r)$, and since we know the interface boundary conditions for $P(r)$ in limits from within the matter region, we are in a position to impose the boundary conditions on $\nu(r)$ across the interfaces, even without having available the complete solution for $\nu_m(r)$. To this end, let us note that from Eq. (27) we have that $\nu_m(r_1) = \nu_m(r_2) = \nu_1$. At the interface r_1 the condition of the continuity of $\nu(r)$ implies that we must have $\nu_i(r_1) = \nu_m(r_1)$, which from Eqs. (20) and (27) gives us the following relation between ν_1, A and the parameters,

$$\nu_1 = A + \frac{1}{2}\,\ln\!\left(\frac{r_1 + r_\mu}{r_1}\right). \tag{34}$$

In addition to this, the condition of the continuity of $\nu(r)$ at the interface r_2 implies that we must have $\nu_m(r_2) = \nu_s(r_2)$, which from Eqs. (18) and (27) gives us the following relation between ν_1 and the parameters,

$$\nu_1 = \frac{1}{2}\,\ln\!\left(\frac{r_2 - r_M}{r_2}\right). \tag{35}$$

This last condition gives us the integration constant ν_1 in terms of the parameters of the problem, and its difference with the previous one determines the integration constant A,

$$A = \frac{1}{2}\,\ln\!\left(\frac{1 - r_M/r_2}{1 + r_\mu/r_1}\right). \tag{36}$$

Note that we have that $A < 0$ for any positive values of r_M and r_μ. This completes the determination of the solution for both $\nu(r)$ and $\lambda(r)$ in the inner vacuum region, for which we now have

$$\lambda_i(r) = -\frac{1}{2}\,\ln\!\left(\frac{r + r_\mu}{r}\right), \tag{37}$$

$$\nu_i(r) = \frac{1}{2}\,\ln\!\left(\frac{1 - r_M/r_2}{1 + r_\mu/r_1}\right) + \frac{1}{2}\,\ln\!\left(\frac{r + r_\mu}{r}\right), \tag{38}$$

with r_μ given by Eq. (31). We also have the following form for the solution for $\nu(r)$ within the matter region, still in terms of $P(r)$,

$$\nu_m(r) = \frac{1}{2}\ln\left(\frac{r_2 - r_M}{r_2}\right) - \ln\left[\frac{\rho_0 + P(r)}{\rho_0}\right]. \tag{39}$$

At this point the situation is as follows, in regard to the complete solution of the problem. Given values of r_1, r_2 and r_M, which completely characterize the geometrical and physical nature of the object under study, we have the complete solution for both $\lambda(r)$ and $\nu(r)$ in the outer vacuum region. We also have the complete solution for both $\lambda(r)$ and $\nu(r)$ in the inner vacuum region, except for the determination of the parameter ρ_0. We have as well the complete solution for $\lambda(r)$ in the matter region, again up to the determination of the parameter ρ_0. The one element of the solution still missing is the complete solution for $\nu(r)$ in the matter region. However, since we have $\nu(r)$ determined in terms of $P(r)$ in this region, this can also be accomplished by the complete determination of $P(r)$ in this region, which is the task we tackle next. Let us emphasize that the parameter ρ_0 is not a free input parameter of the problem, since it must be chosen so that the given value of r_M results, that is, the local value of the energy density must be chosen so that the given value of the asymptotic gravitational mass M results at radial infinity.

2.4 The equation for the pressure

The equation determining the pressure $P(r)$ in the matter region can be obtained by eliminating $\nu'(r)$ from Eqs. (5) and (7), which gives us

$$\rho_0 + P(r) - 2\left[rP'(r)\right] = \mathrm{e}^{2\lambda_m(r)}\left[1 + \kappa r^2 P(r)\right][\rho_0 + P(r)]. \tag{40}$$

In this equation the quantity $\exp[2\lambda_m(r)]$ is a known function, since we have already determined $\lambda(r)$ in the matter region. This is a first-order non-linear differential equation determining $P(r)$, with the boundary conditions $P(r_1) = 0$ and $P(r_2) = 0$. Since the equation is first-order and there are two boundary conditions to be satisfied, it is clear that the parameter ρ_0 will have to be adjusted so that the second condition can be satisfied. This will therefore determine the parameter ρ_0 in terms of the other parameters of the problem. This equation can be simplified by a series of transformations on the variables and parameters. First we define the parameter $\Upsilon_0 = 1/r_0$, which has dimensions of inverse length and is such that

$$\Upsilon_0^2 = \kappa\rho_0, \tag{41}$$

and the dimensionless pressure

$$p(r) = \frac{P(r)}{\rho_0}, \tag{42}$$

in terms of which Eq. (40) becomes

$$\left[rp'(r)\right] = \frac{1}{2}\left[1 + p(r)\right]\left\{1 - \mathrm{e}^{2\lambda_m(r)}\left[1 + \Upsilon_0^2 r^2 p(r)\right]\right\}. \tag{43}$$

Substituting the known value of $\lambda_m(r)$ from Eq. (32) we get

$$p'(r) = \frac{1}{2r}\left[1 + p(r)\right]\frac{\Upsilon_0^2\left(r_2^3 - r^3\right) - 3r_M - 3\Upsilon_0^2 r^3 p(r)}{\Upsilon_0^2\left(r_2^3 - r^3\right) + 3\left(r - r_M\right)}. \tag{44}$$

This has the form of a Ricatti equation, and can be linearized by the transformation of variables

$$p(r) = \frac{1}{z(r)} - 1, \tag{45}$$

thus resulting in the equation for $z(r)$,

$$z'(r) + \frac{\Upsilon_0^2\left(r_2^3 + 2r^3\right) - 3r_M}{2r\left[\Upsilon_0^2\left(r_2^3 - r^3\right) + 3\left(r - r_M\right)\right]}z(r) = \frac{3\Upsilon_0^2 r^3}{2r\left[\Upsilon_0^2\left(r_2^3 - r^3\right) + 3\left(r - r_M\right)\right]}. \tag{46}$$

This equation has an integrating factor given by $\exp[F(r)]$, where $F(r)$ is defined as an integral of the coefficient of the second term from r_2 to some arbitrary r within $[r_1, r_2]$,

$$\begin{aligned} F(r) &= \int_{r_2}^{r} ds\, \frac{\Upsilon_0^2\left(r_2^3 + 2s^3\right) - 3r_M}{2s\left[\Upsilon_0^2\left(r_2^3 - s^3\right) + 3\left(s - r_M\right)\right]} \\ &= \frac{1}{2}\int_{r_2}^{r} ds\, \frac{1}{s} - \frac{1}{2}\int_{r_2}^{r} ds\, \frac{-3\Upsilon_0^2 s^2 + 3}{\Upsilon_0^2\left(r_2^3 - s^3\right) + 3\left(s - r_M\right)}. \end{aligned} \tag{47}$$

One can see now that both integrals can be done, and thus we obtain

$$e^{F(r)} = \sqrt{\frac{r}{r_2}}\sqrt{\frac{3\left(r_2 - r_M\right)}{\Upsilon_0^2\left(r_2^3 - r^3\right) + 3\left(r - r_M\right)}}, \tag{48}$$

in terms of which Eq. (46) for $z(r)$ can be written as

$$\left[\mathrm{e}^{F(r)} z(r)\right]' = \frac{3}{2}\frac{\Upsilon_0^2 r^2\, \mathrm{e}^{F(r)}}{\Upsilon_0^2\left(r_2^3 - r^3\right) + 3\left(r - r_M\right)}, \tag{49}$$

which can then be integrated over the interval $[r, r_2]$ giving

$$z(r) = e^{-F(r)} + \frac{3}{2}\, e^{-F(r)} \int_{r_2}^{r} ds\, \frac{\Upsilon_0^2 s^2 e^{F(s)}}{\Upsilon_0^2 \left(r_2^3 - s^3\right) + 3\,(s - r_M)}, \tag{50}$$

where we used the fact that by definition $F(r_2) = 0$, and the fact that $P(r_2) = 0$ implies $z(r_2) = 1$.

