São Paulo Journal of Mathematical Sciences
https://doi.org/10.1007/s40863-021-00248-x

SPECIAL ISSUE COMMEMORATING THE GOLDEN JUBILEE OF THE INSTITUTE OF MATHEMATICS AND STATISTICS OF THE UNIVERSITY OF SÃO PAULO

A retrospect of the research in nonassociative algebras in IME-USP

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Accepted: 19 June 2021

Abstract
The aim of this paper is to give an overview of the participation of our research group in the development of nonassociative algebras.

Keywords Nonassociative algebras · Superalgebras · Loops

Algebraic structures, not necessarily associative, appear naturally in connection with other areas of Mathematics and even more in other branches of Science, such as Biology, Physics, etc. Apart from Lie algebras, which are the most known and investigated nonassociative structures, many other classes of nonassociative algebras have been widely studied in the world. For instance, Jordan algebras originated in 1934 in order to formalize algebraic properties of observables in quantum mechanics. Alternative algebras appeared even before. The octonions were discovered by Graves in 1843 and, independently, by Cayley in 1845. Alternative algebras also were used by Zorn and Moufang in problems related to projective geometry.

The main results of the research group of nonassociative algebras in IME-USP, through the collaboration of its members with researchers from other centers, graduate and pos-doc students, are briefly mentioned in the sections that follow.

Communicated by Vyacheslav Futorny.

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Published online: 14 July 2021

The investigation on nonassociative algebras in IME-USP started in 1980 by Roberto C. F. Costa. The research group grew up with professors formed in the Institute itself and the coming of external researchers. In this paper, we describe part of the studies developed by this group. The research on Lie Algebras is reported in another paper in the same volume of this journal. Besides the authors of this paper and our colleagues who work in the area of Lie algebras, the members of the research group in the University of São Paulo are Alexandre Grishkov, Henrique Guzzo Jr., Iryna Kashuba and Juan Carlos Gutiérrez Fernández. The group has many collaborators around the world. In 1985 the first postgraduate student was formed. Since then, there have been 23 Master and 32 PhD students formed by this part of the group.

1 Structure and representations

1.1 Alternative and Jordan algebras and superalgebras

We will remind the main definitions used in the paper.

An algebra is called *alternative* if it satisfies the identities

$$(x,x,y)=0,\ (x,y,y)=0,$$

where $(x,y,z)=(xy)z-x(yz)$ denotes the *associator* of the elements x, y, z. A classical example of nonassociative alternative algebra is the algebra of octonions.

An algebra is called *Jordan* if it satisfies the identities

$$xy-yx=0,\ (x,y,x^2)=0.$$

A typical example of Jordan algebra is the algebra $A^{(+)}$ obtained from an associative algebra A over a field of characteristic $\neq 2$ by introducing a new multiplication $a\cdot b=\frac{1}{2}(ab+ba)$. A Jordan algebra J is called *special* if it is isomorphic to a subalgebra of the algebra $A^{(+)}$ for a certain associative algebra A. Otherwise it is called *exceptional*. The most known example of an exceptional Jordan algebra is the *Albert algebra* of 3×3 Hermitian matrices over octonions (with the symmetric product $a\cdot b$).

A *superalgebra* is a $\mathbb{Z}_2$-graded algebra $A=A_0\oplus A_1$, that is, $A_iA_j\subseteq A_{i+j\ (\mathrm{mod}\ 2)}$. A representative example of a superalgebra is the *Grassmann algebra* G generated over a field F by elements $1,e_1,\dots,e_n,\dots$ with $e_ie_j=-e_je_i$. The products

$$1,\quad e_{i_1}e_{i_2}\dots e_{i_k},\quad i_1<i_2<\dots<i_k,$$

form a basis of G over F. Denote by G_0 and by G_1 the subspaces generated respectively by products of even and odd length; then $G=G_0\oplus G_1$ is the direct sum of these subspaces which provides G with a superalgebra structure.

Let now $A=A_0+A_1$ be a superalgebra over F. Consider the tensor product of F-algebras $G\otimes A$. The subalgebra

$$G(A)=G_0\otimes A_0+G_1\otimes A_1$$

is called the *Grassmann envelope* of the superalgebra A.

