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The structure of residue class fields of the Colombeau ring of generalized numbers

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Introduction

In [5], *Ërdos*, *Gillman* and *Henriksen* proved an important isomorphism theorem for real-closed fields. They applied the theorem to a class of fields that appears as non-archimedean residue class fields of maximal ideals in rings of continuous functions on completely regular topological spaces.

In this paper we study the residue class fields of the Colombeau ring of generalized numbers $\overline{\mathbf{K}}$ (where $\mathbf{K}$ is the field of real numbers $\mathbf{R}$ or the field of complex numbers $\mathbf{C}$). This ring can be thought of as the ring of moderate (see definition below) germs of functions (not necessarily continuous) at the right of zero in $]0,1]$. The ring $\overline{\mathbf{K}}$ is provided with a natural *filtration* ν and an associated ultrametric norm (in the sense of [1]), that turns it into a complete non-archimedean pseudometric space. This filtration induces in a natural way a filtration in every residue class field of $\overline{\mathbf{K}}$, that is, in every quotient of $\overline{\mathbf{K}}$ by a maximal ideal J. We prove in Section 1 that the induced filtration is in fact a *valuation* V of the field $\overline{\mathbf{K}}/J$. It turns out that the residue fields $\overline{\mathbf{K}}/J$ are in fact complete non-archimedean valued fields.

In Section 2 we state some of Kaplansky's results on maximal fields with valuations [7], to prove that $\overline{\mathbf{K}}/J$ is in fact a maximal field (Section 3, Theorem 3.4). As a consequence of this fact we can apply a structure theorem for such residue class fields: if k_J is the residue field – in the sense of valuation theory – of the valued field $(\overline{\mathbf{K}}/J, V)$, then

$$\overline{\mathbf{C}}/J \cong k_J((t^{\mathbf{R}})),$$

and

$$\overline{\mathbf{R}}/J \cong k_J((t^{\mathbf{R}}, c_{\alpha,\beta})),$$

where $k_J((t^{\mathbf{R}}, c_{\alpha,\beta}))$ is the field of formal power series $\sum_{\gamma\in S} a_\gamma t^\gamma$ with S a well ordered subset of $\mathbf{R}$, $a_\gamma \in k_J$ and $t^\alpha t^\beta = c_{\alpha,\beta} t^{\alpha+\beta}$ (for some factor set $c_{\alpha,\beta}$). The field $k_J((t^{\mathbf{R}}))$ is the same but with $c_{\alpha,\beta} = 1$.

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In order to decide if all the residue class fields are isomorphic we had to introduce the analog of the zero set of a continuous function. In the ring $\overline{\mathbf{K}}$ the situation is different from the usual context of rings of continuous functions, because we have *germs* of arbitrary (moderated) functions. Some (adapted) machinery of [6] worked out in our case. We proved that all residue class fields $\overline{\mathbf{C}}/J$ are isomorphic (Theorem 6.10), but we left the real case ($\mathbf{K} = \mathbf{R}$) open. The real case seems to be more delicate because the fields $\overline{\mathbf{R}}/J$ are nonarchimedean real-closed fields (Section 5) of power $\mathfrak{c}$ that are not η_1-sets (see [9]).

1. Preliminary results

Let $\mathbf{K}$ denote the field of real numbers or the field of complex numbers. We define the ring $\mathcal{E}_M(\mathbf{K})$ as the ring (pointwise operations) of all functions $f\colon]0,1] \longrightarrow \mathbf{K}$ that are *moderate*, that is, that verify the moderation condition

$$\exists \sigma \in \mathbf{R}, \quad \lim_{\varepsilon \to 0+} \varepsilon^{-\sigma}|f(\varepsilon)| = 0. \tag{1.1}$$

We will write (1.1) using the o-symbol simply as $|f(\varepsilon)| = o(\varepsilon^\sigma)$. For a given $f \in \mathcal{E}_M(\mathbf{K})$ we define

$$\nu(f) = \sup\{\sigma \in \mathbf{R} : |f(\varepsilon)| = o(\varepsilon^\sigma)\}$$

It is clear that $\nu(f) \in \mathbf{R} \cup \{\infty\}$ and that ν verifies the following properties:

LEMMA 1.1.
(1) $\nu(\lambda f) = \nu(f)$ *for all* $\lambda \in \mathbf{K}^*$ *and all* $f \in \mathcal{E}_M(\mathbf{K})$.
(2) $\nu(fg) \geq \nu(f) + \nu(g)$.
(3) $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$.
(4) $\nu(\varepsilon^r f) = r + \nu(f)$ *for all* $r \in \mathbf{R}$ *and all* $f \in \mathcal{E}_M(\mathbf{K})$.
(5) If $\tau < \sigma$ *and* $|f(\varepsilon)| = o(\varepsilon^\sigma)$ *then* $|f(\varepsilon)| = o(\varepsilon^\tau)$
(6) If $n \geq 1$ *is an integer,* $\nu(f^n) = n\nu(f)$.

The function ν is called a *filtration* (see [1]) of the ring $\mathcal{E}_M(\mathbf{K})$. We define $|\ |\colon \mathcal{E}_M(\mathbf{K}) \longrightarrow \mathbf{R}_+$ by

$$|f| := e^{-\nu(f)}$$

with the convention $e^{-\infty} = 0$. Then we have the immediate properties:

LEMMA 1.2. *The function* $|\ |\colon \mathcal{E}_M(\mathbf{K}) \longrightarrow \mathbf{R}_+$ *is an ultrametric function (see* [1]*) in* $\mathcal{E}_M(\mathbf{K})$ *and*
(1) $|-f| = |f|$,
(2) $|f+g| \leq \max\{|f|, |g|\}$,
(3) $|f+g| = \max\{|f|, |g|\}$ *if* $|f| \neq |g|$.
(4) $|\varepsilon^r f| = e^{-r}|f|$.

(5) $|fg| \leq |f||g|$

In the terminology of [1], the ring $\mathcal{E}_M(\mathrm{K})$ with the function $|\,|$ is called a *semi-normed* ring. The function $|\,|$ defines a *pseudometric topology* by means of

$$d(f,g) := |f-g|,$$

which makes $\mathcal{E}_M(\mathrm{K})$ into a topological ring. We define

$$ker|\,| = \{f \in \mathcal{E}_M(\mathrm{K}) \ : \ |f| = 0\},$$

and it is not dificult to prove that $ker|\,|$ is in fact an ideal of $\mathcal{E}_M(\mathrm{K})$ that we will denote by $\mathcal{N}(\mathrm{K})$ or simply by $\mathcal{N}$. The elements of $\mathcal{N}$ will be called null-functions.

REMARK 1.3. *By Lemma 1.1 (6) it follows that $\mathcal{N}$ is equal to its radical*

$$\sqrt{\mathcal{N}} = \{f(\varepsilon) \in \mathcal{E}_M(\mathrm{K}) \ : \ f(\varepsilon)^r \in \mathcal{N} \textit{ for some integer } r > 0\}.$$

The ring of Colombeau generalized numbers is defined by

$$\overline{\mathrm{K}} := \mathcal{E}_M(\mathrm{K})/\mathcal{N}.$$

The filtration ν has the property that $\nu(f) = \infty$ if, and only if, $f \in \mathcal{N}$ and ν is constant in each equivalence class module $\mathcal{N}$. Hence we can define ν in $\overline{\mathrm{K}}$ and we can also naturally define

$$|\,| : \overline{\mathrm{K}} \longrightarrow \mathrm{R}_+.$$

The ring $\overline{\mathrm{K}}$ with this $|\,|$ is a *normed* ring, [1]. It is in fact a complete (see [2]) normed ring, the completion of the semi-normed ring $\mathcal{E}_M(\mathrm{K})$ in the sense of [1].

The K-algebra $\overline{\mathrm{K}}$ can be thought of as the algebra of moderate germs of functions at 0^+. A germ f is moderate if (1.1) hols for some (hence all) representative $f(\varepsilon)$.