Note that once more the existence of the solutions for $F(r)$ and for $z(r)$ is conditioned by the strict positivity of the same cubic polynomial that we discussed before in Eq. (33), which we can now write as

$$\Upsilon_0^2 \left(r_2^3 - r^3\right) + 3\,(r - r_M) > 0, \tag{51}$$

for all $r \in [r_1, r_2]$. Substituting the value of $\exp[F(r)]$ we have the solution for $z(r)$ written in terms of a real integral,

$$\begin{aligned} z(r) = &\sqrt{\frac{\Upsilon_0^2 \left(r_2^3 - r^3\right) + 3\,(r - r_M)}{r}} \\ &\times \left\{ \sqrt{\frac{r_2}{3\,(r_2 - r_M)}} + \frac{3}{2} \int_{r_2}^{r} ds\, \frac{\Upsilon_0^2 s^{5/2}}{\left[\Upsilon_0^2 \left(r_2^3 - s^3\right) + 3\,(s - r_M)\right]^{3/2}} \right\}. \end{aligned} \tag{52}$$

In most cases this remaining integral is elliptic and therefore cannot be written in terms of elementary functions, so that in general this remaining last step of the resolution procedure has to be performed by numerical means. However, as we are going to show in Sect. 4, for some values of the parameters it is possible to express this integral in terms of elementary functions.

After determining $z(r)$ in the matter region, Eq. (45) allows us to calculate the dimensionless pressure $p(r)$ which, according to Eq. (42), is equal to the pressure divided by the energy density ρ_0,

$$p(r) = \frac{1}{z(r)} - 1 \quad \Longrightarrow \tag{53}$$

$$P(r) = \frac{\rho_0}{z(r)} - \rho_0. \tag{54}$$

Note that $z(r)$ also determines $\nu(r)$ in the matter region, since in Eq. (39) we have $\nu_m(r)$ in terms of $P(r)$, and therefore we have for the exponential of $\nu_m(r)$,

$$\mathrm{e}^{\nu_m(r)} = \sqrt{\frac{r_2 - r_M}{r_2}}\, \frac{\rho_0}{\rho_0 + P(r)}, \tag{55}$$

which, using Eq. (54), implies that

$$\mathrm{e}^{\nu_m(r)} = \sqrt{\frac{r_2 - r_M}{r_2}}\, z(r), \tag{56}$$

so that, up to a constant factor, $z(r)$ turns out to be the square root of the temporal coefficient of the metric. This completes the determination of the solution in all three regions, in terms of the parameters of the problem. Given certain values of r_1, r_2 and r_M, one must still find a value of the parameter ρ_0, and hence of Υ_0, such that the boundary conditions for $P(r)$ at the two interfaces are satisfied. One can obtain an equation determining this value of Υ_0 by considering the value of $z(r_1)$. Since $P(r_1) = 0$, we have that $z(r_1) = 1$, so that from Eq. (52) we get

$$\sqrt{\frac{r_2}{3\,(r_2 - r_M)}} = \sqrt{\frac{r_1}{\Upsilon_0^2\left(r_2^3 - r_1^3\right) + 3\,(r_1 - r_M)}} + \frac{3}{2}\int_{r_1}^{r_2} dr\, \frac{\Upsilon_0^2 r^{5/2}}{\left[\Upsilon_0^2\left(r_2^3 - r^3\right) + 3\,(r - r_M)\right]^{3/2}}. \tag{57}$$

The solution of this algebraic equation gives the value of Υ_0, and hence the value of ρ_0, for which the two interface boundary conditions for $P(r)$ will be satisfied. The solution of this equation necessarily includes the consistency check of the solution obtained, since the calculation of the integral is dependent on the strict positivity of the polynomial in Eq. (51), for all r within $[r_1, r_2]$. This is the same condition that guarantees the consistency of the results for $F(r)$ and $z(r)$, and hence the consistency of the results for $P(r)$ and $\nu(r)$ within the matter region.

3 Main properties of the solution

In this section we will state and prove a few important properties of the solution. We will assume that, given certain values of r_1, r_2 and r_M, the corresponding solution exists. In other words, we are assuming that a solution of Eq. (57) for Υ_0 can be found, thus determining ρ_0, which includes establishing the strict positivity of the cubic polynomial within the square roots in the denominators, and that a corresponding function $z(r)$ is therefore determined via Eq. (52). This then implies that the solutions for both $\lambda(r)$ and $\nu(r)$, as well as for $P(r)$, are all determined, with all the boundary conditions duly satisfied. A simpler way to put this is to say that we are establishing the most important properties of all existing solutions of the problem. For easy reference, we state the complete solution explicitly in Table 1, where we have that ρ_0 is determined algebraically via Eq. (57), $z(r)$ is determined by Eq. (52), and r_μ is given by Eq. (31). We will start by the discussion of the presence of the singularity at the origin.

3.1 Existence of the singularity at the origin

First of all, we should clarify that at this point we are not trying to characterize the singularity at the origin in differential-geometric terms. This will be done later, in Sect. 3.4, where we will show that it is in fact a curvature singularity. For now we are simply determining its existence, in the sense of the singularities in the theory of analytic functions, in the relevant functions that characterize the spacetime metric in

Table 1 Summary of the results

$$\lambda(r)=\begin{cases}-\frac{1}{2}\ \ln\left(\frac{r+r_\mu}{r}\right) & \text{for}\quad 0<r\le r_1,\\ -\frac{1}{2}\ \ln\left[\frac{\kappa\rho_0\left(r_2^3-r^3\right)+3\left(r-r_M\right)}{3r}\right] & \text{for}\quad r_1\le r\le r_2,\\ -\frac{1}{2}\ \ln\left(\frac{r-r_M}{r}\right) & \text{for}\quad r_2\le r<\infty,\end{cases}$$

$$\nu(r)=\begin{cases}\frac{1}{2}\ \ln\left(\frac{1-r_M/r_2}{1+r_\mu/r_1}\right)+\frac{1}{2}\ \ln\left(\frac{r+r_\mu}{r}\right) & \text{for}\quad 0<r\le r_1,\\ \frac{1}{2}\ \ln\left(\frac{r_2-r_M}{r_2}\right)+\ln[z(r)] & \text{for}\quad r_1\le r\le r_2,\\ \frac{1}{2}\ \ln\left(\frac{r-r_M}{r}\right) & \text{for}\quad r_2\le r<\infty.\end{cases}$$

Schwarzschild coordinates. We may however note that, since we have that $T_{\mu\nu}=0$ in the inner vacuum region, which implies that $R_{\mu\nu}=0$ there, we have for the scalar curvature that $R=0$ everywhere within that region, including at the origin. In any case, the singularity may be physically characterized as one that does *not* lead to an infinite concentration of matter at that point.

The existence of the singularity at the origin is equivalent to the statement that $r_\mu>0$, because the only way to avoid that singularity would be to have $r_\mu=0$. If we put $r_\mu=0$ then, as we will see, we are forced to make $r_1=0$, so that we no longer have a matter shell, and we obtain instead the Schwarzschild interior solution for a filled sphere. It is important to note that the argument that follows depends on the condition that $r_1>0$, because otherwise there would be no inner vacuum region, and hence no interface boundary conditions at r_1, so that all the derivations done so far would cease to be valid.