Let $\mathcal{M}$ be a variety of algebras over F. A superalgebra $A = A_0 + A_1$ is called an $\mathcal{M}$-*superalgebra* if the Grassmann envelope $G(A)$ belongs to $\mathcal{M}$. In this way, one can define alternative, Jordan, Lie, etc. superalgebras.

If $A = A_0 + A_1$ is an associative superalgebra, then the superalgebra A^{+s} obtained by introducing on A a super-symmetric multiplication $a \cdot b = \frac{1}{2}(ab + (-1)^{\bar{a}\bar{b}}ba)$ is a Jordan superalgebra. Here, $\bar{a}$ denotes *the index of parity* of the element $a \in A_0 \cup A_1 : \bar{a} = i$ if $a \in A_i$. Similarly to the case of algebras, we have notions of special and exceptional Jordan superalgebras.

1.1.1 Speciality of Jordan superalgebras defined by brackets

An important class of Jordan superalgebras was introduced by I. Kantor [110]. Let A be an associative commutative algebra with a *bracket*, that is, an anticommutative product $\{a, b\}$. Let $\bar{A} = \{\bar{a} : a \in A\}$ be a linear copy of A, consider the direct sum of vector spaces $\mathrm{Kan}(A) = A \oplus \bar{A}$, and define on it a multiplication $\cdot$ by the rules

$$a \cdot b = ab,\ a \cdot \bar{b} = \bar{a} \cdot b = \overline{ab},\ \bar{a} \cdot \bar{b} = \{a, b\},$$

where $a, b \in A$ and ab is the product in A. Define a $\mathbb{Z}_2$-grading on $J = \mathrm{Kan}(A)$ by putting $J_0 = A$, $J_1 = \bar{A}$. Then J becomes a commutative superalgebra. If J is Jordan then the bracket $\{a, b\}$ is called a *Jordan bracket*, and the superalgebra J is called a *Jordan superalgebra of brackets*. An important example of Jordan bracket is the *Poisson bracket* (see Sect. 4.1.1).

A natural question that arises is: "Whether and when is the Jordan superalgebra of brackets special?" McCrimmon proved that a simple superalgebra of the classical Poisson bracket is not special. In 1994, I. Shestakov proved that for any Poisson algebra A, the superalgebra $\mathrm{Kan}(A)$ is *i-special*, that is, it is a homomorphic image of a special algebra. In [137], it was proved that any superalgebra of Jordan brackets can be embedded into a Jordan superalgebra of Poisson brackets, hence it is *i*-special. In [175], I. Shestakov presents a condition that is necessary for such a superalgebra be special. The sufficiency of this condition, however, is still open.

1.1.2 Irreducible representations of the Jordan superalgebra of Grassmann Poisson bracket

The definition of Jordan algebra of brackets $\mathrm{Kan}(A)$ may be easily extended to the case when $A = A_0 + A_1$ is a commutative superalgebra; in this case one has to define $J(A) = (A_0 + A_1) \oplus \overline{(A_0 + A_1)}$ with the grading $J(A)_0 = A_0 + \overline{A_1}$, $J(A)_1 = A_1 + \overline{A_0}$, and a natural control over signs in the products. For the Grassmann Poisson superalgebra on n odd generators, $\mathrm{Kan}(n)$, A. Stern, C. Martínez and E. Zelmanov obtained a classification of irreducible representations in zero characteristic and when $n \geq 5$, using the Tits-Kantor-Koecher functor (TKK-functor), which associates to a Jordan (super)algebra J a Lie (super)algebra $\mathrm{TKK}(J)$. In [194], I. Shestakov and O. Folleco Solarte generalize this previous classification, obtaining the description of irreducible bimodules over $\mathrm{Kan}(n)$, over any algebraically closed field of characteristic not

2 and $n \geq 2$. Their proof is direct and does not use the structure of Lie modules over the Lie superalgebra TKK(Kan(n)).

