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J. Veloso and J. Verderesi

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THREE DIMENSIONAL CAUCHY-RIEMANN MANIFOLDS

by

J. Veloso and J. Verderesi

Departamento de Matematica
Universidade de S. Paulo
CP 20570, CEP 05508
S. Paulo, Brasil.

Introduction.

Three dimensional Cauchy-Riemann manifolds (or real hypersurfaces of $\mathbb{C}^2$) were studied by E. Cartan [1]. The generalization to higher dimensions of part of Cartan results was done by Chern and Moser [2] and Tanaka [4]. In [2], a G-structure and a Cartan connection is associated to a CR-manifold, which solves the question when a real hypersurface of $\mathbb{C}^n$ is biholomorphically equivalent to S^{2n+1}. Cartan also studied a class of CR-manifolds, which was called in [2] nonumibilic, and proved several results in this case. These theorems were not generalized up to date.

In this paper, we derive some of Cartan results studying the overdetermined systems of partial differential equations (SPDE) associated to a three dimensional CR-manifold M. We get a intrinsically defined curvature tensor R on M, and M is called umbilic if $R = 0$ and nonumbilic if R

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never vanishes. In case M is nonumbilic, we associate to M a Z_2-structure, which allowed to define three complex functions intrinsically defined on M, the fundamental curvatures associated to a nonumbilic CR-manifolds. We use these curvatures to caracterize nonumbilic homogeneous three dimensional CR-manifolds.

We skip some lenghty calculations and details, which will appear in a forthcoming paper.

1. CR-manifolds and CR-diffeomorphisms

In this paper, M and M' always denote manifolds of dimension 3, TM and TM' the tangent bundles, T^*M and T^*M' the cotangent bundles.

We call M a Cauchy-Riemann manifold (CR-manifold) if TM contains a distribution Δ of codimension 1, such that:

i) there exists a complex structure on Δ, i.e., $J:\Delta\to\Delta$ linear such that $J^2 = -I$;

ii) if Θ is a (local) 1-form on M such that $\ker\Theta = \Delta$, then $\Theta \wedge d\Theta$ never vanishes on M.

We denote by Δ', J' and so on, the correspondents concepts on M'.

A diffeomorphism $f:M\to M'$ is a CR-diffeomorphism if $f_*(\Delta) = \Delta'$ and $f_*\circ J = J'\circ f_*$.

Let $\Delta_{\mathbb{C}} \subset T_{\mathbb{C}}M$ the complexified of Δ. Then $\Delta_{\mathbb{C}} = \Delta^{1,0} \oplus \Delta^{0,1}$, where $\Delta^{1,0} = \{v \in \Delta_{\mathbb{C}} : Jv = iv\}$ and $\Delta^{0,1} = \overline{\Delta^{1,0}}$. We can easily verify that f is a CR-diffeomorfism if and only if $f_*(\Delta^{1,0}) = \Delta'^{1,0}$, where the extension f_* to $T_{\mathbb{C}}M$ is the natural one. From now on, we will work with the complexified tangent bundles.

The most natural examples of CR-manifolds are real hypersurfaces (strictly pseudo convex) of $\mathbb{C}^2$; and restriction of biholomorphic transformations of $\mathbb{C}^2$ to real hypersurfaces, CR-diffeomorphisms.

2. Coordinate sistem on $J^k(M,M')$.

Let $U \subset M$ be an open set, such that there exists a (complex) vector field $Z_1 \in \Delta^{1,0}|U$, different of zero in each point of U. Because Δ is 2-dimensional, Z_1 is a base of $\Delta^{1,0}|U$. We define

(2.1) $$Z_{\bar{1}} = \bar{Z}_1 \in \Delta^{0,1}|U$$

and

(2.2) $$Z_o = -i[Z_1, Z_{\bar{1}}].$$

Observe that Z_o is a real vector field. It follows from the condition $\theta \wedge d\theta$ never vanishes, if $\ker \theta = \Delta$, that $Z_o(p), Z_1(p), Z_{\bar{1}}(p)$ is a basis of $(T_{\mathbb{C}}M)_p$ for every $p \in U$.