We start with a preliminary lemma for a matter shell with $r_1>0$, in which we will prove that the following combination of parameters

$$\frac{1}{3}\ \Upsilon_0^2\left(r_2^3-r_e^3\right)-r_M>0, \tag{58}$$

is strictly positive, where r_e is the position of the maximum of the dimensionless pressure $p(r)$ within the interval $[r_1,r_2]$. In order to do this, we consider the equation for $p(r)$ given in Eq. (44). Since $p(r)$ is a positive function that is the solution of a first-order differential equation within the strict interior (r_1,r_2) of the interval, it must be a continuous and differentiable function there. Therefore, given that due to the interface boundary conditions it is zero at both ends and hence always increases as we go to the interior of the interval, it must have a point of maximum r_e somewhere in the strict interior of the interval, where we will have that $p'(r_e)=0$. Using the differential equation for $p(r)$ given by Eq. (44) at this point we thus obtain

$$\frac{1}{2r_e}\left[1+p(r_e)\right]\frac{\Upsilon_0^2\left(r_2^3-r_e^3\right)-3r_M-3\Upsilon_0^2r_e^3p(r_e)}{\Upsilon_0^2\left(r_2^3-r_e^3\right)+3\left(r_e-r_M\right)}=0. \tag{59}$$

This can only be zero if the numerator is zero, so we have that

$$\Upsilon_0^2 r_e^3 p(r_e) = \frac{1}{3}\Upsilon_0^2\left(r_2^3 - r_e^3\right) - r_M. \tag{60}$$

Since $\Upsilon_0^2 > 0$ and at its maximum we must have $p(r_e) > 0$ for the dimensionless pressure, we conclude that our lemma holds,

$$\frac{1}{3}\Upsilon_0^2\left(r_2^3 - r_e^3\right) - r_M > 0. \tag{61}$$

Let us now consider the result for r_μ in terms of the given parameters of the problem, as shown in Eq. (31), which we can write as

$$r_\mu = \frac{1}{3}\Upsilon_0^2\left(r_2^3 - r_1^3\right) - r_M. \tag{62}$$

By adding and subtracting terms to this equation, we can write it as

$$r_\mu = \left[\frac{1}{3}\Upsilon_0^2\left(r_2^3 - r_e^3\right) - r_M\right] + \frac{1}{3}\Upsilon_0^2\left(r_e^3 - r_1^3\right). \tag{63}$$

The quantity within square brackets is the one we just proved to be strictly positive in our lemma. The other term is also strictly positive because we certainly have that $r_e > r_1$. Therefore, we have our theorem,

$$r_\mu > 0. \tag{64}$$

It is important to observe that what we have proved here can be stated as

$$r_1 > 0 \implies r_\mu > 0, \tag{65}$$

which then implies that we must also have, as we mentioned before, that

$$r_\mu = 0 \implies r_1 = 0. \tag{66}$$

As a consistency check, we can easily calculate the derivative $p'(r)$ at the two ends of the interval. Since $p(r)$ is a positive function that is zero at both ends, it follows that its derivative at r_2 must be negative, and that its derivative at r_1 must be positive. Applying Eq. (44) at r_2, since we have that $p(r_2) = 0$, we get for the derivative at the right end of the matter interval,

$$p'(r_2) = -\frac{r_M}{2r_2\left(r_2 - r_M\right)}. \tag{67}$$

Since by hypothesis we have that $r_2 > r_M$ and that $r_M > 0$, we conclude that the derivative $p'(r_2)$ is strictly negative, as expected. In addition to this, applying that same equation at r_1 and using Eq. (62) in order to substitute in favor of r_μ, since we also have that $p(r_1) = 0$, we get for the derivative at the left end of the matter interval,

$$p'(r_1) = \frac{r_\mu}{2r_1\left(r_1 + r_\mu\right)}, \tag{68}$$

which is positive, as expected, since $r_\mu > 0$. Therefore we see that our result is consistent with a positive pressure within the whole matter interval.

In this way we see that every solution of the problem that exists at all is bound to have a singularity at the origin, which is characterized by the factor

$$\ln\left(\frac{r + r_\mu}{r}\right), \tag{69}$$

that appears with a negative sign in $\lambda_i(r)$ and with a positive sign in $\nu_i(r)$. This implies that at this singular point we have that

$$\lim_{r\to 0} \lambda_i(r) = -\infty, \tag{70}$$

$$\lim_{r\to 0} \mathrm{e}^{\lambda_i(r)} = 0, \tag{71}$$

$$\lim_{r\to 0} \nu_i(r) = \infty, \tag{72}$$

$$\lim_{r\to 0} \mathrm{e}^{\nu_i(r)} = \infty. \tag{73}$$

Note that this singularity does not have any disastrous consequences, since it does not imply infinite concentrations of matter. In fact, we have $\rho(r) = 0$ in the whole inner vacuum region, including at the origin. For the proper lengths in the radial direction, it just implies that they get progressively more *contracted* as we approach the origin, rather than being expanded with respect to the corresponding variations of the radial coordinate r, as is the case in the outer vacuum region. For the proper times it just means that we get progressively more severe *red* shifts as we approach the origin, rather than the blue shifts that we get as we approach the event horizon from the outer vacuum region.

As a corollary to the proof that $r_\mu > 0$, note that this fact guarantees the positivity of the cubic polynomial in Eq. (33). This is so because the second derivative of that polynomial is given by $-6\kappa\rho_0 r$, being therefore negative for all $r \in [r_1, r_2]$. This means that the graph of the cubic polynomial has a concavity turned downward throughout this interval. In addition to this, it is easy to see that at $r = r_2$ the polynomial is given by $3\,(r_2 - r_M)$, which is strictly positive so long as $r_2 > r_M$. Finally, at $r = r_1$ the polynomial is given by

$$\kappa\rho_0\left(r_2^3 - r_1^3\right) + 3\,(r_1 - r_M) = 3\left(r_1 + r_\mu\right), \tag{74}$$

where we used Eq. (31), which is also strictly positive since $r_\mu > 0$. As a consequence of this, we may conclude that, so long as the conditions $r_2 > r_M$ and $r_\mu > 0$ hold, as they must for physically sensible solutions, the polynomial is strictly positive for all $r \in [r_1, r_2]$.

3.2 Nature of the inner gravitational field

The physical interpretation of the function $\nu(r)$ is that the proper time interval at the radial position r, between two events occurring at the same spatial point, is given by $d\tau = \exp[\nu(r)]dt$, where dt is the time interval between the two events as seen at spatial infinity, where spacetime is flat. If we consider a photon traveling in the radial direction, either inwards or outwards, this means that the proper frequency $f(r)$ of the photon changes with position, between a first point r_a and a second point r_b, according to

$$f(r_a) = \mathrm{e}^{-\nu(r_a)} f_\infty, \tag{75}$$
$$f(r_b) = \mathrm{e}^{-\nu(r_b)} f_\infty, \tag{76}$$

where f_∞ is the frequency of the photon at radial infinity. Dividing these two equations and making the two points very close together, so that $r_a = r$ and $r_b = r_a + \delta r$, we have

$$\frac{f(r+\delta r)}{f(r)} = \mathrm{e}^{-[\nu(r+\delta r)-\nu(r)]}. \tag{77}$$