NOTATION 1.4. *In this paper we will always denote elements of $\overline{\mathrm{K}}$ by letters f, g, h, and so on. For a representative of f we will write $f(\varepsilon)$ or $\hat{f}$. An element of $\overline{\mathrm{K}}/J$ will be denoted by $[f]$. The quotient $\overline{\mathrm{K}}/J$ will also be denoted by $\overline{\mathrm{K}}_J$.*

If J is a maximal ideal of $\overline{\mathrm{K}}$ then it is easy to see that the multiplicative group of units $\overline{\mathrm{K}}^\times$ is open, and consequently J must be closed (see [1], p. 27, corollary 5). We can define a filtration:

$$\nu([f]) = \sup\{\nu(f+h) \ : \ h \in J\}, \tag{1.2}$$

and an ultrametric ([1], p. 17) function:

$$|[f]|_{res} := \inf\{|f+h| \ : \ h \in J\}.$$

We have the following properties:

LEMMA 1.5 ([1], pp. 16 - 17). *Since J is closed in $\overline{\mathbb{K}}$, the function $|\ |_{res}$ is a norm in $\overline{\mathbb{K}}_J$ and the corresponding topology is the quotient topology. Since $\overline{\mathbb{K}}$ is complete, $\overline{\mathbb{K}}_J$ is also complete.*

We are going to see that the filtration (1.2) is in fact a real valuation of the field $\overline{\mathbb{K}}_J$. We will need the following criterium:

LEMMA 1.6 ([1], p. 43, Proposition 1). *Let $(A, |\ |)$ be a normed ring with the following properties:*

(1) For each $a \in A$, $a \neq 0$, there exists a multiplicative element $m \in A$ and an expoent $s \in \mathbb{N}$ such that $|ma^s| = |m||a|^s = 1$,

(2) $A^{\sim} = B_1'/B_1$ is an integral domain,

where $A^{\sim}$ is the residue ring of A, defined by the quotient of the closed unit ball $B_1' = \{x \in A : |x| \leq 1\}$ by the open unit ball $B_1 = \{x \in A : |x| < 1\}$. An element $m \in A$ is called a multiplicative element if $m \notin ker|\ |$ and if

$$|mx| = |m||x| \qquad \forall x \in A.$$

THEOREM 1.7. *If J is a maximal ideal of $\overline{\mathbb{K}}$ then the induced filtration ν in $\overline{\mathbb{K}}_J$, defined by (1.2) is in fact a valuation, that is*

(1) $\nu([f]) = \infty$ if, and only if $f \in J$.

(2) $\nu([f][g]) = \nu([f]) + \nu([g])$.

Proof. To prove (1), suppose that $\nu([f]) = \infty$. Then $\sup\{\nu(f+h) : h \in J\} = \infty$ and there is a sequence $\{f_n\}$ with $f_n \in [f]$ for all $n \geq 1$ such that $\lim_{n\to\infty} \nu(f_n) = \infty$. Hence, for any $g \in [f]$ we have $g - f_n \in J$ and so $\lim(g - f_n) = g \in J$ because J is closed. This proves that $[g] = [f] = 0$.

To use Lemma 1.6, we must note that the functions $\varepsilon \mapsto \varepsilon^r$ (that we denote simply by ε^r) are multiplicative elements of $\overline{\mathbb{K}}$ by Lemma 1.2 (4), and consequently are multiplicative elements of $\overline{\mathbb{K}}_J$. Besides, the whole ring $\overline{\mathbb{K}}$ verifies $|f^n| = |f|^n$ for all natural numbers $n \geq 1$ (see Lemma 1.1 (6). This property is called *power-multiplicativity* in [1]). To verify (1) of Lemma 1.6, we consider any $[f] \in \overline{\mathbb{K}}_J$ and put $\alpha := \nu([f])$. Then, defining $m = \varepsilon^{-\alpha}$ we have that

$$\nu([m][f]) = -\alpha + \nu([f]) = 0,$$

and so $|[m][f]| = |[m]||[f]| = 1$. To verify (2), consider the elements $[x], [y] \in B_1'$ such that $[xy] \in B_1$. Then $\nu([xy]) > 0$. We will prove that the only way of having $\nu([x][y]) > 0$ is having $\nu([x]) > 0$ or $\nu([y]) > 0$. Let us chose representatives $\hat{x}, \hat{y}$ of $[x], [y]$ such that $\nu(\hat{x}\hat{y}) > 0$. Since $\nu(x) \leq \nu([x])$ and $\nu(y) \leq \nu([y])$, if by absurdum $\nu([x]) = \nu([y]) = 0$ we would have $\nu(\hat{x}) \leq 0$, $\nu(\hat{y}) \leq 0$ and $\nu(\hat{x}\hat{y}) > 0$.

Choose any $0 < b < \nu(\hat{x}\hat{y})$. Then, it is not true that $|\hat{x}(\varepsilon)|/\varepsilon^{b/2} \to 0$ if $\varepsilon \to 0^+$. This means that there is an $M > 0$ such that the set

$$A := \{\varepsilon \in]0,1] \, : \, \frac{|\hat{x}(\varepsilon)|}{\varepsilon^{b/2}} > M\}$$

is not empty and $0 \in \overline{A}$ (the closure of A). Let $\hat{\chi}_A$ denote the characteristic function of the set A and $\hat{\chi}_{A^c}$ the characteristic function of its complement set A^c in $]0,1]$ (they are clearly moderate functions). Then $\hat{\chi}_A\hat{\chi}_{A^c} = 0$ and so, since J is a maximal ideal, we must have $\hat{\chi}_A \in J$ or $\hat{\chi}_{A^c} \in J$. And so, if we write

$$\hat{x} = \hat{x}\hat{\chi}_A + \hat{x}\hat{\chi}_{A^c}, \qquad \hat{y} = \hat{y}\hat{\chi}_A + \hat{y}\hat{\chi}_{A^c}$$

we have four possibilities:

(1) $\hat{x} \equiv \hat{x}\hat{\chi}_A \; mod \; J$ and $\hat{y} \equiv \hat{y}\hat{\chi}_A \; mod \; J$,

(2) $\hat{x} \equiv \hat{x}\hat{\chi}_A \; mod \; J$ and $\hat{y} \equiv \hat{y}\hat{\chi}_{A^c} \; mod \; J$,

(3) $\hat{x} \equiv \hat{x}\hat{\chi}_{A^c} \; mod \; J$ and $\hat{y} \equiv \hat{y}\hat{\chi}_A \; mod \; J$,

(4) $\hat{x} \equiv \hat{x}\hat{\chi}_{A^c} \; mod \; J$ and $\hat{y} \equiv \hat{y}\hat{\chi}_{A^c} \; mod \; J$,

In case (1), since $|\hat{x}(\varepsilon)\hat{y}(\varepsilon)|/\varepsilon^b \to 0$ when $\varepsilon \to 0^+$, and we have

$$\frac{|\hat{x}(\varepsilon)|}{\varepsilon^{b/2}}\frac{|\hat{y}(\varepsilon)\chi_A(\varepsilon)|}{\varepsilon^{b/2}} \geq M\frac{|\hat{y}(\varepsilon)\hat{\chi}_A(\varepsilon)|}{\varepsilon^{b/2}} \geq 0,$$

it follows that $|\hat{y}(\varepsilon)\hat{\chi}_A(\varepsilon)|/\varepsilon^{b/2} \to 0$. This means that $\nu(\hat{y}\hat{\chi}_A) \geq b/2$ and so $\nu([y\chi_A]) = \nu([y]) > 0$.

Cases (2) and (3) are simpler because in both cases we have

$$\hat{x}\hat{y} \equiv \hat{x}\hat{\chi}_A\hat{y}\hat{\chi}_{A^c} \equiv 0 \; mod \; J,$$

and so $[x][y] = 0$, that is, either $x \in J$ or $y \in J$, and so or $\nu([x]) = \infty$ or $\nu([y]) = \infty$. In case (4) we have $\hat{x} \equiv \hat{x}\hat{\chi}_{A^c}$ and so

$$|\hat{x}(\varepsilon)\hat{\chi}_{A^c}(\varepsilon)| \leq M\varepsilon^{b/2},$$

that is, $\nu([x]) = \nu([x\chi_{A^c}]) \geq b/2$. This proves the theorem. □

2. Pseudo-convergence and maximality

In this section we recall some definitions and results concerning some aspects of valuation theory that will be needed in the sequel.

Let K be a field with a valuation V, with value group $\mathbf{R}$ and k its residue field.

DEFINITION 2.1. *A well ordered set $\{a_\rho\}$ of elements of K, without a last element, is said to be pseudo-convergent if whenever $\rho < \sigma < \tau$,*

$$V(a_\sigma - a_\rho) < V(a_\tau - a_\sigma). \tag{2.1}$$

LEMMA 2.2 ([7]). *If $\{a_\rho\}$ is pseudo-convergent, then either*
(1) $V(a_\rho) < V(a_\sigma)$ for all pairs ρ, σ with $\rho < \sigma$, or $V(a_\rho) = V(a_\sigma)$ from some point on, i.e., for all $\rho, \sigma \geq \lambda$.
(2) $V(a_\sigma - a_\rho) = V(a_{\rho+1} - a_\rho)$ for all $\rho < \sigma$.