We call $\{Z_o, Z_1, Z_{\overline{1}}\}$ adapted basis on the open set U.

We define complex valued functions A, B and C on U, by

(2.3) $$[Z_1, Z_o] = A\,Z_1 + \overline{B}\, Z_{\overline{1}} + C\, Z_o .$$

From Jacobi identity, we get

(2.4)
$$B_1 - A_{\overline{1}} + A\,\overline{C} - B\,C = 0,$$
$$(\overline{C})_1 - C_{\overline{1}} + i(A + \overline{A}) = 0$$

where, if F is a complex valued function on U, we denote by

$$F_i = Z_i(F), \quad i = 0,1,\overline{1} .$$

Let $U' \subset M'$ be an open set, with adapted base $\{Z'_o, Z'_1, Z'_{\overline{1}}\}$, and define the correspondent complex functions A', B' and C' on U'.

By $J^k(M,M')$ we denote the space of jets of (local) diffeomorphisms, from M to M', and by $\alpha : J^k(M,M') \to M$ the projection $\alpha(j^k_x f) = x$, and $\beta : J^k(M,M') \to M'$ the projection $\beta(j^k_x f) = f(x)$, and $\Pi^k_h : J^k(M,M') \to J^h(M,M')$ is defined as $\Pi^k_h(j^k_x f) = j^h_x f$.

We introduce a coordinate system on the open set

$J^1(U,U') = \alpha^{-1}(U) \cap \beta^{-1}(U')$ of $J^1(M,M')$. If $f:U \to U'$ is a diffeomorphism, we define coordinates

$$p^i_j : J^1(U,U') \longrightarrow \mathbb{C} \;,\; i,j = 0,1,\bar{1}$$

by

$$(2.5) \qquad (f*)_x(Z_i(x)) = p^o_i(j^1_x f).Z'_o(f(x)) + p^1_i(j^1_x f).Z'_1(f(x)) + \\ + p^{\bar{1}}_i(j^1_x f).Z_{\bar{1}}(f(x)).$$

The coordinates p^i_j are not all independents. It follows from (2.1) and (2.2) that

$$(2.6) \qquad \overline{p^i_j} = p^{\bar{i}}_{\bar{j}} \;,\; i,j = 0,1,\bar{1},$$

where we assume the convention $\bar{o} = o$, and $\bar{\bar{1}} = 1$.

The coordinates on $J^2(U,U')$ are introduced by

$$(2.7) \qquad p^j_{ki}(j^2_x f) = Z_k(p^j_i(j^1 f))(x) \;,\; i,j,k = 0,1,\bar{1},$$

and again $\overline{p^i_{jk}} = p^{\bar{i}}_{\bar{j}\,\bar{k}}$. These are not the only relations between the p^i_{jk}'s. From $f*[Z_i,Z_j] = [f*Z_i,f*Z_j]$, we get

$$(2.8)\qquad \sum_k A^k_{ij}(x)\,p^m_k(j^1_x f) = p^m_{ij}(j^2_x f) - p^m_{ji}(j^2_x f) + \sum_{r,s} p^r_i(j^1_x f)\,.\,p^s_j(j^1_x f)\,.\,A'^m_{rs}(\,f(x))$$

where

$$[Z_i, Z_j] = \sum_k A^k_{ij}\; Z_k\,, \qquad i,j,k = 0,1,\bar{1},\,.$$

and the A^k_{ij} are defined in the same way

For example,

$$(2.9)\qquad p^o_{1\bar{1}} - p^o_{\bar{1}1} = i(p^o_o - p^1_1\, p^{\bar{1}}_{\bar{1}} + p^1_{\bar{1}}\, p^{\bar{1}}_1) + C'(p^1_1\, p^o_{\bar{1}} - p^o_1\, p^1_{\bar{1}}) + \bar{C}'(p^{\bar{1}}_1\, p^o_{\bar{1}} - p^{\bar{1}}_{\bar{1}}\, p^o_1)\,.$$

The coordinates for $J^3(U,U')$ are defined by

$$p^j_{mki}(j^3_x f) = Z_m(p^j_{ki}(j^2 f))(x)$$

and so on.