For sufficiently small δr we may write the variation of the function $\nu(r)$ in terms of its derivative $\nu'(r)$, so that we get

$$\frac{f(r+\delta r)}{f(r)} \simeq \mathrm{e}^{-\delta r\, \nu'(r)}. \tag{78}$$

Since the energy $hf(r)$ of a photon, h being the Planck constant, is proportional to its frequency, we have an interpretation of the red and blue shifts of the frequency of the photons as decreases or increases in their energies, respectively. We thus observe that, if a photon is going outward, so that $\delta r > 0$, and if the derivative $\nu'(r)$ is positive, then we will have that $f(r+\delta r) < f(r)$, and therefore a red shift in the frequency. If it is going outward but the derivative is negative, then we will have that $f(r+\delta r) > f(r)$ and hence a blue shift. On the other hand, if the photon is going inward, so that $\delta r < 0$, and the derivative is positive, then we will have a blue shift, and finally, if it is going inward and the derivative is negative, then we will have a red shift. Let us write down the derivative of $\nu(r)$ in the inner and outer vacuum regions,

$$\nu'(r) = \begin{cases} -\dfrac{1}{2}\dfrac{r_\mu}{r(r+r_\mu)} & \text{for} \quad 0 < r \leq r_1, \\ \dfrac{1}{2}\dfrac{r_M}{r(r-r_M)} & \text{for} \quad r_2 \leq r < \infty. \end{cases} \tag{79}$$

Let us now consider the consequences of Eq. (78) in more detail in each one of these two regions, starting with the outer vacuum region. As one can see above, in the outer vacuum region, since we have that $r > r_2 > r_M > 0$, the derivative $\nu'(r)$ is always positive. Therefore, photons traveling outward undergo red shifts, while those traveling inward undergo blue shifts. This can be interpreted in energetic terms as the statement that when traveling inward the photons gain energy from the gravitational field, and when traveling outward they lose energy to it. This is characteristic of a gravitational field that is attractive towards the origin.

However, in the inner vacuum region the situation is reversed. Since we have that $r_\mu > 0$, the derivative is everywhere *negative* in that region. This means that photons traveling outward within this region are *blue* shifted, and therefore *gain* energy from the gravitational field, while photons traveling inward within this region are *red* shifted, and therefore *lose* energy to the gravitational field. This is characteristic of a gravitational field that is repulsive, driving matter and energy away from the origin. This is the exact opposite of what happens in the outer vacuum region. It is important to note that this repulsion is not from the matter in itself, but from the *origin*, consisting therefore of an outward *attraction* towards the shell of matter.

3.3 The energy conditions

It is possible to determine, up to a certain point, when the weak and dominant energy conditions [2] are satisfied for the solutions that we present here. Let us start by pointing out that in our case here both $g_{\mu\nu}$ and $T_\mu{}^\nu$ are diagonal, with

$$\mathrm{diag}[g_{\mu\nu}] = \left[\mathrm{e}^{2\nu(r)}, -\mathrm{e}^{2\lambda(r)}, -r^2, -r^2 \sin^2(\theta) \right], \tag{80}$$

$$\mathrm{diag}[T_\mu{}^\nu] = [\rho(r), -P(r), -P(r), -P(r)] \,. \tag{81}$$

Given an arbitrary vector field

$$V^\mu = (V_0, V_1, V_2, V_3) \,, \tag{82}$$

it is not difficult to show that under these conditions the weak energy condition is always satisfied,

$$T_\nu{}^\mu V_\mu V^\nu \geq 0, \tag{83}$$

regardless of any properties that V^μ may or may nor have. Therefore the only relevant energy condition is the dominant energy condition, which consists of first defining the vector field

$$W^\mu = T_\nu{}^\mu V^\nu, \tag{84}$$

and then showing that, if V^μ is a future-pointing causal vector field, with $V_0 > 0$ and

$$V_\mu V^\mu \geq 0, \tag{85}$$

then W^{μ} is a future-pointing causal vector field, with $W_0 > 0$ and

$$W_{\mu} W^{\mu} \geq 0. \tag{86}$$

In fact, it can be shown without too much difficulty that the condition holds so long as one has that

$$\rho(r) \geq P(r), \tag{87}$$

for all r within the matter region. All this can be done in general, under the hypotheses stated, without discriminating any special forms of $\rho(r)$ and $P(r)$, or any specific relations between them.

In our specific case here we have that $\rho(r) = \rho_0$, so that the condition above reduces to a condition on the dimensionless pressure $p(r)$ defined in Eq. (42), namely the condition that

$$p(r) \leq 1, \tag{88}$$

for all r within the matter region. While the particular examples given in Figs. 1 and 3 in this paper show that this is indeed the case in those examples, and although there seems to be many more examples in which this holds, so far we do not have an analytic determination of the complete subset of the parameter space of the model within which the dominant energy condition holds. However, we can certainly state that there is a region in the parameter space where the dominant energy condition is satisfied.

3.4 The curvature scalars

We have established that the solution contains a singularity at the origin $r = 0$. However, we have not yet determined completely the exact nature of this singularity. While we know that it is a *repulsive* singularity, since the gravitational field around it is repulsive with respect to the origin, which in itself is a somewhat unexpected property, its geometric nature is far from clear. We also know that this singularity is related to the time, since we have shown that photons approaching the singularity undergo ever larger amounts of red shift. However, the fact remains that we cannot yet be sure of whether or not this is a curvature singularity in the invariant differential-geometric sense of the term, or something that depends fundamentally on the coordinate system used.

The singularity is contained in the sector of the solution for the inner vacuum, which is known analytically, as shown in Eqs. (19) and (20). The most serious aspect of this singularity is that, as shown in Eq. (20), the temporal coefficient $\exp[2\nu(r)]$ of the metric diverges to infinity as we make $r \to 0$, behaving in fact as $1/r$. This is true so long as $r_{\mu} > 0$, that is, so long as there is an inner vacuum region at all, since we have shown that $r_{\mu} = 0 \Rightarrow r_1 = 0$, and also that $r_{\mu} > 0 \Rightarrow r_1 > 0$. This is a strong divergence, similar to the Coulomb divergence of the potential of a point charge in Electrodynamics. Note that in the case of the radial component $\exp[2\lambda(r)]$

of the metric there is no divergence, but a convergence to zero instead, for any $r_\mu > 0$, as shown in Eq. (19).

In order to bring the nature of this singularity into proper perspective, we calculated several curvature invariants [2, 12] within the inner vacuum region, so as to be able to subsequently examine their $r \to 0$ limits. We will give here an outline of this calculation, with all the results written for the inner vacuum region. The diagonals of the metric tensor and of its inverse are given by

$$\mathrm{diag}[g_{\mu\nu}] = \left[A\,\frac{r+r_\mu}{r}, -\frac{r}{r+r_\mu}, -r^2, -r^2\sin^2(\theta)\right], \tag{89}$$

$$\mathrm{diag}[g^{\mu\nu}] = \left[\frac{1}{A}\,\frac{r}{r+r_\mu}, -\frac{r+r_\mu}{r}, -\frac{1}{r^2}, -\frac{1}{r^2\sin^2(\theta)}\right], \tag{90}$$

where A is the constant given in Eq. (36). The Riemann curvature tensor, in its geometric-definition form, has the 24 non-zero components given by

$$\begin{aligned}
R^0{}_{1(10)} &= +\frac{r_\mu}{r^2\left(r+r_\mu\right)},\\
R^1{}_{0(10)} &= +A\,\frac{r_\mu\left(r+r_\mu\right)}{r^4},\\
R^0{}_{2(20)} &= -\frac{1}{2}\,\frac{r_\mu}{r},\\
R^2{}_{0(20)} = R^3{}_{0(30)} &= -\frac{1}{2}\,A\,\frac{r_\mu\left(r+r_\mu\right)}{r^4},\\
R^0{}_{3(30)} &= -\frac{1}{2}\,\frac{r_\mu}{r}\,\sin^2(\theta),\\
R^1{}_{2(12)} &= +\frac{1}{2}\,\frac{r_\mu}{r},\\
R^2{}_{1(21)} = R^3{}_{1(31)} &= +\frac{1}{2}\,\frac{r_\mu}{r^2\left(r+r_\mu\right)},\\
R^1{}_{3(13)} &= +\frac{1}{2}\,\frac{r_\mu}{r}\,\sin^2(\theta),\\
R^2{}_{3(23)} &= -\frac{r_\mu}{r}\,\sin^2(\theta),\\
R^3{}_{2(32)} &= -\frac{r_\mu}{r},
\end{aligned} \tag{91}$$