As a consequence of Lemma 2.2 we can unambiguously put γ_ρ for $V(a_\sigma - a_\rho)$ $(\rho < \sigma)$. We note that by inequality (2.1), $\{\gamma_\rho\}$ is a monotone increasing set of elements of $\mathbf{R}$. In fact, if $\rho < \sigma$ then $\rho < \sigma < \sigma + 1$ and

$$\gamma_\rho = V(a_\sigma - a_\rho) < V(a_{\sigma+1} - a_\sigma) = \gamma_\sigma.$$

DEFINITION 2.3. *An element $x \in K$ is said to be a limit of the pseudo-convergent set $\{a_\rho\}$ if $V(x - a_\rho) = \gamma_\rho$ for all ρ.*

DEFINITION/LEMMA 2.4. *The set of all elements $y \in K$ such that $V(y) > \gamma_\rho$ for all ρ forms an (integral or fractionary) ideal $\mathfrak{A}$ in the valuation ring B, called the breadth of $\{a_\rho\}$.*

The limit of a pseudo-convergent set is by no means unique; however, given one limit, it is easy to describe the totality of limits:

LEMMA 2.5 ([7]). *Let $\{a_\rho\}$ be a pseudo-convergent, with breadth $\mathfrak{A}$, and let x be a limit of $\{a_\rho\}$. Then an element is a limit of $\{a_\rho\}$ if, and only if, it is of the form $x + y$, with $y \in \mathfrak{A}$.*

Let the field L be an extension of K, with a valuation that is an extension of V. If the group and residue class field of L coincide with $\mathbf{R}$ and k, respectively, we say that L is an *immediate* extension of K. If K admits no proper immediate extensions, K is said to be *maximal*.

THEOREM 2.6 (Kaplansky, [7]). *A field with a valuation is maximal if, and only if it contains a limit for each of its pseudo-convergent sets.*

LEMMA 2.7 ([3], p. 90). *Let $\{a_\rho\}_{\rho\in T}$ be a pseudo-convergent set and $T' \subset T$ a well ordered cofinal subset. Then $\{a_\rho\}_{\rho\in T'}$ is a pseudo-convergent set with the same breadth and the same set of limits of $\{a_\rho\}_{\rho\in T}$.*

We need some results concerning the structure of a maximal field F with value group $\mathbf{R}$ and residue class field k, in order to have a unique maximal extension of a given valued field. We have

THEOREM 2.8 ([4], p. 226). *Let the maximally complete field F have value group Γ and residue class field k, such that F and k have characteristic* *zero. Suppose that every element of k has an n-th root in k for all integers n. Then F is analytically isomorphic to $k(((t^{\Gamma}))$, where this last field is the field of formal power series $\sum_{\gamma \in S} a_\gamma t^\gamma$ with $a_\gamma \in k$ and S a well ordered subset of Γ.*

REMARK 2.9. *We will use this result in the case $F = \overline{\mathbf{C}}/J$ (that we will prove to be maximal), where $k = k_J$, the residue field of $\overline{\mathbf{C}}/J$, and $\Gamma = \mathbf{R}$.*

If the residue field k_J does not contain the n-th roots of its elements – as is the case of $\overline{\mathbf{R}}/J$ – then a factor set $c_{\alpha,\beta}$ appears:

THEOREM 2.10 ([7], Theorem 6, p. 317). *Let K be a maximal field with value group Γ and residue field k. If K and k are both of characteristic zero, then K is analytically isomorphic to a power series field $k((t^\Gamma, c_{\alpha,\beta}))$, where $t^\alpha \cdot t^\beta = c_{\alpha,\beta} t^{\alpha+\beta}$ and $c_{\alpha,\beta}$ is a factor set.*

3. Maximality of residue class fields

Now we work again in the ring $\overline{\mathbf{K}}$ of Colombeau generalized numbers. We will denote the filtration ν simply by V, because in the quotient $\overline{\mathbf{K}}/J$ it is really a valuation.

LEMMA 3.1. *Let $\{v_n(\varepsilon)\}_{n\geq 1}$ be a sequence of moderate functions such that each v_n is zero in $[1/2^n, 1]$ and $V(v_n) \geq \gamma_n$. Suppose that $\{\gamma_n\}$ is a monotone increasing sequence of positive real numbers. Then the series*

$$\phi_1(\varepsilon) = \sum_{n\geq 1} v_n(\varepsilon) \tag{3.1}$$

defines a moderate function.

Proof. First notice that the series (3.1) makes sense because for each ε, there is only a finite number of indices n such that $v_n(\varepsilon)$ can possibly be different of zero.

If we define $g_m^{m+k} = v_m + v_{m+1} + \cdots + v_{m+k}$ then g_m^{m+k} is moderate and $V(g_m^{m+k}) \geq \min\{V(v_{m+j}) \ : \ 0 \leq j \leq k\} \geq \gamma_m$ by hypothesis. To prove the lemma, it is enough to prove that there is a real number b such that

$$\frac{|\sum_{n\geq 2} v_n(\varepsilon)|}{\varepsilon^b} = \frac{|\lim_{k\to\infty} g_2^{2+k}(\varepsilon)|}{\varepsilon^b} \longrightarrow 0 \qquad (\varepsilon \to 0^+). \tag{3.2}$$

We choose $b = \gamma_1 - \delta_1$, for some $\delta_1 > 0$. If (3.2) is false, then, there is an $\epsilon^* > 0$ such that for every $\delta > 0$ we can find some $0 < \varepsilon^* < \delta$ with

$$\frac{|\lim_{k\to\infty} g_2^{2+k}(\varepsilon^*)|}{(\varepsilon^*)^{\gamma_1 - \delta_1}} > \epsilon^*.$$

But this means that for $k \geq N_0$ sufficiently great we must have

$$\frac{|g_2^{2+k}(\varepsilon^*)|}{(\varepsilon^*)^{\gamma_1-\delta_1}} > \epsilon^*,$$

which is a contradiction because if we let $\delta \to 0$, we already know that $V(g_2^{2+k}(\varepsilon)) \geq \gamma_2$ and, by Lemma 1.1 (5),

$$\frac{g_2^{2+k}(\varepsilon)}{(\varepsilon)^{\gamma_1-\delta_1}} \longrightarrow 0.$$

□

REMARK 3.2. *Notice that in the situation of the above lemma we can conclude*

$$V(\sum_{j\geq m} v_j(\varepsilon)) \geq \gamma_m.$$

In fact, if we had considered g_m^{m+k} instead of g_2^{2+k} we could have chosen $b = \gamma_m - \delta_m$ and the proof would have been the same.

THEOREM 3.3. *Let $\{[a_\rho]\}_{\rho\in T}$ be a pseudo-convergent set in $\overline{\mathbf{K}}/J$. Then there is a cofinal subset $T' \subset T$ and a pseudo-convergent set $\{a'_\rho\}_{\rho\in T'}$ in $\overline{\mathbf{K}}$ such that $[a'_\rho] = [a_\rho]$ for all $\rho \in T'$.*

Proof. Since the value group Γ of the valued field $\overline{\mathbf{K}}/J$ is $\Gamma = \mathbf{R}$ and T is a well ordered subset of Γ without a last element, we have that T is enumerable and hence, can be written as

$$T = \{\rho_1 < \rho_2 < \cdots < \rho_\omega < \rho_{\omega+1} < \cdots < \rho_{2\omega} < \rho_{2\omega+1} < \cdots\},$$

where ω is the first non finite ordinal. We will choose a well ordered cofinal subset T' of T and then apply Lemma 2.7. Then we can find a convenient lifting to a pseudo-convergent set in $\overline{\mathbf{K}}$.

First we consider the cofinal subset

$$T'' = T - \{\rho_\omega, \rho_{2\omega}, \rho_{3\omega}, \ldots\}.$$

Notice that he set T'' can be divided into *segments*, the first being $\{\rho_1 < \rho_2 < \cdots\}$, the second $\{\rho_{\omega+1}, \rho_{\omega+2}, \cdots\}$, and so on. Then it is clear that we can have either

(1) T'' has only a finite number k of segments, or

(2) T'' has an infinite number of segments.

In the first case we define $T' = T''$ if $k = 1$; $T' = \{\rho_{\omega+1}, \rho_{\omega+2}, \cdots\}$ if $k = 2$; and, for general $k \geq 2$,

$$T' = \{\rho_{(k-1)\omega+1}, \rho_{(k-1)\omega+2}, \rho_{(k-1)\omega+3}, \cdots\}.$$

In the second case we define

$$T' = \{\rho_1, \rho_{\omega+1}, \rho_{2\omega+1}, \rho_{3\omega+1}, \cdots\}.$$

It is clear in both cases that T' is a well ordered cofinal subset of T and by Lemma 2.7, $\{[a_\rho]\}_{\rho\in T'}$ is a pseudo-convergent set with the same limits of the initial one.