3. SPDE associated to CR-diffeomorphisms.

We have seen that a necessary and sufficient condition such that $f:U \to U'$ is a CR-diffeomorphism is $f_*(\Delta^{1,0}) = \Delta'^{1,0}$. In an adapted basis, this condition becomes

$$(3.1) \qquad f_* Z_1 = p_1^1(j^1 f)\, Z_1'$$

or

$$(3.2) \qquad p_1^{\bar{1}}(j^1 f) = p_1^{o}(j^1 f) = 0 .$$

Let $S^1 \subset J^1(M,M')$ defined by

$$S^1|_{U\times U'} = \{X \in J^1(U,U'): p_1^{\bar{1}}(X) = p_1^{o}(X) = 0\} .$$

We denote $S^1|_{U\times U'}$ briefly as

$$(3.3) \qquad S^1 : \{p_1^{\bar{1}} = p_1^{o} = 0.$$

Then $f:M \to M'$ is a CR-diffeomorphism if and only if $j_x^1 f \in S^1$ for every $x \in M$.

Next, this SPDE will be prolonged until geometric information on CR-manifolds, invariant by CR-diffeomorphisms, is obtained .

If $S^k \subset J^k(M,M')$ is a SPDE of order k, defined by

$$(3.4) \qquad S^k|_{U \times U'} = \{X \in J^1(U,U') : F_r(X) = 0,\ 1 \leq r \leq a\},$$

then $S^{k+1} \subset J^{k+1}(M,M')$, the <u>prolongation</u> of S^k, is given by

$$S^{k+1}|_{U \times U'} = \{X \in J^{k+1}(U,U') : F_r(\Pi_k^{k+1} X) = 0,$$

$$(\partial_{Z_i} F_r)(X) = 0,\ 1 \leq r \leq a,\ i = 0,1,\bar{1}\},$$

where

$$(\partial_{Z_i} F_r)(j_x^{k+1} f) = Z_i(F_r(j^k f))(x).$$

It follows easily from the definition that

$$(3.5) \qquad S^2 : \begin{cases} p_1^{\bar{1}} = p_1^{o} = 0 \\ \bar{p}_{11}^{\bar{1}} = p_{\bar{1}1}^{\bar{1}} = p_{o1}^{\bar{1}} = 0 \\ p_{11}^{o} = p_{\bar{1}1}^{o} = p_{o1}^{o} = 0 \end{cases}$$

Conjugating $p_{\bar{1}1}^{o} = 0$, we get $p_{1\bar{1}}^{o} = 0$, then using.

formulas (2.9) and (3.5), we can replace Im $p^{o}_{\overline{1}1} = 0$ in (3.5), by.

$$p^{o}_{o} - p^{1}_{1}\, p^{\overline{1}}_{\overline{1}} = 0,$$

a new equation of first order. Calling $\tilde{S}^{1} = \Pi^{2}_{1}(S^{2})$, we get

$$(3.6) \qquad \tilde{S}^{1}: \{p^{\overline{1}}_{1} = p^{o}_{1} = p^{o}_{o} - p^{1}_{1}\, p^{\overline{1}}_{\overline{1}} = 0.$$

Then

$$(3.7) \qquad \tilde{S}^{2}: \begin{cases} \text{equations (3.5)} \\ p^{o}_{oo} - p^{\overline{1}}_{\overline{1}}\, p^{1}_{o1} - p^{1}_{1}\, p^{\overline{1}}_{o\overline{1}} = 0 \\ \dfrac{p^{\overline{1}}_{\overline{1}\overline{1}}}{p^{\overline{1}}_{1}} - 2i\, \dfrac{p^{1}_{o}}{p^{1}_{1}} - (\overline{C} - \overline{C}'\, p^{\overline{1}}_{1}) = 0 \end{cases}$$