1.1.3 Engel theorem for Jordan superalgebras

It is well known that finite dimensional Jordan nil algebras are nilpotent. This fact does not hold for Jordan superalgebras [170]. Nevetherless, I. Shestakov and K. Okunev [180] proved that, over an infinite field of characteristic not 2, a finite dimensional Jordan superalgebra J which satisfies the *Engel condition* on homogeneous elements (that is, the right multiplication R_a is nilpotent, for any homogeneous element $a \in J$) is nilpotent.

The question remains open for superalgebras over finite fields.

1.1.4 Wedderburn Principal Theorem for Jordan superalgebras

The validity of an analogue of the Wedderburn Principal Theorem for finite dimensional Jordan superalgebras is investigated by F. Gomez González in [63, 64] under the supervision of I. Shestakov. The theorem is proved in the case $\mathcal{J}$ is a finite dimensional Jordan superalgebra over a field of characteristic zero, with radical N such that $N^2 = 0$, and

- $\mathcal{J}/N \cong M_{n|m}(F)^{(+)}$, and no homomorphic image of N, considered as $\mathcal{J}/N$-bimodule, contains a subbimodule isomorphic to the $M_{n|m}(F)^{(+)}$-regular bimodule [63];
- $\mathcal{J}/N$ is a simple Jordan superalgebra of one of the types: $\mathcal{K}_{10}$, the 10-dimensional Kac superalgebra, $\mathcal{K}_3$, the 3-dimensional Kaplansky superalgebra, the Jordan superalgebra of a superform or a superalgebra in the 1-parametric family $\mathcal{D}_t$ of 4-dimensional superalgebras, over an algebraically closed field [64]. As in the first case, some additional conditions on N must be imposed.

In all cases, it is shown that the restrictions made on N are needed, by means of counterexamples.

1.1.5 Commuting U-operators in Jordan algebras

In [177] I. Shestakov presents an example of Jordan algebra with elements a, b such that $ab = 0$ but the quadratic operators U_a and U_b do not commute. This gives an answer to the open question published by J. Anquela, T. Cortés and H. Petersson in [4]. Moreover, he gives a simpler proof of the main result (in characteristic not 2) of that paper: in any non degenerate Jordan algebra, the condition $ab = 0$ implies $[U_a, U_b] = 0$. In [5] J. Anquela, T. Cortés and I. Shestakov generalize the construction in [177] and, as a consequence, obtain examples of Jordan algebras which cannot be imbedded in nondegenerate algebras.

1.1.6 Finitely presented Jordan algebras

It is known that if R is a finitely generated associative algebra over a field F of characteristic not 2, then the Jordan algebra $R^{(+)}$ is finitely generated as well. If the algebra R has an involution $*$ then the space of symmetric elements $H(R,*)$ is also finitely generated. Finite presentability of the Jordan algebras $R^{(+)}$ and $H(R,*)$ are considered in [189]: I. Shestakov and E. Zelmanov provide an example of a finitely presented associative algebra R for which the Jordan algebra $R^{(+)}$ is not finitely presented. Also, they prove that if R is a unital associative finitely presented algebra with involution that has at least 3 orthogonal connected symmetric idempotents then the Jordan algebra $H(R,*)$ is finitely presented. In particular, if A is a finitely presented associative algebra then the Jordan matrix algebra $M_3(A)^{(+)}$ is finitely presented. The corresponding question is open for matrices of order 2.