The first case. We can suppose without loss of generality that

$$T' = \{\rho_1, \rho_2, \rho_3, \cdots\}.$$

Defining $V([a_{\rho+1}]-[a_\rho]) = \gamma_\rho$ $(\rho \in T')$ we have (see Lemma 2.2) that $\{\gamma_\rho\}_{\rho\in T'}$ is a monotone increasing sequence of real numbers without a greatest element. Let us now construct the convenient lifting. Choose a'_{ρ_2}, a'_{ρ_1} representatives of $[a_{\rho_2}], [a_{\rho_1}]$ such that $V(a'_{\rho+1} - a'_\rho) = \gamma'_{\rho_1}$ with $\gamma'_{\rho_1} < \gamma_{\rho_1}$. Choose a'_{ρ_3} a representative of $[a_{\rho_3}]$ such that $V(a'_{\rho_3} - a'_{\rho_2}) = \gamma'_{\rho_2}$ with $\gamma_{\rho_1} < \gamma'_{\rho_2} < \gamma_{\rho_2}$. If we continue in this way we have a similar construction of all $\{a'_{\rho_j}\}$, for $1 \leq j$, with the property:

$$\gamma'_{\rho_1} < \gamma_{\rho_1} < \gamma'_{\rho_2} < \gamma_{\rho_2} < \gamma'_{\rho_3} < \gamma_{\rho_3} < \cdots$$

Let us prove that $\{a'_\rho\}_{\rho\in T'}$ is a pseudo-convergent set in $\overline{\mathbb{K}}$. Consider the set of elements $\rho < \sigma < \tau$ of T'. We must prove that

$$V(a'_\sigma - a'_\rho) < V(a'_\tau - a'_\sigma).$$

Since $\{\gamma'_\rho\}_{\rho\in T'}$ is monotone increasing, it is enough to prove that

$$V(a'_\sigma - a'_\rho) = V(a'_{\rho+1} - a'_\rho)$$

for all pairs $\rho < \sigma$. If $\sigma = \rho + k$ for some $k \geq 2$, then

$$a'_{\rho+k} - a'_\rho = \left(a'_{\rho+k} - a'_{\rho+(k-1)}\right) + \left(a'_{\rho+(k-1)} - a'_{\rho+(k-2)}\right) + \cdots + \left(a'_{\rho+1} - a'_\rho\right)$$

and since $\gamma'_\rho < \gamma'_{\rho+1} < \cdots < \gamma'_{\rho+(k-1)}$ we have that

$$V(a'_\sigma - a'_\rho) = \min\{V(a'_{\rho+j} - a'_{\rho+(j-1)}) \ : \ 1 \leq j \leq k\} = \gamma'_\rho.$$

This proves that $\{a'_\rho\}_{\rho\in T'}$ is a pseudo-convergent set in $\overline{\mathbb{K}}$.

The second case. Now $T' = \{\rho_1, \rho_{\omega+1}, \rho_{2\omega+1}, \rho_{3\omega+1}, \cdots\}$, and, in the same way we did above, we can construct liftings $a'_{\rho_{j\omega+1}}$ for all $j \geq 0$ such that if we put $V([a_\tau] - [a_{\rho_{j\omega+1}}]) = \gamma_j$ $(\tau \in T'$ and $\rho_{j\omega+1} < \tau)$ then

$$V(a'_{\rho_{(j+1)\omega+1}} - a'_{\rho_{j\omega+1}}) = \gamma'_j < \gamma_j,$$

$$\gamma'_1 < \gamma_1 < \gamma'_2 < \gamma_2 < \gamma'_3 < \gamma_3 < \cdots.$$

The remainder of the proof is the same. This proves the theorem. □

THEOREM 3.4. *The quotient $\overline{\mathbb{K}}_J$ of the Colombeau ring $\overline{\mathbb{K}}$ by a maximal ideal J is a maximal field.*

Proof. Let $\{[a_\rho]\}_{\rho\in T}$ be a pseudo-convergent set in $\overline{\mathbb{K}}_J$. We must construct an element $[f] \in \overline{\mathbb{K}}_J$ such that $V([f]-[a_\rho]) = \gamma_\rho$ for all ρ. Since the sequence of real numbers $\{\gamma_\rho\}_{\rho\in T}$ is monotone increasing, multiplying each a_ρ by a ε^r (for an appropriate fixed r, independent of ρ) we can suppose that $\gamma_\rho > 0$ for all ρ. It is clear that if $[f]$ is a limit for $\{[\varepsilon^r a_\rho]\}_{\rho\in T}$ then $[\varepsilon^{-r} f]$ will be a limit for the original pseudo-convergent set.

By Theorem 3.3 we can suppose without loss of generality that there is a set of representatives $\{a'_\rho\}_{\rho\in T}$ of the classes $\{[a_\rho]\}_{\rho\in T}$ such that if we set $V(a'_\tau - a'_\rho) = \gamma'_\rho$ then

$$\gamma'_{\rho_1} < \gamma_{\rho_1} < \gamma'_{\rho_2} < \gamma_{\rho_2} < \gamma'_{\rho_3} < \gamma_{\rho_3} < \cdots \tag{3.3}$$

It does not matter if we are in the first or the second case. We will use tha notation of the first case for its simplicity.

If we can find a "limit" $f \in \overline{\mathbb{K}}$ for this set $\{a'_\rho\}_{\rho\in T}$, in the sense that

$$V(f - a'_\rho) = \gamma'_\rho, \qquad \forall \rho \in T$$

then

$$V([f]-[a'_\rho]) = V([f]-[a'_{\rho+1}]+[a'_{\rho+1}]-[a'_\rho])$$

but $V([a'_{\rho+1}]-[a'_\rho]) = \gamma_\rho$ and $V([f]-[a'_{\rho+1}]) \geq V(f-a'_{\rho+1}) = \gamma'_{\rho+1} > \gamma_\rho$ by (3.3) So,

$$V([f]-[a'_\rho]) = \min\{V([a'_{\rho+1}]-[a'_\rho]), V([f]-[a'_{\rho+1}])\} = \gamma_\rho.$$

This proves that $[f]$ is a limit for the pseudo-convergent set $\{[a_\rho]\}_{\rho\in T}$. Let us find this limit f in $\overline{\mathbb{K}}$.

Let us write T as $T = \{\rho_1 < \rho_2 < \cdots\}$. By hypothesis, for every $\rho \in T$ we have $V(a'_{\rho+1} - a'_\rho) = \gamma'_\rho$ and we can find some representatives $a'_{\rho+1}(\varepsilon), a'_\rho(\varepsilon) \in \mathcal{E}_M(\mathbb{K})$, in order to define: $v_n(\varepsilon) = a'_{\rho_n+1}(\varepsilon) - a'_{\rho_n}(\varepsilon)$, for $\varepsilon < 1/2^n$ and $v_n(\varepsilon) = 0$ for $\varepsilon \geq 1/2^n$. Define $g_n = a'_{\rho_1} + v_1 + v_2 + \cdots + v_n$. Then g_n and a'_{ρ_n+1} define the same class in $\overline{\mathbb{K}}$. Notice that since $1/2^n \to 0$, the series

$$f(\varepsilon) = a'_{\rho_1}(\varepsilon) + \sum_{n\geq 1} v_n(\varepsilon)$$

makes sense, because, for each given ε it consists only in a finite sum. In fact it is a moderate function by Lemma 3.1.

Besides, considering the projections in $\overline{\mathbb{K}}$, we have:

$$f - a_{\rho_n} = a'_{\rho_1} + \sum_{j\geq 1} v_n - \left(a'_{\rho_1} + v_1 + v_2 + \cdots + v_{n-1}\right) = \sum_{j\geq n} v_j$$

and so

$$V(f - a'_{\rho_n}) = V(a'_{\rho_n+1} - a'_{\rho_n} + \sum_{j\geq n+1} v_j) =$$

$$= \min\{V(a'_{\rho_n+1} - a'_{\rho_n}), V(\sum_{j\geq n+1} v_j)\} = \gamma'_{\rho_n},$$

by Remark 3.2. This proves the theorem.

□

To use Theorems 2.8 and 2.10 above, we notice that if k_J is the residue field of $\overline{K}_J$ (it is clear that the characteristic of k_J is zero and that $\mathbf{K} \subset k_J$), then, if $\mathbf{K} = \mathbb{C}$, every element of k_J has an n-th root in k_J for all integers $n \geq 1$. It follows from Theorem 3.4 that

$$\overline{\mathbb{C}}/J \cong k_J((t^{\mathbf{R}})).$$

This is no longer true if $\mathbf{K} = \mathbf{R}$. In this case we have that ([7], p. 317)

$$\overline{\mathbf{R}}/J \cong k_J((t^{\mathbf{R}}, c_{\alpha,\beta})),$$

where $k_J((t^{\mathbf{R}}, c_{\alpha,\beta}))$ is the field of formal power series $\sum_{\gamma \in S} a_\gamma t^\gamma$ with S a well ordered subset of $\mathbf{R}$ and

$$t^\alpha t^\beta = c_{\alpha,\beta} t^{\alpha+\beta},$$

with $c_{\alpha,\beta} \in k_J$ a factor set.

4. A partial order in $\overline{\mathbf{R}}$

In this and the next section section we shall construct a total order in the residue class fields $\overline{\mathbf{R}}/J$. This order was introduced by the authors of [8], and most of the results of this and the next sections, except Lemma 4.1 and Theorem 5.2, are due to D. Scarpalezos in a personal communication to the authors of [8].

The partial order and the lattice structure of the set of all functions $\mathbf{R}^{]0,1]}$ induce naturally a partial order on the subring of the moderate functions $\mathcal{E}_M(\mathbf{R})$, which is also a sublattice. To pass these structures to $\overline{\mathbf{R}} = \mathcal{E}_M(\mathbf{R})/\mathcal{N}$ some care is needed. For the function $-exp(-1/\varepsilon)$ is negative in $\mathcal{E}_M(\mathbf{R})$ and zero in $\overline{\mathbf{R}}$.