It is a verification that $\Pi^{2}_{1}(\tilde{S}^{2}) = \tilde{S}^{1}$, and if we put $\tilde{\tilde{S}}^{2} = \Pi^{3}_{2}(\tilde{S}^{3})$, we get

$$(3.8) \qquad \tilde{\tilde{S}}^{2}: \begin{cases} \text{equations (3.7)} \\ \dfrac{p^{\overline{1}}_{o\overline{1}}}{p^{\overline{1}}_{\overline{1}}} - \dfrac{p^{o}_{oo} + 3i p^{1}_{o}\, p^{\overline{1}}_{o}}{2\, p^{o}_{o}} + \frac{1}{2}(D - D' p^{o}_{o}) - \frac{1}{2}(C' p^{1}_{o} - \overline{C}' p^{\overline{1}}_{o}) \end{cases}$$

where

$$(3.9) \qquad D = \frac{1}{2}\left(C_{\bar{1}} + i(A - 2\bar{A})\right)$$

and D' is defined similarly.

Again, if $\hat{S}^2 = \Pi^3_2(\tilde{\tilde{S}}^3)$, then

$$(3.10) \qquad \hat{S}^2 : \begin{cases} \text{equation (3.8)} \\ \dfrac{p^{\bar{1}}_{oo}}{p^{\bar{1}}_{\bar{1}}} - \dfrac{p^{\bar{1}}_{o} \; p^{o}_{oo}}{p^{\bar{1}}_{\bar{1}} \; p^{o}_{o}} - (K - K' \; p^1_1 \; p^o_o) - i\dfrac{p^1_o (p^{\bar{1}}_o)^2}{p^{\bar{1}}_{\bar{1}} \; p^o_o} - \\ \qquad - C' \dfrac{p^1_o \; p^{\bar{1}}_o}{p^{\bar{1}}_{\bar{1}}} - \bar{B}' \; p^1_1 \; p^1_o + (i\,D' - \bar{A}')p^1_1 \; p^{\bar{1}}_o = \end{cases}$$

where

$$(3.11) \qquad K = -\frac{i}{3}(C_o - i\,D_1 + i\,C\,D + A\,C - \bar{B}\,\bar{C}).$$

Finally,

$$(3.12) \qquad \Pi^3_2(\hat{S}^3) : \begin{cases} \text{equation (3.10)} \\ p^{\bar{1}}_{\bar{1}}\, R - p^1_1 (p^o_o)^2 \, R' = 0 \end{cases}$$

where

$$(3.13) \qquad R = K_1 - \overline{B}_o - 2CK - \overline{B}(A + \overline{A} - i\,D).$$

Theorem 3.1. If $\mathcal{R}$ is defined by

$$(3.14) \qquad \mathcal{R}|_U = R\, Z_1^* \wedge Z_o^* \otimes Z_o^* \otimes Z_{\overline{1}} + \overline{R}\, Z_{\overline{1}}^* \wedge Z_o^* \otimes Z_o^* \otimes Z_1,$$

then $\mathcal{R}$ is a tensor on M, i.e, $\mathcal{R} \in \Gamma(\Lambda^2 T^*M \otimes T^*M \otimes TM)$.

Proof. Let U, U' open sets of M, with $U \cap U' \neq \psi$. In coordinate sistem given by adapted basis $\{Z_o, Z_1, Z_{\overline{1}}\}$ on U, and $\{Z_o', Z_1', Z_{\overline{1}}'\}$ on U', call $A_j^i = p_j^i(j^1 id)$.

Then, by (2.5),

$$Z_i = id_*(Z_i) = \sum_j A_i^j Z_j' \quad , \quad i,j = 0,1,\overline{1}.$$

Because id is a CR-diffeomorphism of M, the A_i^j satisfy equations (3.6) and the new equation in (3.12), i.e.,

$$(3.15) \qquad \begin{aligned} Z_1 &= A_1^1 Z_1' , \\ Z_o &= A_o^1 Z_1' + A_o^{\overline{1}} Z_{\overline{1}}' + A_o^o Z_o' . \end{aligned}$$

and

(3.16) $A^{\bar{1}}_{\bar{1}} R - A^1_1 (A^o_o)^2 R' = 0.$

From (3.15), we have

(3.17) $(Z'_1)^* = A^1_1 Z^*_1 + A^1_o Z^*_o$

$(Z'_o)^* = A^o_o Z^*_o$.