where the index notation (01) means that the equation holds for the two indices shown in either order, but with the corresponding changes of sign, given the antisymmetry of the various curvature tensors by the interchange of the two indices in the first pair, or of the two indices in the second pair. And the same holds for any pair of indices other than (01), of course. This means that each result written with two such parenthesized pairs of indices corresponds in fact to four tensor components. With this notation the Riemann curvature tensor, in its fully covariant form, has the 24 non-zero components given by

$$
\begin{aligned}
R_{(01)(10)} &= +\frac{r_\mu}{r^3}, \\
R_{(02)(20)} &= -\frac{1}{2}\,A\,\frac{r_\mu(r+r_\mu)}{r^2}, \\
R_{(03)(30)} &= -\frac{1}{2}\,A\,\frac{r_\mu(r+r_\mu)}{r^2}\,\sin^2(\theta), \\
R_{(12)(21)} &= +\frac{1}{2}\,\frac{r_\mu}{r+r_\mu}, \\
R_{(13)(31)} &= +\frac{1}{2}\,\frac{r_\mu}{r+r_\mu}\,\sin^2(\theta), \\
R_{(23)(32)} &= -r_\mu r\,\sin^2(\theta).
\end{aligned} \tag{92}
$$

In order to calculate the invariant curvature scalars, we start by considering the Riemann curvature tensor in the mixed form $R^{\mu\nu}{}_{\lambda\sigma}$, which has the 24 non-zero components given by

$$
\begin{aligned}
R^{(01)}{}_{(01)} &= +\frac{r_\mu}{r^3}, \\
R^{(02)}{}_{(02)} = R^{(03)}{}_{(03)} &= -\frac{1}{2}\,\frac{r_\mu}{r^3}, \\
R^{(12)}{}_{(21)} = R^{(13)}{}_{(31)} &= +\frac{1}{2}\,\frac{r_\mu}{r^3}, \\
R^{(23)}{}_{(32)} &= -\frac{r_\mu}{r^3}.
\end{aligned} \tag{93}
$$

We also consider the left dual curvature tensor in the form $*R^{\mu\nu}{}_{\lambda\sigma}$, defined by

$$
*R^{\mu\nu}{}_{\lambda\sigma} = -\frac{1}{\sqrt{-g}}\,\epsilon^{\mu\nu\alpha\beta}\,R_{\alpha\beta\lambda\sigma}, \tag{94}
$$

of which there are also 24 non-zero components, given by

$$
\begin{aligned}
*R^{(01)}{}_{(23)} &= -2\,\frac{1}{\sqrt{A}}\,\sin(\theta)\,\frac{r_\mu}{r}, \\
*R^{(02)}{}_{(13)} &= -\frac{1}{\sqrt{A}}\,\sin(\theta)\,\frac{r_\mu}{r^2\left(r+r_\mu\right)}, \\
*R^{(03)}{}_{(12)} &= +\frac{1}{\sqrt{A}}\,\frac{1}{\sin(\theta)}\,\frac{r_\mu}{r^2\left(r+r_\mu\right)}, \\
*R^{(12)}{}_{(03)} &= -\sqrt{A}\,\sin(\theta)\,\frac{r_\mu\left(r+r_\mu\right)}{r^4}, \\
*R^{(13)}{}_{(02)} &= -\sqrt{A}\,\frac{1}{\sin(\theta)}\,\frac{r_\mu\left(r+r_\mu\right)}{r^4}, \\
*R^{(23)}{}_{(01)} &= +2\sqrt{A}\,\frac{1}{\sin(\theta)}\,\frac{r_\mu}{r^5}.
\end{aligned} \tag{95}
$$

Finally we consider the double dual curvature tensor in the form $*R*^{\mu\nu}{}_{\lambda\sigma}$, defined by

$$*R*^{\mu\nu}{}_{\lambda\sigma} = -\,\epsilon^{\mu\nu\alpha\beta}\,R^{\gamma\delta}{}_{\alpha\beta}\,\epsilon_{\gamma\delta\lambda\sigma}\,, \tag{96}$$

of which there are also 24 non-zero components, given by

$$\begin{aligned} *R*^{(01)}{}_{(01)} &= -4\,\frac{r_\mu}{r^3},\\ *R*^{(02)}{}_{(02)} = *R*^{(03)}{}_{(03)} &= +2\,\frac{r_\mu}{r^3},\\ *R*^{(12)}{}_{(21)} = *R*^{(13)}{}_{(31)} &= -2\,\frac{r_\mu}{r^3},\\ *R*^{(23)}{}_{(32)} &= +4\,\frac{r_\mu}{r^3}. \end{aligned} \tag{97}$$

Given these three tensors, the following six invariant curvature scalars can be defined. We start by the three invariants which are linear on the curvature tensors, the scalar curvatures

$$R = R^{\mu\nu}{}_{\nu\mu}, \tag{98}$$

$$*R = *R^{\mu\nu}{}_{\nu\mu}, \tag{99}$$

$$*R* = *R*^{\mu\nu}{}_{\nu\mu}, \tag{100}$$

and add to this the three invariants which are quadratic on the curvature tensors, namely the Kretschmann scalar K_1, the Chern–Pontryagin scalar K_2 and the Euler scalar K_3,

$$K_1 = R^{\mu\nu}{}_{\lambda\sigma}\,R^{\lambda\sigma}{}_{\mu\nu}, \tag{101}$$

$$K_2 = *R^{\mu\nu}{}_{\lambda\sigma}\,R^{\lambda\sigma}{}_{\mu\nu}, \tag{102}$$

$$K_3 = *R*^{\mu\nu}{}_{\lambda\sigma}\,R^{\lambda\sigma}{}_{\mu\nu}. \tag{103}$$

Performing the calculations we get for the two invariant scalars associated to $R^{\mu\nu}{}_{\lambda\sigma}$,

$$R \equiv 0, \tag{104}$$

as is to be expected, since the Einstein equations are satisfied within the inner vacuum region, where they reduce to $R^{\mu}{}_{\nu} = 0$, and hence imply that $R = 0$, and

$$K_1 = 12\,\frac{r_\mu^2}{r^6}. \tag{105}$$

In the case of the invariant scalars associated to the left-dual tensor $*R^{\mu\nu}{}_{\lambda\sigma}$, due to the index structure of the non-zero components shown in Eq. (95), there are no non-zero terms on either of the contractions, so that we immediately get

$$*R \equiv 0, \tag{106}$$

$$K_2 \equiv 0. \tag{107}$$

In fact, these two results hold even in the general static and spherically symmetric case, that is, for *all* static and spherically symmetric geometries. Therefore, they imply very little about our solution in particular. Performing the calculation of the last two invariant scalars, those associated to $*R*^{\mu\nu}{}_{\lambda\sigma}$, we get

$$*R* \equiv 0, \tag{108}$$

which is a consequence of the fact that the two tensors in Eqs. (93) and (97) are proportional to each other, thus implying that their traces are proportional as well, and

$$K_3 = -48\,\frac{r_\mu^2}{r^6}, \tag{109}$$

which not unexpectedly is proportional to K_1.

Having calculated all the six invariant curvature scalars, we are now in a position to consider their $r \to 0$ limits, as well as the same limit of the individual tensor components, in each case. All the linear curvature scalars turn out to be identically zero within the inner vacuum region, although in the case of $*R$ this retains precious little relation to our particular solution. The same comment applies to the quadratic curvature scalar K_2. Of course it then follows that their $r \to 0$ limits are zero as well.