1.1.7 Representations of alternative and Jordan superalgebras of characteristic 3

Simple alternative superalgebras were classified by I. Shestakov and E. Zelmanov. It was proved by I. Shestakov that a simple alternative superalgebra which is not just a $\mathbb{Z}_2$-graded alternative algebra should have characteristic 3 and is isomorphic to one of the superalgebras: $B(1, 2)$ of dimension 3 and $B(4, 2)$ of dimension 6 (see [171]). Irreducible superbimodules and superbimodules with superinvolution over these superalgebras are classified by M.C. López Diaz and I. Shestakov in [131]. They prove that, besides a certain two-parametric series of bimodules $V(\lambda,\mu)$ over $B(1, 2)$, all other unital irreducible superbimodules for these superalgebras are regular or opposite to them. Also, they prove that every unital $B(4, 2)$-superbimodule is completely reducible. Moreover, they obtain an analogue of the Kronecker factorization theorem for alternative superalgebras B containing $B(4, 2)$ as a unital subsuperalgebra: $B = B(4,2)\tilde{\otimes}U$, for a convenient associative commutative superalgebra U. In the case of superbimodules with superinvolution, they obtained the same superbimodules for $B(4, 2)$; however, the family $V(\lambda,\mu)$ does not occur for $B(1, 2)$. As a consequence, they obtain that every unital supermodule with a superinvolution with symmetric elements in the nucleus over $B(1, 2)$ is completely reducible and admits a Kronecker factorization as above.

Representations of simple Jordan superalgebras of 3×3 hermitian matrices over the exceptional simple alternative superalgebras $B(1, 2)$ and $B(4, 2)$ of characteristic 3 are described in [132]. It is proved that every irreducible bimodule over these superalgebras is either a regular bimodule or opposite to them and every unital supermodule over these superalgebras is completely reducible. They also obtain a Kronecker factorization theorem for Jordan superalgebras that contain $H_3(B(1,2))$ and $H_3(B(4,2))$.

1.1.8 Representations of alternative algebras and superalgebras

The complete classification of irreducible alternative bimodules of arbitrary dimension and characteristic as well as finite dimensional irreducible alternative superbimodules for any characteristic over an algebraically closed field were obtained by

I. Shestakov and M. Trushina. They proved that if M is a nonassociative irreducible bimodule over an alternative algebra A of arbitrary dimension then A is simple, central, has finite dimension over its center and satisfies one of the following cases

- A is a quaternion algebra and M is the Cayley bimodule over A;
- A is an octonion algebra and M is the regular bimodule over A.

In the case of superalgebras, they proved that if a superalgebra $A = A_0 + A_1$ has an irreducible superbimodule M then $A_0 = 0$ or A is a prime superalgebra. In the case $A_0 = 0$, M has dimension 2 or 4 over its centroid and is isomorphic to one of the two bimodules considered in [170]. When $A_0 \neq 0$ and A is prime, they obtained a classification of irreducible A-superbimodules in the following cases

- A is finite dimensional;
- A has characteristic $\neq 3$;
- A has characteristic 3 and is simple.

It remains only the case when A is a non simple prime superalgebra of characteristic 3. These results can be found in [181].

1.1.9 Associative representations of nonassociative algebras

Let L be a Lie algebra. The Poincaré–Birkhoff–Witt Theorem states the existence of an associative algebra $U(L)$ and an injective linear map $\mu : L \to U(L)$ such that $\mu(xy) = [\mu(x), \mu(y)]$, for $x, y \in L$. In order to generalize this situation, the notion of associative representation for algebras was defined by I. Shestakov and A. Kornev in [125]: for an algebra A and an associative algebra with involution $(B, *)$, a linear map $\mu : A \to B$ is called an *associative representation* of A in $(B, *)$ if there exists a bilinear polynomial $f(x, y)$ such that $\mu(ab) = f\big(\mu(a), \mu(b)\big)$, for $a, b \in A$. If $\ker \mu = 0$, the representation is called *faithful*. They prove the existence of faithful associative representations of any alternative, Malcev and Poisson algebras for the polynomials $f(x, y) = xy + \frac{1}{2}[x^*, y] + \frac{1}{2}[x, y^*]$, $g(x, y) = [x, y] + [x^*, y] + [x, y^*]$, and $p(x, y) = xy - \frac{1}{4}yx + \frac{1}{4}yx^* + \frac{1}{4}y^*x - \frac{1}{4}y^*x^*$, respectively. They also exhibit an explicit associative representation for the Cayley-Dickson algebra in the 8×8 matrix algebra over a field of characteristic not 2 or 3.