So, replacing (3.17) below, and using (3.16),

$$R'(Z'_1)^* \wedge (Z'_o)^* \otimes (Z'_o)^* \otimes Z'_{\bar{1}} + \bar{R}'(Z'_{\bar{1}})^* \wedge (Z'_o)^* \otimes (Z'_o)^* \otimes Z'_1 =$$

$$R' A^1_1 (A^o_o)^2 (A^{\bar{1}}_{\bar{1}})^{-1} Z^*_1 \wedge Z^*_o \otimes Z_o \otimes Z_{\bar{1}} + \bar{R}' A^{\bar{1}}_{\bar{1}} (A^o_o)^2 (A^o_o)^2 (A^1_1)^{-1} Z^*_{\bar{1}} \wedge$$

$$\wedge Z^*_o \otimes Z^*_o \otimes Z_1 = R Z^*_1 \wedge Z^*_o \otimes Z_o \wedge Z_{\bar{1}} + \bar{R} Z^*_{\bar{1}} \otimes Z^*_o \otimes Z_1,$$

we get that the definition (3.44) independs of the coordinate sistem.

We call R the (first) <u>curvature tensor</u> associated to the CR-manifold M. We call M <u>umbilic</u> if $R = 0$, and <u>nonumbilic</u> if R never vanishes.

<u>Example:</u> Let be $Q = \{(z1,z2) \in \mathbb{C}^2 : \frac{z2-\bar{z}2}{2i} - z1\,\bar{z}1 = 0\}$

Choosing $Z_1 = \frac{i}{2}\frac{\partial}{\partial z1} - \overline{z1}\frac{\partial}{\partial z2} \in \Delta^{1,0}$, then

$$[Z_1, Z_{\overline{1}}] = -\frac{1}{2}\left(\frac{\partial}{\partial z2} + \frac{\partial}{\partial \overline{z2}}\right), \text{ i.e, } Z_o = -\frac{1}{2}\left(\frac{\partial}{\partial z2} + \frac{\partial}{\partial \overline{z2}}\right).$$

It follows $[Z_1, Z_o] = 0$, i.e., $A = \overline{B} = C = 0$ and this implies that $R \equiv 0$, i,e, Q is umbilic

Theorem 3.2. M is locally CR-diffeomorphic to Q if and only if $R_M \equiv 0$.

Proof. Suppose first that M is locally diffeomorphic to Q, i,e, given $x \in M$, there exists a nbd U of x, and a CR-diffeomorphism $f: U \longrightarrow U' \subset Q$. Then f satisfies equation (3.12), where $R' = 0$, i.e., $p_{\overline{1}}^{\overline{1}} R \equiv 0$, what implies $R \equiv 0$.

Let's suppose now that $R_M \equiv 0$. Then the new equation in (3.12), $p_{\overline{1}}^{\overline{1}} R - p_1^1 (p_o^o)^2 R' \equiv 0$, because R_M and R_Q are identically zero. This implies that $\Pi_2^3(\hat{S}^3) = \hat{S}^2$. Let $\hat{g}^k \subset S^k T^* M \otimes_{\hat{S}^k} TQ$ the symbol of $\hat{S}^k$ (cf. [3] for concepts used in this proof) Then.

$$\hat{g}^2 = \left[Z_o^* \vee Z_o^* \otimes \left(Z_o' + \frac{p_o^1}{p_o^o} Z_1' + \frac{p_o^{\overline{1}}}{p_o^o} Z_{\overline{1}}'\right) + \frac{1}{2} Z_o^* \vee \left(\frac{1}{p_{\overline{1}}^{\overline{1}}} Z_1^* \otimes Z_1' + \frac{1}{p_1^1} \cdot Z_{\overline{1}}^* \otimes Z_{\overline{1}}'\right) \right]$$

and $\hat{g}^k = 0$, if $k \geq 3$.