The two remaining quadratic curvature scalars K_1 and K_3 both diverge fast to infinity when we make $r \to 0$. Since these are invariants, the fact that they diverge to infinity when we approach a given point is not dependent on the coordinate system used. Besides this, most of the non-zero components of all the three forms of the curvature tensor, Riemann, left dual and double dual, with any sets of upper and lower indices, also diverge to infinity in the $r \to 0$ limit. The linear scalars R and $*R*$, that turn out to be zero, vanish because of cancellations among divergent components of the corresponding tensors, which have the same absolute value and opposite signs. In the case of the quadratic scalars K_1 and K_3, the negative signs are eliminated, the divergent components add up and we get results that diverge when we approach the origin.

While establishing the regularity of the curvature tensor at the origin would require the calculation of a complete set of curvature invariants, with results that would have to be finite, it is enough that one such invariant results divergent at the origin in order to establish that the singularity is in fact a curvature singularity. This is so because the values of an invariant scalar at any given points are not changed by any transformations of coordinates. The description of the points may depend on the coordinate system, but the values of the scalar function cannot. Hence, if there is a sequence of points within a neighborhood of the singular point, along which the invariant curvature scalar diverges to infinity in one coordinate system, then it will so diverge in any coordinate system. Besides, since this invariant curvature scalar is an algebraic combination of the components of the curvature tensor, it follows at once that in any coordinate system at least some of the components of that tensor will diverge to infinity.

Therefore we may conclude that there is in fact an invariant singularity at the origin, in the differential-geometric sense of the term. The singularity is *not*, therefore, an artifact of the particular system of coordinates that we use, but a real geometrical singularity with an invariant meaning. Still, from the point of view of the physics, possibly the most important property of this singularity is that it is a repulsive rather than attractive one.

4 Examples of specific solutions

In order to calculate $z(r)$ either analytically or numerically it is convenient to define a dimensionless variable ξ such that

$$\xi \equiv \Upsilon_0 r \quad \Longrightarrow \tag{110}$$

$$\frac{d}{dr} = \Upsilon_0 \frac{d}{d\xi}, \tag{111}$$

where $\Upsilon_0 = 1/r_0$. In terms of ξ, Eq. (46), that determines $z(r)$, becomes

$$z'(\xi) + \frac{\eta + 2\xi^3}{2\xi(\eta + 3\xi - \xi^3)} z(\xi) = \frac{3\xi^3}{2\xi(\eta + 3\xi - \xi^3)}, \tag{112}$$

where the primes indicate now derivatives with respect to ξ, and where we define

$$\eta \equiv \xi_2^3 - 3\xi_M, \tag{113}$$

$$\xi_1 \equiv \Upsilon_0 r_1, \tag{114}$$

$$\xi_2 \equiv \Upsilon_0 r_2, \tag{115}$$

$$\xi_M \equiv \Upsilon_0 r_M. \tag{116}$$

Thus ξ_1, ξ_2 and ξ_M correspond respectively to the internal radius r_1, the external radius r_2 and the Schwarzschild radius r_M, expressed in terms of the new variable ξ. The solution of Eq. (112) is obtained by writing Eq. (52) in terms of ξ,

$$z(\xi) = \sqrt{\frac{\eta + 3\xi - \xi^3}{\xi}} \left[\sqrt{\frac{\xi_2}{3(\xi_2 - \xi_M)}} + \frac{3}{2} \int_{\xi_2}^{\xi} d\chi \, \frac{\chi^{5/2}}{(\eta + 3\chi - \chi^3)^{3/2}} \right], \tag{117}$$

where, in order to remain within the matter region, we must have $\xi_1 \leq \xi \leq \xi_2$. If we multiply both the numerator and the denominator in the integrand of the integral in

Eq. (117) by $\chi^{3/2}$, define the polynomial $Q(\chi) = \chi\left(\eta + 3\chi - \chi^3\right)$ and the rational function $S(\chi, Q) \equiv \chi^4/Q^3$, then the integral in Eq. (117) can be rewritten as

$$\int_{\xi_2}^{\xi} d\chi \, \frac{\chi^{5/2}}{(\eta + 3\chi - \chi^3)^{3/2}} = \int_{\xi_2}^{\xi} d\chi \, S\Big[\chi, \sqrt{Q(\chi)}\Big]. \tag{118}$$

The expression on the right-hand side of Eq. (118) is by definition an elliptic integral [13] and cannot be expressed in terms of elementary functions except in two cases: (1) $S\big(\chi, Q^{1/2}\big)$ contains no odd powers of χ; in our case this happens when $\eta = 0$ and leads to the Schwarzschild interior solution; (2) the polynomial $Q(\chi)$ has two equal roots; this leads to the explicit solutions that we discuss next.

4.1 A family of explicit solutions

The integral in Eq. (117) contains a cubic polynomial. The nature of its three roots depends on the value of its discriminant Δ [14]. For cubic polynomials of the form $a\xi^3 + c\xi + d$ we have $\Delta = -4ac^3 - 27a^2d^2$. If $\Delta > 0$ the polynomial has three distinct real roots, if $\Delta = 0$ it has three real roots but two of them are equal, and if $\Delta < 0$ it has one real and two complex roots which are conjugate to each other. In our case we have $a = -1$, $c = 3$, $d = \eta$ and therefore $\Delta = 27(4 - \eta^2)$.

The value $\Delta = 0$ corresponds to the case where the solution for $z(\xi)$ can be expressed in terms of elementary functions. Note that we have $\Delta = 0$ when $\eta = \pm 2$, which corresponds to $\xi_2^3 = \pm 2 + 3\xi_M$. For $\eta = -2$ the polynomial in the integral in Eq. (117) is non-positive for $\xi \geq 0$. Therefore, we must choose $\eta = 2$. For this value of η the polynomial is strictly positive in the interval $[0, 2)$ and can be factored as

$$2 + 3\chi - \chi^3 = (2 - \chi)(\chi + 1)^2. \tag{119}$$

In this case we can express the integral in Eq. (117) in terms of elementary functions. The calculation can be considerably simplified using a new integration variable u defined by $u = \sqrt{\chi/(2 - \chi)}$. The final result, up to an integration constant, is

$$\begin{aligned} \mathcal{I}(\chi) &\equiv \int d\chi \, \frac{\chi^{5/2}}{(2 - \chi)^{3/2}(\chi + 1)^3} \\ &= \frac{2\chi^2 + 15\chi + 10}{18\,(\chi + 1)^2}\sqrt{\frac{\chi}{2 - \chi}} - \frac{5\sqrt{3}}{27}\arctan\left(\sqrt{\frac{\chi}{2 - \chi}}\right). \end{aligned} \tag{120}$$

Thus, in terms of $\mathcal{I}(\chi)$ Eq. (117) reads

$$z(\xi) = \sqrt{\frac{2 + 3\xi - \xi^3}{\xi}} \left\{ \sqrt{\frac{\xi_2}{3(\xi_2 - \xi_M)}} + \frac{3}{2}\left[\mathcal{I}(\xi) - \mathcal{I}(\xi_2)\right] \right\}. \tag{121}$$

Note that, in order to guarantee that the cubic polynomial for $\eta = 2$ shown in Eq. (119) is always positive, we need to have $\chi < 2$. Therefore, since we already know that the polynomial is positive, the arguments of the square roots in Eq. (120) are always positive.