We will put a partial order in $\overline{\mathbf{R}}$ as follows: if $f, g \in \overline{\mathbf{R}}$, then $f \geq 0$ if for any representative $f(\varepsilon)$ of f we have: for every $b > 0$ there is an $0 < \eta_b < 1$ such that $f(\varepsilon) \geq -\varepsilon^b$ in $]0, \eta_b]$.

We say that $f \geq g$ if $f - g \geq 0$.

Lemma 4.1. *If $f \in \overline{\mathbf{R}}$ then $f \geq 0$ if, and only if, there exists a representative $f(\varepsilon)$ of f such that $f(\varepsilon) \geq 0$ for all $\varepsilon \in]0,1]$.*

Proof. If there is some representative $f(\varepsilon)$ of f as above, then all representatives are of the form

$$f(\varepsilon) + \xi(\varepsilon)$$

for $\xi(\varepsilon) \in \mathcal{N}$. Since for every $b > 0$ we have $-\varepsilon^b \leq \xi(\varepsilon) \leq \varepsilon^b$ for all ε in some $]0, \eta_b]$, then $-\varepsilon^b \leq f(\varepsilon) + \xi(\varepsilon)$ for $0 < \varepsilon \leq \eta_b$. Then $f \geq 0$. Conversely, let

$f_1(\varepsilon)$ be an arbitrary fixed representative of f. Then, for every $b > 0$ there is an η_b such that $f_1(\varepsilon) \geq -\varepsilon^b$ in $]0, \eta_b]$. Let us choose $b = n$, for all natural numbers $n \geq 1$ and put $\eta_b = \eta_n$ (that we can choose to verify: $\eta_{n+1} < \eta_n$) Define:

$$A_n = \{\varepsilon \in]0, \min\{1/2^n, \eta_n\}] \ : \ f_1(\varepsilon) < 0\}.$$

Then $A_{n+1} \subset A_n$ and $A_n \subset]0, 1/2^n]$ for all $n \geq 1$. If there is some $N \geq 1$ such that A_N is empty, then we can choose $f(\varepsilon)$ as being $f_1(\varepsilon)$ in $]0, 1/2^{N+1}]$ and 0 in $]1/2^{N+1}, 1]$. Then it is clear that $f = f_1$ in $\overline{\mathbf{R}}$ and that $f(\varepsilon) \geq 0$ for all ε. If A_n is not empty for every $n \geq 1$, we define

$$f(\varepsilon) = \begin{cases} f_1(\varepsilon) & if\ f_1(\varepsilon) \geq 0 \\ \varepsilon^n & if\ \varepsilon \in A_n \setminus A_{n+1} \end{cases}$$

Then $f(\varepsilon)$ is a moderate function and $f(\varepsilon) - f_1(\varepsilon)$ is equal to zero or to $\varepsilon^n - f_1(\varepsilon)$ in $A_n \setminus A_{n+1}$. But then, since in $A_n \setminus A_{n+1}$ we have $-\varepsilon^n \leq f_1(\varepsilon) < -\varepsilon^{n+1}$, we must have

$$0 < \varepsilon^n + \varepsilon^{n+1} < \varepsilon^n - f_1(\varepsilon) \leq 2\varepsilon^n, \qquad in\ A_n \setminus A_{n+1}.$$

But this implies that $f(\varepsilon) - f_1(\varepsilon)$ belongs to $\mathcal{N}$. This proves the lemma. □

Lemma 4.2. *The relation $\geq$ is a partial order in $\overline{\mathbf{R}}$.*

Proof. It is clear that if $f \geq g$ then $f + h \geq g + h$. From Lemma 4.1 it is clear that if $f \geq 0$ and $g \geq 0$ we must have $fg \geq 0$. Besides, if $f \geq g$ and $g \geq f$ then $f = g$. In fact, fixed $b > 0$, we must have

$$f(\varepsilon) - g(\varepsilon) \geq -\varepsilon^b \qquad g(\varepsilon) - f(\varepsilon) \geq -\varepsilon^b \qquad \varepsilon \in]0, \eta]$$

Thus, $|f(\varepsilon) - g(\varepsilon)| < \varepsilon^b$, and hence $f - g$ is in $\mathcal{N}$. □

We can define the absolute value of $f \in \overline{\mathbf{R}}$: it is the class $|f|$ of the function defined by

$$|f|(\varepsilon) := |f(\varepsilon)|.$$

It is clearly a positive moderate function. The class $|f|$ does not depend on the representative; for if f_1 is another representative,

$$||f(\varepsilon)| - |f_1(\varepsilon)|| \leq |f(\varepsilon) - f_1(\varepsilon)| = o(\varepsilon^b)$$

for all $b \in \mathbf{R}$. Thus, $|f| - |f_1| \in \mathcal{N}$.

Lemma 4.3 (Convexity of ideals). *If J is an arbitrary ideal of $\overline{\mathbf{R}}$, then $f \in J$ if, and only if, $|f| \in J$. More generally, if $0 \leq |g| \leq |f|$ and $f \in J$, then $g \in J$.*

Proof. We fix a representative $f(\varepsilon)$ of f in what follows. Let $u(\varepsilon)$ be the function such that $u(\varepsilon) = 1$ if $f(\varepsilon) \geq 0$ and $u(\varepsilon) = -1$ if $f(\varepsilon) < 0$. Then $u(\varepsilon)$ is a moderate function and its class belongs to $\overline{\mathbb{R}}$.

$$|f(\varepsilon)| = u(\varepsilon)f(\varepsilon) \qquad \forall \varepsilon \in]0,1]$$

and so, since $u^{-1}(\varepsilon) = u(\varepsilon)$, $f \in J \iff |f| \in J$. If $0 \leq |g| \leq |f|$ and $f \in J$ then we can choose representatives $f(\varepsilon)$ and $g(\varepsilon)$ of the classes f and g such that $|g(\varepsilon)| \leq |f(\varepsilon)|$ (notice that if $f(y) = 0$ then $g(y) = 0$) for ε in some $]0, \eta]$. Define the function $u(\varepsilon)$ for $\varepsilon \in]0, \eta]$ as

$$u(\varepsilon) = \begin{cases} g(\varepsilon)/f(\varepsilon) & if\ f(\varepsilon) \neq 0 \\ 1 & if\ f(\varepsilon) = 0 \end{cases}$$

and $u(\varepsilon) = 1$ for $\varepsilon > \eta$. It is clear that u is moderate, and so, since

$$g(\varepsilon) = u(\varepsilon)f(\varepsilon) \qquad \forall \varepsilon \in]0, \eta]$$

we have the lemma. □

We can define the positive and negative parts of an element $f \in \overline{\mathbb{R}}$:

$$f^+ = \frac{f + |f|}{2} \qquad f^- = \frac{f - |f|}{2}$$

then $f = f^+ + f^-$, $f^+ \geq 0$, $f^- \leq 0$.

LEMMA 4.4. *If J is a prime ideal of $\overline{\mathbb{R}}$ and f is an element not in J, then either f^+ or f^- belongs to J.*

Proof. Choose a representative $f(\varepsilon)$ of f. Then, it is clear that $f^+(\varepsilon)$ is a representative of f^+ and $f^-(\varepsilon)$ is a representatvive of f^-. Besides

$$f^+(\varepsilon) \cdot f^-(\varepsilon) = 0, \qquad \forall \varepsilon \in]0,1].$$

Since J is a prime ideal, we must have $f^+ \in J$ or $f^- \in J$. □

5. Total order in $\overline{\mathbb{R}}/J$

We have seen that $\overline{\mathbb{R}}$ is a partially ordered ring and every maximal ideal is convex (Lemma 4.3 above). To put an order in the residue field $\overline{\mathbb{R}}/J$ we recall a general fact (see [9], Chap. 5, Theorem 5.2): If I is an ideal in a partially ordered ring A, then, in order that A/I be a partially ordered ring according to the definition

$$[f] \geq 0 \Leftrightarrow \exists g \in A,\ with\ g \geq 0\ \&\ g \equiv f\ (mod\ J),$$

it is necessary and sufficient that I be convex.[1]

So we define the order in $\overline{\mathbf{R}}/J$ in this way and it follows from this definition ([9], Chap. 5, Theorem 5.3) that $[f] \geq 0$ if, and only if, $f \equiv |f|$ $(mod\ J)$. We could also say that $[f] \geq 0$ if, and only if, $f \equiv f^+$ $(mod\ J)$ (or equivalently: $[f] \geq 0$ if, and only if $f^- \in J$).