The symbol $\hat{g}^2$ is 2-acyclic if the sequence

$$T^*M \otimes \hat{g}^{k+1} \xrightarrow{\delta} \Lambda^2 T^*M \otimes \hat{g}^k \xrightarrow{\delta} \Lambda^3 T^*M \otimes \hat{g}^{k-1}$$

is exact for $k \geq 2$. Because $\hat{g}^k = 0$, if $k \geq 3$, we only have to show that the sequence

$$0 \xrightarrow{\delta} \Lambda^2 T^*M \otimes \hat{g}^2 \xrightarrow{\delta} \Lambda^3 T^*M \otimes T^*M \otimes TQ$$

is exact, or, δ is one to one. This is an easy calculation. It follows from i) $\hat{S}^3 \longrightarrow \hat{S}^2 \longrightarrow 0$ is exact; ii) $\hat{g}^2$ is 2-acyclic, that $\hat{S}^2$ is formally integrable. As $\hat{g}^3 = 0$, this equation is integrable, and for each integral jet of $\hat{S}_2$ there is a solution, which is unique. So, given $x \in M$ and $y \in Q$, there exists open sets $x \in U \subset M$ and $y \in U' \subset Q$, and a CR-diffeomorphism $f: U \longrightarrow U'$.

Observation. As long as we know, $\hat{g}^2$ is the second example in the literature of a symbol which is 2-acyclic, but not involutive.

4. Nonumbilic CR-manifolds

In this section, M denotes a nonumbilic CR-manifold. Call

$$(4.1) \qquad \Lambda = \sqrt{R/\sqrt[4]{R\,\overline{R}}}\ .$$

Then, if we put $E_1 = \varepsilon \Lambda^{-1} Z_1$, with $\varepsilon = \pm 1$, it is an easy verification that the only adapted basis where $R \equiv 1$ are $\{E_o, E_1, E_{\bar{1}}\}$ and $\{E_o, -E_1, -E_{\bar{1}}\}$ where we define $E_{\bar{1}}$ and E_o as before. These basis define a canonical Z_2-structure on M. So, we proved.

Theorem 4.1. There exists only two adapted basis on M, such that $R \equiv 1$. If one is $\{E_o, E_1, E_{\bar{1}}\}$, the other is $\{E_o, -E_1, -E_{\bar{1}}\}$. In this way, to a nonumbilic CR-manifold M, we associate a canonical Z_2-structure.

From now on, we choose adapted basis $\{E_o, E_1, E_{\bar{1}}\}$ on $U \subset M$ and $\{E'_o, E'_1, E'_{\bar{1}}\}$ on $U' \subset M'$, as in Theorem 4.1. Then equation (3.14) becomes $p_1^1 - p_{\bar{1}}^{\bar{1}}(p_o^o)^2 = 0$, and from $p_o^o = p_1^1 \cdot p_{\bar{1}}^{\bar{1}}$; it follows that

$$p_1^1 = \varepsilon, \quad \varepsilon = \pm 1. \tag{4.2}$$

Then, equation (3.1) becomes

$$f_* E_1 = \varepsilon E'_1 ,$$

and from $[E_1, E_{\bar{1}}] = i E_o$, we get

$$f_* E_o = -i[f_* E_1, f_* E_{\bar{1}}] = E'_o$$

We have proved.

<u>Proposition 4.1.</u> If M and M' are nonumbilic CR-manifolds, then $f:M \to M'$ is a CR-diffeomorphism if and only if is a solution of

$$(4.3) \qquad S: \begin{cases} p_1^1 - \varepsilon = p_1^{\bar{1}} = p_1^o = 0, & \varepsilon = \pm 1. \\ p_o^o - 1 = p_o^1 = 0 \end{cases}$$

<u>Corollary 4.1.</u> The group of automorphisms of a nonumbilic CR-manifold M has dimension at most three.