4.2 Examples of numerical solutions

In our numerical approach here, we assume that the external radius $\xi_2 = \Upsilon_0 r_2$ is given. In order to complete the calculation we have to determine the interior radius ξ_1. This can be done recalling that the dimensionless pressure $p(\xi)$ is zero for $\xi = \xi_2$ and $\xi = \xi_1$. Since according to Eq. (53) $p(\xi) = 1/z(\xi) - 1$, this is equivalent to the determination of the values of ξ for which $z(\xi) = 1$. By the determination of ξ_1 we would have solved the problem in the entire matter region. Note that since $\xi = \Upsilon_0 r = \sqrt{\kappa\rho_0}\ r$ we have obtained a family of solutions parametrized by two parameters, the external radius r_2 and the parameter η.

If the discriminant $\Delta \neq 0$ the integral in Eq. (117) is expressed in terms of elliptic integrals and the result is not very transparent. It is more convenient to integrate the differential Eq. (112) using the fourth-order Runge–Kutta algorithm (RK4) [15]. We start by choosing a value of $\xi = \xi_2$ for which the cubic polynomial is positive and we put $z(\xi_2) = 1$. This determines the outer radius of the matter shell. We then iterate the differential equation given in Eq. (112) in the decreasing ξ direction until we reach the first point for which the value of z returns to 1. This point is chosen as ξ_1. If a value for ξ_1 cannot be found, we conclude that there is no solution to the problem with the given values of ξ_2 and ξ_M. A good test for the efficiency of the algorithm is to compare the exact analytic result given in Eq. (121) with the result from the numerical integration in that same case. These results are shown in Fig. 1. On any current 64-bit desktop computer one can easily reach a high degree of precision with little numerical effort. After iterating the RK4 algorithm from ξ_2 to ξ_1 the difference between the exact and the numerical results for $z(\xi)$ stays below 1.03536×10^{-29} for an iteration step of $\delta\xi \approx 10^{-7}$.

In the comments that follow $\xi_\mu \equiv \Upsilon_0 r_\mu$, where r_μ is the integration constant that results from the solution of the Einstein equations in the inner vacuum region, given in Eq. (31). In the matter region the input parameters are η and ξ_2. The parameter ξ_1 is obtained from the integration of Eq. (112). The value of ξ_M that is necessary for plotting the curves is given in Eq. (116). The expressions for $\lambda(\xi)$ and $\nu(\xi)$ are given in Table 1. Figure 2 shows the plots of the functions $\nu(\xi)$ and $\lambda(\xi)$ for $\eta = 2.0$ and $\xi_2 = 5^{1/3}$. The curves were obtained analytically using Eq. (121) and the expressions in Table 1, but using the numerically calculated parameters $\xi_1 = 0.594881$ and $\xi_\mu = 0.596494$.

In Fig. 3 we plot the dimensionless pressure $p(\xi)$ as a function of ξ, in a case in which there is no analytic expression in terms of elementary functions and the calculation is performed numerically. The parameters are $\xi_1 = 1.24050$ and $\xi_\mu = 1.03035$. Comparing Figs. 1 and 3, that depict the dimensionless pressure $p(\xi)$ as a function of ξ for $\eta = 2.0$ and $\eta = 5.0$, one notes that the two graphs are similar but for larger values of η the graph becomes less symmetric. Note that in both cases we have that $p(\xi) < 1$.

Figure 4 shows the plots of the functions $\nu(\xi)$ and $\lambda(\xi)$, for $\eta = 5.0$ and $\xi_2 = 2.0$. In this case there are no analytical solutions in terms of elementary functions available

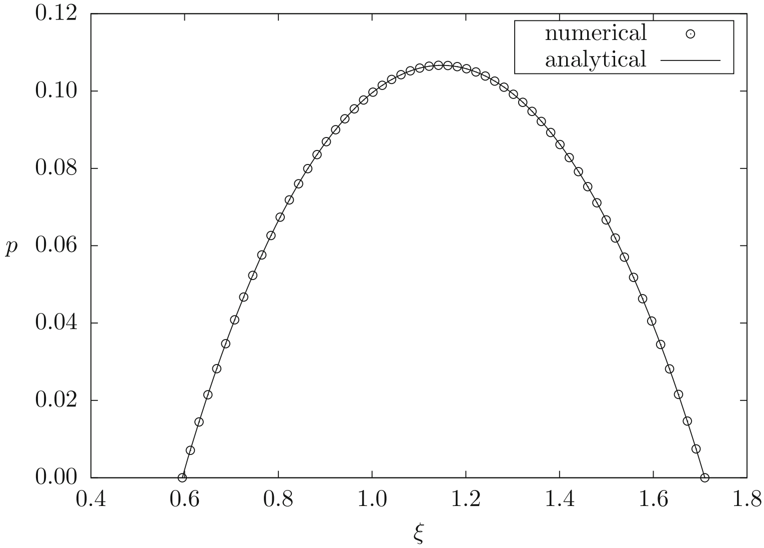

Fig. 1 Comparison between the dimensionless pressure $p(\xi)$ calculated analytically and numerically using the Runge–Kutta fourth-order algorithm for $\eta = 2.0$, $\xi_2 = 5^{1/3}$ and $\xi_M = 1.0$, resulting in $\xi_1 = 0.594881$ and $\xi_\mu = 0.596494$

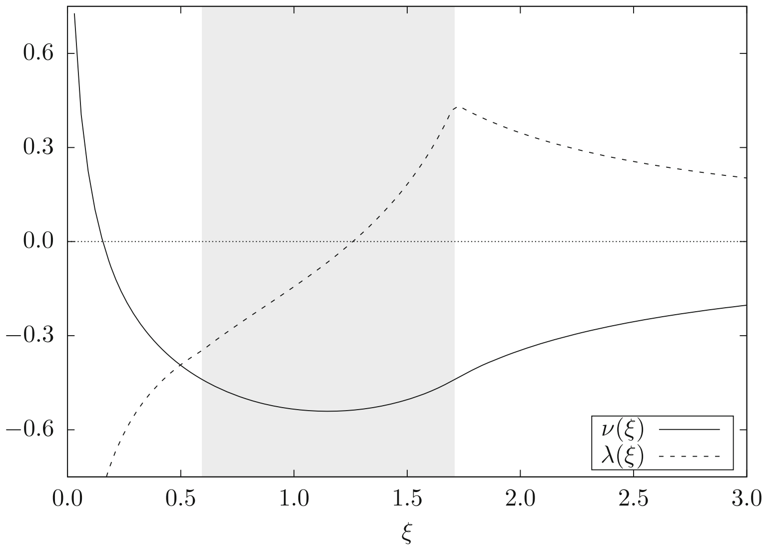

Fig. 2 The functions $\nu(\xi)$ and $\lambda(\xi)$ for $\eta = 2.0$, $\xi_2 = 5^{1/3}$ and $\xi_M = 1.0$. The shaded area indicates the matter region, to its right is the outer vacuum and to its left is the inner vacuum. Here we have $\xi_1 = 0.594881$ and $\xi_\mu = 0.596494$

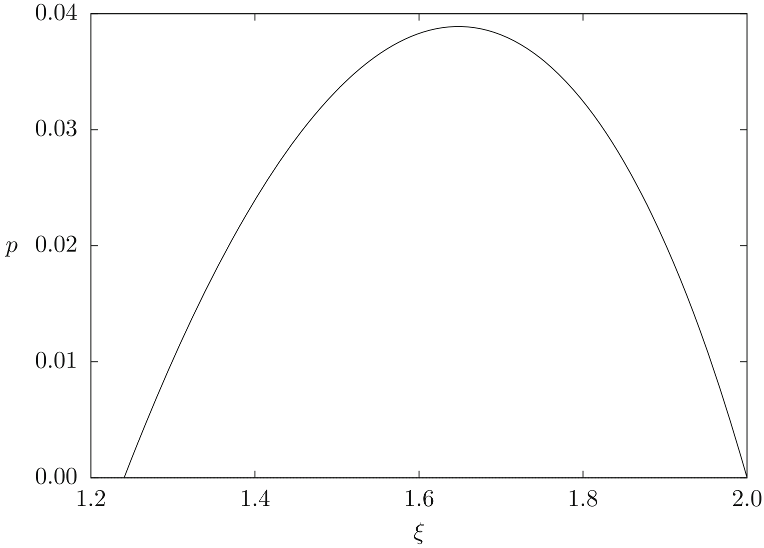

Fig. 3 The dimensionless pressure $p(\xi)$ calculated numerically for $\eta = 5.0$, $\xi_2 = 2.0$ and $\xi_M = 1.0$. Here we have $\xi_1 = 1.24050$ and $\xi_\mu = 1.03035$

in the matter region and the values of $\nu(\xi)$ and $\lambda(\xi)$ were obtained numerically. In the vacuum regions we used the analytical expressions given in Table 1 with the parameters $\xi_1 = 1.24050$ and $\xi_\mu = 1.03035$.