THEOREM 5.1. *The field $\overline{\mathbf{R}}_J$ is totally ordered by the above order.*

Proof. We saw in Lemma 4.4 that if $f \in \overline{\mathbf{R}}$ then either $f^+ \in J$ or $f^- \in J$. This means that every element is comparable with zero. It is also clear that if $[f] \leq [g]$ and $[g] \leq [f]$ then $[f] = [g]$, because $f - g \equiv |f - g|$ $(mod\ J)$ and $g - f \equiv |g - f|$ $(mod\ J)$ imply $f - g \equiv g - f$ $(mod\ J)$ and so, $f \equiv g$ $(mod\ J)$. □

Besides, this order is compatible with the order of $\mathbf{R} \subset \overline{\mathbf{R}}_J$.

THEOREM 5.2. *If J is a maximal ideal of $\overline{\mathbf{R}}$ then $\overline{\mathbf{R}}_J$ is a real-closed field.*

Proof. We will prove that every positive element has a square root and every polynomial of odd degree has a root in the field $\overline{\mathbf{R}}_J$. If $0 < [f]$ is a positive element of $\overline{\mathbf{R}}_J$, then $f \equiv f^+ (mod\ J)$ and by Lemma 4.1 we can suppose that there is a representative such that $f(\varepsilon) \geq 0$ for $\varepsilon \in]0, \eta]$. Then, we consider $g(\varepsilon) := \sqrt{f(\varepsilon)}$ for $0 < \varepsilon < \eta$ and 0 for $\eta \leq \varepsilon \leq 1$. Thus g is a moderate function and $g^2 = f$ in $\overline{\mathbf{R}}$. If

$$X^n + [f_{n-1}]X^{n-1} + \cdots + [f_1]X + [f_0]$$

is a polynomial with odd degree an coefficients in $\overline{\mathbf{R}}_J$. We consider the polynomial equation

$$X^n + f_{n-1}(\varepsilon)X^{n-1} + \cdots + f_1(\varepsilon)X + f_0(\varepsilon) = 0 \tag{5.1}$$

obtained by choosing representatives $f_j(\varepsilon) \in \mathcal{E}_M(\mathbf{R})$ of the classes f_j in $\overline{\mathbf{R}}$.

For each fixed $\varepsilon \in]0, 1]$, (5.1) is a polynomial equation with real coefficients and odd degree. Therefore, there is a real root $r(\varepsilon)$ which we can suppose, by standard calculus, to verify:

$$|r(\varepsilon)| \leq \max\{1, 2n|f_{n-1}(\varepsilon)|, \ldots, 2n|f_0(\varepsilon)|\}. \tag{5.2}$$

This defines a function $r:]0, 1] \to \mathbf{R}$. Since the right side of (5.2) is a moderate function it follows that $r(\varepsilon)$ is also moderate. If we had considered a general polynomial

$$[f_n]X^n + [f_{n-1}]X^{n-1} + \cdots + [f_1]X + [f_0]$$

[1] In [9], convexity is defined in a different way. What we call *convexity* in Lemma 4.3 is called *absolute convexity* there. Absolute convexity is stronger then convexity.

with $[f_n] \neq 0$, then dividing by $[f_n]$ we would have obtained a monic polynomial with the same roots.

□

REMARK 5.3. *Since $\overline{\mathbb{C}} = \overline{\mathbb{R}} + i\overline{\mathbb{R}}$ (where $i = \sqrt{-1}$), if J_c is a maximal ideal of $\overline{\mathbb{C}}$, then*

$$J = \{f \in \overline{\mathbb{R}} \; : \; f = \Re(h), \; for\ some\ h \in J_c\}$$

(where $\Re(h)$ is the "real part" of the germ h) is a subset of J_c that generates J_c in $\overline{\mathbb{C}}$ and is a maximal ideal of $\overline{\mathbb{R}}$. Then $J_c = J + iJ$ and

$$\overline{\mathbb{R}}_J(i) = \overline{\mathbb{R}}[i]/(J[i]) = \overline{\mathbb{C}}/J_c,$$

and so, since $\overline{\mathbb{R}}/J$ is a real-closed field, $\overline{\mathbb{C}}/J_c$ is algebraically closed.

6. Isomorphism classes of residue class fields

Of course the next question is: are the fields $\overline{\mathbb{K}}_J$ isomorphic? We will answer this question in the case $\mathbb{K} = \mathbb{C}$. For the real case $\mathbb{K} = \mathbb{R}$, although $\overline{\mathbb{R}}/J$ is a real-closed field we cannot apply the main theorem of [5], because it is not a η_1-set (see [5] or [9] for a definition), as can be seen by choosing $A = \{[0]\}$ and $B = \{[\varepsilon^n] \; : n \geq 1\}$. It is clear that there is no $[h] \in \overline{\mathbb{R}}/J$ such that

$$[0] < [h] < [\varepsilon^n] \qquad \forall n \geq 1.$$

LEMMA 6.1. *If J is a maximal ideal of $\overline{\mathbb{K}}$ then the degree of transcendency of $\overline{\mathbb{K}}_J$ over the field $\mathbb{K}$ is at least $\mathfrak{c}$.*

Proof. It is clear that $\mathbb{K} \subset \overline{\mathbb{K}}_J$. Consider the subfield of $\overline{\mathbb{K}}_J$ defined by

$$\mathfrak{L} := \mathbb{K}(\{[\varepsilon^r] \; : \; r \in \mathbb{R}\}).$$

We will prove that the set $A = \{[\varepsilon^r] \; : \; r \in \mathbb{R}\}$ is a transcendency base of $\mathfrak{L}$ over $\mathbb{K}$. Since $\varepsilon^b \cdot \varepsilon^d = \varepsilon^{b+d}$ and $(\varepsilon^b)^m = \varepsilon^{mb}$ for all integers $m \geq 1$, it is enough to prove that the set A is linearly independent over $\mathbb{K}$. Suppose that

$$\alpha_1[\varepsilon^{b_1}] + \alpha_2[\varepsilon^{b_2}] + \cdots + \alpha_k[\varepsilon^{b_k}] = [0],$$

with each $\alpha_j \neq 0$ and $b_1 < b_2 < \cdots < b_k$. This means that $\alpha_1\varepsilon^{b_1} + \alpha_2\varepsilon^{b_2} + \cdots + \alpha_k\varepsilon^{b_k} \in J$, which is impossible unless all α_j's are zero, because $\alpha_1\varepsilon^{b_1} + \alpha_2\varepsilon^{b_2} + \cdots + \alpha_k\varepsilon^{b_k}$ is a unit of $\overline{\mathbb{K}}$. □

For every $\hat{f} \in \mathcal{E}_M(\mathbb{K})$ we write $Z(\hat{f}) = \{\varepsilon \in X \; : \; \hat{f}(\varepsilon) = 0\}$; $Z(\hat{f})$ is called the *zero-set* of $\hat{f}$. We have the following characterization of the elements of $\overline{\mathbb{K}}$:

LEMMA 6.2 ([8]). *Let f be any non-zero element of $\overline{\mathbb{K}}$. Then:*

(1) If f is a unit, then every representative $\hat{f}$ of f verifies $0 \notin \overline{Z(\hat{f})}$.

(2) If f is not a unit, there is a representative $\hat{f}$ of f such that $0 \in \overline{Z(\hat{f})}$, where $\overline{X}$ denotes the closure of a subset $X \subset]0,1]$ in $[0,1]$.

DEFINITION 6.3. *If f is a non-unit of $\overline{\mathbb{K}}$, a representative $\hat{f}$ of f such that (2) of Lemma 6.2 holds will be called a generic representative of f.*

Let I be any ideal of $\overline{\mathbb{K}}$. We define the set (we use the convention: if $\hat{f} \in \mathcal{E}_M(\mathbb{K})$ then its projection in $\overline{\mathbb{K}}$ will be denoted simply by f)

$$\mathcal{Z}(I) = \{Z(\hat{f}) \ : \ \forall \hat{f} \text{ generic}, \ f \in I, \}.$$

We also denote by $\mathcal{Z}(]0,1])$ the set of all $Z(\hat{f})$ for every generic $\hat{f} \in \mathcal{E}_M(\mathbb{K})$.

We say that a subfamily $\mathcal{A} \subset \mathcal{Z}(]0,1])$ is *generically closed* (for short g-closed) if for every pair $Z(\hat{f})$, $Z(\hat{g})$ in $\mathcal{A}$, there are generic elements $\hat{f}_1$ and $\hat{g}_1$ such that

(1) $f = f_1$ and $g = g_1$ in $\overline{\mathbb{K}}$ and

(2) $Z(\hat{f}_1) \cap Z(\hat{g}_1) \in \mathcal{A}$.

LEMMA 6.4. *The family $\mathcal{Z}(I)$, I a proper ideal of $\overline{\mathbb{K}}$, satisfies the following properties:*

(1) The empty set is not a member of $\mathcal{Z}(I)$.

(2) $\mathcal{Z}(I)$ is g-closed.

(3) Let $A \in \mathcal{Z}(I)$ and $B = Z(\hat{g})$ for some generic $g \in \mathcal{E}_M(\mathbb{K})$ such that $B \supset A$. Then $B \in \mathcal{Z}(I)$.