1.1.10 Basic superranks for varieties of algebras

Let $\mathcal{V}$ be a variety of algebras and $\mathcal{V}_r$ be a subvariety of $\mathcal{V}$ generated by the free $\mathcal{V}$ -algebra of rank r. If the chain

$$\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_r \subseteq \cdots \subseteq \mathcal{V}$$

stabilizes and r is the minimal number with the property that $\mathcal{V}_r = \mathcal{V}$, then r is called the *basic rank* of the variety $\mathcal{V}$. Otherwise, we say that $\mathcal{V}$ has infinite basic rank. It is known that the varieties of associative, Lie and special Jordan algebras have basic rank 2 and the varieties of alternative, Malcev and $(-1, 1)$ algebras have infinite basic rank. Also, proper subvarieties of varieties with finite basic rank can have infinite basic rank; for instance, the associative variety generated by the Grassmann algebra on infinite number of generators has infinite basic rank.

This notion of basic rank for varieties of algebras was generalized to the concept of *basic superrank* by I. Shestakov and A. Kuzmin [127]: for a variety $\mathcal{V}$ of algebras, a pair of positive integers (m, n) is a *superrank* of $\mathcal{V}$ if this variety coincides with the variety generated by the Grassmann envelope of the free $\mathcal{V}$-superalgebra with m even and n odd generators. It is a consequence of Kemer's proof of the *Specht Problem* (see Sect. 6.4.4) that every subvariety of associative algebras over a field of characteristic zero has a finite basic superrank.

It was proved, for varieties of metabelian alternative, Jordan and Malcev algebras, that the superrank is finite. However, the variety of all metabelian algebras has infinite superrank and, for each pair (m, n), there exists a metabelian variety of algebras with superrank (m, n).

1.1.11 Differentially simple algebras

I. Shestakov, V. Zhelyabin and A. Popov present, in [197], the first examples of differentially simple algebras which are not free modules over their centroid. These examples were constructed in the varieties of associative, alternative, Jordan, Lie and Malcev algebras.

1.1.12 Invariants of G_2 and Spin(7) in positive characteristic

Let F be an infinite field of odd characteristic. Polynomial invariants under the diagonal action of the Chevalley group $G_2(F)$ and $\mathrm{Spin}(7, F)$ over n copies of octonion algebra over F were studied by A. Zubkov and I. Shestakov in [200]. The authors proved that the known generators of degree less than or equal to 4 for the algebra R of polynomial invariants, obtained by G.W. Schwarz over the complex number field $\mathbb{C}$, is also a generating set for the algebra R over the field F.

1.2 Coordinatization theorems

Coordinatization theorems are very useful in structure theory. They state that certain algebras are "rigid" and define a structure of an algebra which contains them: the bigger algebra has a similar structure though not over the basic field but over certain algebra which "coordinatize" it. The first coordinatization theorem was obtained by J.M. Wedderburn, in 1907: "If A is an unital associative algebra which contains a subalgebra isomorphic to $M_n(F)$ (the algebra of $n \times n$ matrices over the field F) with the same unity then $A = M_n(F) \otimes_F B$, for a convenient subalgebra B." In this case, B coordinates the algebra A. There are known, for instance, coordinatization theorems for unital alternative algebras containing matrix algebras of order greater than or equal to 3 (M. Zorn, 1930), for unital alternative algebras containing octonions (I. Kaplansky, 1951 and N. Jacobson, 1954), for unital Jordan algebras containing the exceptional

Albert algebra (N. Jacobson, 1954), and unital Jordan algebras containing $n \times n$ hermitian matrices over the ground field, $n \geq 3$ (N. Jacobson, 1954). Very often, the whole algebra and the subalgebra that coordinates it have equivalent categories of bimodules.