Again, we have $[E_1, E_o] = A\, E_1 + \bar{B}\, E_{\bar{1}} + C\, E_o$, and from $f_*[E_1, E_o] = \varepsilon[E_1', E_o']$, we get

$$(4.4) \qquad A' \circ f = A \ ; \ B' \circ f = B, \ C' \circ f = C.$$

Observe that A, B, C^2 are complex functions intrinsically defined on M. We call these functions the <u>fundamental invariants</u> associated to a nonumbilic CR-manifold M. We have proved

<u>Theorem 4.2.</u>. A necessary condition for $f:M \to M'$ to be a CR-diffeomorphism is that

$$A = A' \circ f ; \ B = B' \circ f \ ; \ C^2 = (C')^2 \circ f$$

5. Nonumbilic homogeneous CR-manifolds .

In this section, M will denote a nonumbilic homogeneous CR-manifold. As a consequence of Theorem 4.2, A,B and C^2 are constant functions, and two homogeneous nonumbilic CR-manifolds are locally CR-diffeomorphic if and only if they have the same constants. These constants must satisfy (2.4) and $R \equiv 1$, that is

$$A\overline{C} - BC = A+\overline{A} = 1 + \frac{3}{2} A\overline{B} - \frac{i}{3} AC^2 = 0 .$$

From these relations we obtain the following complex Lie algebras, generated by $[E_1, E_{\overline{1}}] = i\, E_o$, and

(5.1) $[E_1, E_o] = i\,\lambda\, E_1 + \frac{2i}{3\lambda} E_{\overline{1}}$, $\lambda \in R^*$

(5.2) $[E_1, E_o] = i\,\lambda\, E_1 - i\,\lambda\, E_{\overline{1}} + \sqrt{-\frac{9\lambda}{2} - \frac{3}{\lambda}}\; E_o$, $\lambda < 0$

(5.3) $[E_1, E_o] = i\lambda(E_1 + E_{\overline{1}}) + i\sqrt{\frac{3}{\lambda} - \frac{9\lambda}{2}}\; E_o$, $\lambda < -\sqrt{\frac{2}{3}}$

or $0 < \lambda < \sqrt{\frac{2}{3}}$.

Theorem 5.1. Let be M a non umbilic homogeneous CR-manifold , G his group of automorphisms and L(G) the Lie algebra of G. Two cases can happen:

(i) $M \cong G$ and $L(G)_{\mathbb{C}}$ is isomorphic to a Lie algebra (5.2) or (5.3).

(ii) $M \tilde{=} G/Z_2$, and $L(G)_{\mathbb{C}}$ is isomorphic to a Lie algebra (5.1).

<u>Proof.</u> Let's call $\Theta: G\times M \to M$ the action of G on M Fixing $x \in M$, let be $\Theta_x: G\times M \to M$ defined by $\Theta_x(g) = \Theta(g,x)$ for $g \in G$. Because dim G = 3, there exists a nbd V of e in G such that $\Theta_x: V \to \Theta_x(V)$ is a diffeomorphism. If $A \in L(G)$, then $\tilde{A} = (\Theta_x)_*(A|_V)$ is a vector field on $\Theta_x(V)$ invariant by G. So we can choose $A_1, A_{\overline{1}}, A_o \in L(G)_{\mathbb{C}}$ such that $\tilde{A}_1 = E_1$, $\tilde{A}_{\overline{1}} = E_{\overline{1}}$ and $\tilde{A}_o = E_o$. This implies that $L(G)_{\mathbb{C}}$ is isomorphic to a Lie algebra from (5.1), (5.2) or (5.3).

He have now two cases to consider. Doing M = M' in equation (4.3), it tell us that if G_x is the isotropy of G in the point $x \in M$, then $G_x = \{e\}$ or $G_x \equiv Z_2$. If $C \neq 0$, it folows from (4.4) that $\varepsilon = 1$, and we are in case $G_x = \{e\}$ or, $G \tilde{=} M$. If C = 0, we have $G_x \tilde{=} Z_2$, and so, $M \tilde{=} G/Z_2$. The element in G_x, which is not the identity, transforms $\{E_o(x), E_1(x), E_{\overline{1}}(x)\}$ to $\{E_o(x), -E_1(x), -E_{\overline{1}}(x)\}$.

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