Proof. Since I is a proper ideal, (1) is clear from the definition of $\mathcal{Z}(I)$.

If $Z(\hat{f})$ and $Z(\hat{g})$ are members of $\mathcal{Z}(I)$ then we have $Z(\hat{f}) \cap Z(\hat{g}) = Z(|\hat{f}| + |\hat{g}|)$. But although $|f| + |g| \in I$ (see Lemma 4.3) the function $|\hat{f}| + |\hat{g}|$ may be non-generic. Let this be the case; we can then suppose without loss of generality that there is an $\eta > 0$ such that $|\hat{f}(\varepsilon)| + |\hat{g}(\varepsilon)| \neq 0$ for every $\varepsilon \in]0,\eta]$.

But by Lemma 6.2 we can find a generic $\hat{h}$ such that $h = |f| + |g|$ in $\overline{\mathbb{K}}$ and $\hat{h} = |\hat{f}| + |\hat{g}| + \hat{\xi}$, where $\hat{\xi} \in \mathcal{N}$.

Define the set $U = Z(\hat{h})$. Then U is a nonempty subset of $]0,1]$ and $0 \in \overline{U}$. Consider the function

$$\hat{\varphi}(\varepsilon) = \begin{cases} |\hat{f}(\varepsilon)| + |\hat{g}(\varepsilon)| & if\ \varepsilon \in U^c \\ 0 & if\ \varepsilon \in U \end{cases}$$

Then $\hat{\varphi}$ is a moderate function and

$$\hat{h}(\varepsilon) - \hat{\varphi}(\varepsilon) = \begin{cases} \hat{\xi}(\varepsilon) & if\ \varepsilon \in U^c \\ 0 & if\ \varepsilon \in U \end{cases},$$

and this means that the projection φ of $\hat{\varphi}$ is equal to $|f|+|g|$ in $\overline{\mathbf{K}}$. Now define

$$\hat{f}_1(\varepsilon) = \begin{cases} \hat{f}(\varepsilon) & if\ \varepsilon \in U^c \\ 0 & if\ \varepsilon \in U \end{cases} \qquad \hat{g}_1(\varepsilon) = \begin{cases} \hat{g}(\varepsilon) & if\ \varepsilon \in U^c \\ 0 & if\ \varepsilon \in U \end{cases}$$

We will prove that $\hat{f}_1 - \hat{f} \in \mathcal{N}$ and $\hat{g}_1 - \hat{g} \in \mathcal{N}$. To do this, notice that the functions

$$|\hat{f}(\varepsilon)|\hat{\chi}_U(\varepsilon) \qquad |\hat{g}(\varepsilon)|\hat{\chi}_U(\varepsilon)$$

are functions of $\mathcal{N}$; in fact

$$\frac{|\hat{f}(\varepsilon)|\hat{\chi}_U(\varepsilon)}{\varepsilon^b} \leq \frac{(|\hat{f}(\varepsilon)| + |\hat{g}(\varepsilon)|)\hat{\chi}_U(\varepsilon)}{\varepsilon^b} = \frac{|\hat{\xi}(\varepsilon)\hat{\chi}_U(\varepsilon)|}{\varepsilon^b} \to 0$$

for very $b \in \mathbf{R}$. Since we have supposed that $|\hat{f}(\varepsilon)|+|\hat{g}(\varepsilon)| \neq 0$ for every $\varepsilon \in]0,\eta]$ and $0 \in \overline{U}$, the functions $|\hat{g}(\varepsilon)|\hat{\chi}_U(\varepsilon)$ and $|\hat{f}(\varepsilon)|\hat{\chi}_U(\varepsilon)$ are non identically zero. But we may write:

$$|\hat{g}(\varepsilon)|\hat{\chi}_U(\varepsilon) = |\hat{g}(\varepsilon)\hat{\chi}_U(\varepsilon)| \qquad |\hat{f}(\varepsilon)|\hat{\chi}_U(\varepsilon) = |\hat{f}(\varepsilon)\hat{\chi}_U(\varepsilon)|,$$

and since it is clear that $u \in \mathcal{N}$ if and only if $|u| \in \mathcal{N}$, it follows that $\hat{f}(\varepsilon)\hat{\chi}_U(\varepsilon) \in \mathcal{N}$ and $\hat{g}(\varepsilon)\hat{\chi}_U(\varepsilon) \in \mathcal{N}$. This means that $\hat{f}(\varepsilon) - \hat{f}_1(\varepsilon) = \hat{f}(\varepsilon)\hat{\chi}_U \in \mathcal{N}$ and $\hat{g}(\varepsilon) - \hat{g}_1(\varepsilon) = \hat{g}(\varepsilon)\hat{\chi}_U \in \mathcal{N}$. Then $f = f_1$ and $g = g_1$ in $\overline{\mathbf{K}}$, and $Z(\hat{f}_1) \cap Z(\hat{g}_1) = U$. This proves (2). Let $A = Z(\hat{f}) \in \mathcal{Z}(I)$ and $B = Z(\hat{h})$ for some $\hat{h} \in \mathcal{E}_M(\mathbf{R})$, with $B \supset A$. Then, since I is an ideal, $fg \in I$ and $\hat{f}\hat{h}$ is a generic representative of fh. But

$$Z(\hat{f}\hat{h}) = Z(\hat{f}) \cup Z(\hat{h}) = Z(\hat{h}),$$

and this proves (3). □

Lemma 6.5. *Let $\mathcal{A}$ be a family of $\mathcal{Z}(]0,1])$ such that (1), (2) and (3) above are satisfied. Then $\mathcal{A} = \mathcal{Z}(I)$ for some proper ideal I of $\overline{\mathbf{K}}$.*

Proof. Let $I = \{f \in \overline{\mathbf{K}} \ : \ Z(\hat{f}) \in \mathcal{A}\}$. If $f, g \in I$ then by (2), there are generic representatives $\hat{f}$ and $\hat{g}$ such that

$$Z(\hat{f} + \hat{g}) \supset Z(|\hat{f}| + |\hat{g}|) = Z(\hat{f}) \cap Z(\hat{g})$$

Then, since we can suppose that $Z(\hat{f}) \cap Z(\hat{g}) \in \mathcal{A}$, by (3) we have $Z(\hat{f}+\hat{g}) \in \mathcal{A}$ and hence $f + g \in I$.

If $f \in I$ and $\psi \in \overline{\mathbf{K}}$ then we can find a generic representative $\hat{f}$ of f and a representative $\hat{\psi}$ of ψ. Then $\hat{f}\hat{\psi}$ is a generic representative of $f\psi$. Then

$$Z(\hat{f}\hat{\psi}) = Z(\hat{f}) \cup Z(\hat{\psi}) \supset Z(\hat{f}).$$

Therefore, by (3), $Z(\hat{f}\hat{\psi}) \in \mathcal{A}$ and so $f\psi \in I$.

This proves that I is an ideal. Let us prove that it is a proper ideal. If some unit $u \in \overline{\mathbf{K}}$ belongs to I, then $Z(\hat{u}) \in \mathcal{A}$ for some generic representative

$\hat{u}$. But by Lemma 6.2, u does not have any generic representative. This proves the lemma. □

LEMMA 6.6. *Let J be a maximal ideal of $\overline{\mathbb{K}}$ and define*

$$J' = \{f \in \overline{\mathbb{K}} \,:\, \exists \hat{f},\ \textit{generic}\ , Z(\hat{f}) \in \mathcal{Z}(J)\}.$$

Then $J' = J$ and so, if $[f]$ and $[g]$ are elements of the quotient $\overline{\mathbb{K}}/J$ such that there are representatives $\hat{f}$ and $\hat{g}$ that coincide in some $Z \in \mathcal{Z}(J)$, then $[f] = [g]$.

Proof. By Lemmas 6.4 and 6.5 we have that J' is a proper ideal of $\overline{\mathbb{K}}$ and it is clear that $J \subset J'$. By the maximality of J we have that $J = J'$. If $\hat{f}$ and $\hat{g}$ coincide in some $Z(\hat{u}) \in \mathcal{Z}(J)$ then $Z(\hat{f} - \hat{g}) \supset Z(\hat{u})$ and so, by Lemma 6.4 (3) we have $Z(\hat{f} - \hat{g}) \in \mathcal{Z}(J)$. But by the above reasoning, $f - g \in J$, and so $[f] = [g]$. This proves the lemma.