5 Conclusions

In this paper we have given the complete and exact solution of the Einstein field equations for the case of a shell of liquid matter. Although this particular problem can be seen as having a somewhat academic nature, it does lead us to two important and unexpected conclusions. One of them is that all solutions for shells of liquid matter have a singularity at the origin, within the inner vacuum region, that does *not*, however, lead to any kind of pathological behavior involving the matter. The other is that, contrary to what is usually thought, a non-trivial gravitational field does exist within a spherically symmetric central cavity, namely the inner vacuum region.

The geometry within the cavity is associated with a spacetime that is contracted in the radial direction, rather than expanded. It is easy to verify that, unlike what happens in the outer vacuum region, in the inner vacuum region the proper radial length, ℓ_1, say from $r = 0$ to $r = r_1$, is in fact *smaller* than the corresponding radial coordinate r_1. We have that $d\ell_1 = \sqrt{g_{11}}\, dr$, and therefore

$$\ell_1 = \int_0^{r_1} dr \sqrt{\frac{r}{r + r_\mu}}$$
$$< \int_0^{r_1} dr$$

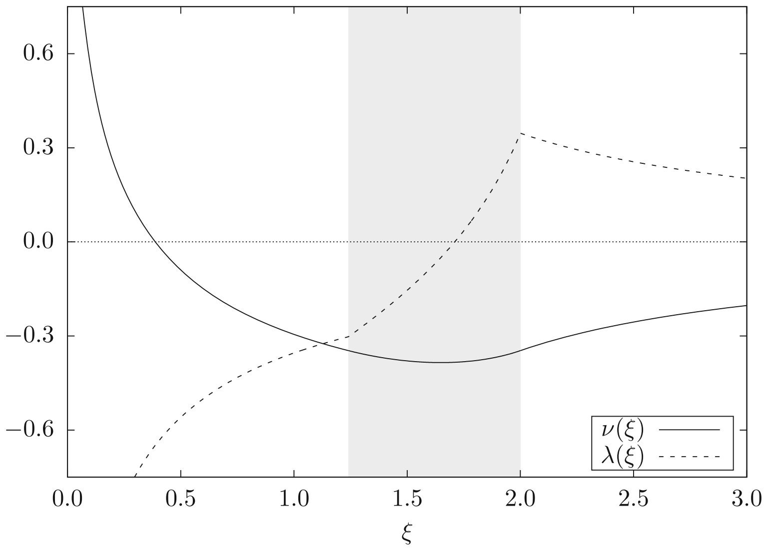

Fig. 4 The functions $\nu(\xi)$ and $\lambda(\xi)$ for $\eta = 5.0$, $\xi_2 = 2.0$ and $\xi_M = 1.0$. The shaded area indicates the matter region, to its right is the outer vacuum and to its left is the inner vacuum. Here we have $\xi_1 = 1.24050$ and $\xi_\mu = 1.03035$

$$= r_1, \tag{122}$$

given that $r_\mu > 0$. This illustrates the fact that the radial lengths within the inner vacuum region are contracted rather than expanded. The true physical volume of the inner vacuum region is therefore correspondingly smaller than the apparent coordinate volume. This renders this inner geometry not embeddable in the illustrative way that is usually employed in the case of the outer vacuum region.

The gravitational field associated to this geometry, inside the inner vacuum region, can be interpreted as a repulsive field with respect to the origin. This can be ascertained from an examination of the sign of the derivative of $\nu(r)$ in the inner and outer vacuum regions, and its interpretation in terms of the energy of a photon traveling in the radial direction. This sign is positive in the outer vacuum region, corresponding to an attractive field towards the origin, and negative in the inner vacuum region, corresponding to an repulsive field away from the origin.

Of course, since $\nu'(r)$ is a continuous function, and since we enter the matter region from the outer vacuum region with a positive derivative for $\nu(r)$, and exit it into the inner vacuum region with a negative derivative, there must be a point within the matter region where $\nu'(r) = 0$, and where the derivative flips sign. This is clearly the point r_e of minimum of $\nu(r)$, which is also the point of minimum of $z(r)$, and hence the point of maximum of the pressure $P(r)$, a point which already had a role to play in our arguments.

The arisal of a spherically symmetric region where the gravitational field is repulsive rather that attractive with respect to the origin may feel contrary to our classical intuition regarding gravity. However, this type of situation can arise even in the context of a Newtonian framework in flat spacetime, if we use a slightly modified potential. One can acquire an intuitive understanding of the unexpected situation in the inner vacuum region by considering the time-honored Newtonian argument for the determination of the gravitational force within a hollow spherically symmetric thin shell of matter, but with a potential that behaves as $1/r^{1+\epsilon}$ for some $|\epsilon| \ll 1$, thus leading to a force that behaves as $1/r^{2+\epsilon}$.

If one considers a test mass at a point in the interior of the hollow shell, at the position $\mathbf{r}$ with respect to the center, it is not difficult to use the usual Newtonian argument to show that, if $\epsilon > 0$, then the resulting gravitational force at that point is oriented outward, in the direction of $\mathbf{r}$, towards the shell of matter. In other words, the attraction by the part of the shell that is closer to the point $\mathbf{r}$ outweighs the attraction from the opposite side, thus leading to a resulting force that repels particles away from the origin. Note that this argument involving a potential behaving in a way other than $1/r$ is the same that can be used to model the precession of the perihelion of orbits in General Relativity using this Newtonian framework. That precession is prograde precisely if $\epsilon > 0$.

It is interesting to note that this configuration of the gravitational field tends to stabilize the shell of liquid matter, since any particle of matter that detaches from the liquid and wanders into one of the vacuum regions will be driven back to the bulk of the liquid by the gravitational field. This can be interpreted as a successful stability test satisfied by all the solutions. The general tendency of the gravitational field is therefore that of compressing the shell of fluid matter, from both sides. This suggests that the same interpretation should be valid in the case of a gaseous fluid.

The singularity at the origin is usually thought to be associated with an infinite concentration of matter there, and thus considered to be an evil that must be avoided at any cost. However, this argument only makes any sense at all if one thinks of that singularity as a point of gravitational attraction, rather than as a point of repulsion of matter. Here we do have the singularity, but not the infinite concentration of matter at the origin, due to the repulsive character of the gravitational field around the origin.

In any case, the existence of the singularity is not a question of choice, of course, since it is required by the field equations and by the interface boundary conditions that follow from them. In the case of the shell solutions one is not at liberty to impose that $r_\mu = 0$ in order to avoid this singularity. The condition $r_\mu = 0$ just selects a particular subset of non-shell solutions. In our case here it selects the interior Schwarzschild solution.

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors hereby certify that there are no actual or potential conflicts of interest of any of the authors in relation to this article.

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