□

DEFINITION 6.7 (Adapted from [6]). *By the minimal cardinal associated with a maximal ideal J of $\overline{\mathbb{K}}$, we shall mean the smallest of the cardinal numbers of the subsets Z, for $Z \in \mathcal{Z}(J)$. By construction it is clear that the smallest minimal cardinal is $\aleph_0$.*

LEMMA 6.8. *Let $\mathfrak{b}$ denote the associated minimal cardinal of a maximal ideal J of $\overline{\mathbb{K}}$. Then the power of $\overline{\mathbb{K}}/J$ is not greater then $2^{\mathfrak{b}}$.*

Proof. By Lemma 6.6, if $[f]$ and $[g]$ have representatives $\hat{f}$ and $\hat{g}$ that coincide in some $Z(\hat{u}) \in \mathcal{Z}(J)$ then $[f] = [g]$. Since there are at most $\mathfrak{c}^{\mathfrak{b}}$ (i.e., $2^{\mathfrak{b}}$, because $\mathfrak{b} \geq \aleph_0$) K-valued functions defined in $Z(\hat{u})$, it follows that there are at most this many mutually incongruent such functions. □

THEOREM 6.9. *Let J be a maximal ideal of $\overline{\mathbb{K}}$. Then the mininal cardinal of J is $\aleph_0$.*

Proof. We will construct a generic function $\hat{f}(\varepsilon)$ such that $Z(\hat{f})$ is enumerable and $f \in J$. Let E be any enumerable subset of $X =]0,1]$ such that $0 \in \overline{E}$. If, for some such E, $\chi_{E^c} \in J$ then we are done. If not, let $E = \{1/n \,:\, n \geq 1\}$. Then $\hat{\chi}_E \in J$. Define $I_n =]1/(n+1), 1/n[$, $n \geq 1$, and

$$\hat{g}(\varepsilon) = \begin{cases} 1 & \textit{if}\ \varepsilon \in E \\ \varepsilon^n & \textit{if}\ \varepsilon \in I_n - \{p_n\} \\ 0 & \textit{if}\ \varepsilon = p_n \end{cases}$$

where p_n is the middle point of I_n. Then $\hat{g}$ is moderate and $\hat{\chi}_E - \hat{g}$ is a function that has the properties:

(1) for all real numbers $b \in \mathbf{R}$ we have:

$$\frac{\hat{\chi}_E(\varepsilon) - \hat{g}(\varepsilon)}{\varepsilon^b} \longrightarrow 0 \qquad \varepsilon \to 0^+,$$

that is, $\chi_E = g$ in $\overline{\mathbf{K}}$, and

(2) $Z(\hat{g}) = \{p_n \; : \; n \geq 1\}$ (and so $\hat{g}$ is generic).

So, $g \in J$ and the cardinality of $Z(\hat{g})$ is $\aleph_0$. This proves the theorem. □

Theorem 6.10. *All quotients $\overline{\mathbf{C}}_J$ have power $\mathfrak{c}$ and so are all isomorphic.*

Proof. By Theorem 6.9, the minimal cardinal associated to every maximal ideal J of $\overline{\mathbf{C}}$ is $\aleph_0$ and so, by Lemma 6.8, the cardinality of $\overline{\mathbf{C}}/J$ is $\leq 2^{\aleph_0} = \mathfrak{c}$. But since we have trivially that $\mathfrak{c} \leq$ the cardinality of $\overline{\mathbf{C}}/J$ we have that all such quotients have cardinality exactly $\mathfrak{c}$. Since all these fields are algebraically closed (see remark 5.3), a classical theorem of Steinitz inplies that all these fields are isomorphic. □

If $\mathbf{K} = \mathbf{R}$ we have already observed that $\overline{\mathbf{R}}/J$ is not of order type η. We will prove the following:

Lemma 6.11. *The order of $\overline{\mathbf{R}}/J$ is unbordered and continuous.*

Proof. A set is unbordered if it is not empty and nas no first and last elements. Considering any $[f] \in \overline{\mathbf{R}}/J$ we must have that for some representative $f(\varepsilon)$ of f: $|f(\varepsilon)| = o(\varepsilon^b)$ (for some $b \in \mathbf{R}$). Then, there is an $0 < \eta_b < 1$ such that $-\varepsilon^b < f(\varepsilon) < \varepsilon^b$ for $0 < \varepsilon < \eta_b$, and so, by the definition of the order in $\overline{\mathbf{R}}$ we have $-\varepsilon^b \leq f \leq \varepsilon^b$, and since the family $\{\varepsilon^r\}_{r \in \mathbf{R}}$ has no first and no last element, $\overline{\mathbf{R}}/J$ is unbordered.

An ordered set, each of whose decompositions is a cut, is said to be continuous ([10], Ch. III, §7). Let us suppose that $< A, B >$ is a decomposition of $\overline{\mathbf{R}}/J$ that is a gap. Then $a < b$ for all $a \in A$ and all $b \in B$ and A has no last and B has no first element. Let $\{[f_n]\}$ and $\{[g_n]\}$ $(n \geq 1)$ be two sequences of elements of $\overline{\mathbf{R}}/J$ cofinal in A and coinitial in B respectively, such that

$$[f_n] < [f_{n+1}] < [g_{m+1}] < [g_m], \tag{6.1}$$

for all $n, m \geq 1$. We will prove that there is an element $[h] \in \overline{\mathbf{R}}/J$ such that

$$[f_n] < [h] < [g_m]$$

for all n, m. This is not possible since $< A, B >$ is a decomposition of $\overline{\mathbf{R}}/J$, and so there are no gaps in $\overline{\mathbf{R}}/J$.

First note that we may assume, without loss of generality, that we can find representatives $f_n(\varepsilon)$ and $f_{n+1}(\varepsilon)$ of f_n and f_{n+1} (and $g_{m+1}(\varepsilon)$, $g_m(\varepsilon)$ of g_{m+1}, g_m) such that for all $\varepsilon \in]0, 1]$ we have:

$$f_n(\varepsilon) \leq f_{n+1}(\varepsilon), \qquad g_{n+1}(\varepsilon) \leq g_n(\varepsilon). \tag{6.2}$$

For if we put

$$f'_n(\varepsilon) = \max\{f_1(\varepsilon), \dots, f_n(\varepsilon)\}, \qquad g'_m(\varepsilon) = \min\{g_1(\varepsilon), \dots, g_m(\varepsilon)\},$$

then these functions are clearly moderate and, by (6.1) (and the definition of the order) we will have that $f'_n = f_n$ and $g'_m = g_m$ in $\overline{\mathbb{R}}$ (that is, they define the same class module $\mathcal{N}$).

Secondly, we note that we may also assume that for all $\varepsilon \in]0,1]$

$$f_n(\varepsilon) \leq g_n(\varepsilon). \tag{6.3}$$

For put $f''_1(\varepsilon) = f'_1(\varepsilon)$, and $g''_1(\varepsilon) = \max\{f''_1(\varepsilon), g'_1(\varepsilon)\}$. If we have defined $f''_1, \dots, f''_n, g''_1, \dots, g''_n$ so that

$$f''_1(\varepsilon) \leq \dots \leq f''_n(\varepsilon) \leq g''_n(\varepsilon) \leq \dots \leq g''_1(\varepsilon),$$

for all $\varepsilon \in]0,1]$, then we put

$$f''_{n+1}(\varepsilon) = \min\{\max\{f''_n(\varepsilon), f'_{n+1}(\varepsilon)\}, g''_n(\varepsilon)\},$$
$$g''_{n+1}(\varepsilon) = \max\{\min\{g''_n(\varepsilon), g'_{n+1}(\varepsilon)\}, f''_{n+1}(\varepsilon)\},$$

then it follows from (6.1) and the order in $\overline{\mathbb{R}}$ that we have $f''_n = f'_n$ and $g''_n = g'_n$ for all $n \geq 1$. Moreover, we have

$$f''_n(\varepsilon) \leq f''_{n+1}(\varepsilon) \leq g''_{n+1}(\varepsilon) \leq g''_n(\varepsilon)$$

for all $\varepsilon \in]0,1]$.

We define $h(\varepsilon) = \sup\{f_n(\varepsilon) \ : \ n \geq 1\}$. Then, by (6.2) and (6.3), $h(\varepsilon)$ is a moderate function and we have $f_n(\varepsilon) \leq h(\varepsilon) \leq g_n(\varepsilon)$ for all $\varepsilon \in]0,1]$. But in fact we have

$$[f_n] < [h] < [g_m]$$

for all $m, n \geq 1$, because if $[f_k] = [h]$ for some k, then, since $f_n(\varepsilon) \leq f_{n+1}(\varepsilon)$, and $|h(\varepsilon) - f_{k+j}(\varepsilon)| \leq |h(\varepsilon) - f_k(\varepsilon)|$ for all $j \geq 1$, the fact that $[h] = [f_k]$ means $h - f_k \in J$, and so, by Lemma 4.3, we would have $[h] = [f_{k+j}]$, contradicting (6.1). In the same way we cannot have $[h] = [g_k]$ for any k. This proves that $\overline{\mathbb{R}}/J$ has a continuous order. □

7. Open questions

(1) Is the order type of $\overline{\mathbb{R}}/J$ the order type of the field of real numbers? That is, in view of the above lemma, can we construct an enumerable dense subset of $\overline{\mathbb{R}}/J$? (See [10], Chap. III, §7, Theorem 3).

(2) Are the fields $\overline{\mathbb{R}}/J$, J a maximal ideal, isomorphic?

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