1.2.1 Jordan superalgebras

C. Martínez, I. Shestakov and E. Zelmanov [138] extend Jacobson's Coordinatization Theorem for unital Jordan superalgebras $\mathcal{J}$ containing at least three strongly connected ortogonal idempotents. (The orthogonal idempotents $e_i, e_j \in \mathcal{J}$ are *strongly connected* if there exists an element a_{ij} in the Peirce component $\mathcal{J}_{ij}$ such that $a_{ij}^2 = e_i + e_j$.) They show that these algebras are coordinated by an alternative superalgebra B with nuclear superinvolution. As in the case of algebras, the problem of classifying irreducible Jordan superbimodules for $\mathcal{J}$ can be reduced to the description of irreducible alternative superbimodules with nuclear superinvolution for B. Using this reduction, they classify Jordan bimodules over simple Jordan superalgebras of types $Q(n)^+$ and $JP(n)$, for $n \geq 3$, where $Q(n) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in M_n(F) \right\}$ are associative superalgebras and $JP(n)$ stands for the Jordan superalgebra of symmetric elements in the associative superalgebra $M_{n+n}(F)$ considered with the superinvolution σ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sigma} = \begin{pmatrix} d^t & -b^t \\ c^t & a^t \end{pmatrix},$$

where a^t is the usual transpose matrix. The case $Q(2)$ is also treated in the paper. The authors use the Tits-Kantor-Koecher construction and representation theory of Lie superalgebras for this remaining case.

1.2.2 Noncommutative Jordan superalgebras

A Coordinatization Theorem for noncommutative Jordan algebras was obtained by K. McCrimmon in 1971. In [160], A. Pozhidaev and I. Shestakov prove an analogous theorem for noncommutative Jordan superalgebras with $n \geq 3$ connected idempotents. Given λ an element of the ground field (of characteristic not two), and a superalgebra $(A, \cdot)$, the λ*-mutation* of A is the superalgebra $A^{(\lambda)} = (A, \cdot_\lambda)$, where

$$x \cdot_\lambda y = \lambda x \cdot y + (-1)^{\bar{x}\bar{y}} y \cdot x.$$

They prove that if U is a strictly noncommutative Jordan superalgebra (that is, there are homogeneous elements $x, y \in U$ such that $xy \neq (-1)^{\bar{x}\bar{y}} yx$), then $U = \mathcal{D}_n^{(\lambda)}$, where $\mathcal{D}_n$ is a superalgebra of $n \times n$ matrices with entries are in an associative superalgebra $\mathcal{D}$.

1.2.3 Alternative algebras

V. López Sólis, under the supervision of I. Shestakov, proved coordinatization theorems for unital alternative algebras and superalgebras [133]. More specifically, it is proved, for any characteristic, that alternative algebras with unity which contains the matrix algebra $M_n(F)$ with the same unity are associative if $n \geq 3$. As a consequence, they obtain that unital alternative bimodules over $M_n(F)$ are associative, if $n \geq 3$. This result was already proved by N. Jacobson if the characteristic of the ground field is different from 2.

The analogous are obtained for superalgebras: if $m + k \geq 3$, alternative superalgebras with unity that contain the associative superalgebra $M_{(m|k)}(F)$ with the same unity are associative and isomorphic to $M_{(m|k)}(F) \otimes_F Z$ where Z is an associative superalgebra. Also, every unital alternative superbimodule over $M_{(m|k)}(F)$ is associative.

Alternative algebras and superalgebras containing the Cayley-Dickson algebra $\mathbb{O}$ with the same unity are also described. In the case of algebras, they complete the previous results obtained by N. Jacobson, for characteristic not 2, and by Kaplansky, for the split Cayley algebra. The complete reducibility of unital alternative bimodules over $\mathbb{O}$ is obtained. For superalgebras, they prove that if the even part of an alternative superalgebra with unity contains $\mathbb{O}$ with the same unity then the superalgebra is isomorphic to $\Omega \otimes_F \mathbb{O}$, where Ω is a supercommutative associative superalgebra.

Finally, V. López Sólis and I. Shestakov [134] have proved a coordinatization theorem for unital alternative algebras containing 2×2 matrix algebra with the same identity element 1. This solves an old problem by Nathan Jacobson on the description of alternative algebras containing a generalized quaternion algebra H with the same 1, for the case when the algebra H is split. In particular, this is the case when the basic field is finite or algebraically closed.

1.2.4 Right alternative algebras

S. Pchelintsev, O. Shashkov and I. Shestakov [145] proved that the coordinatization theorem is verified for unital right alternative algebras which contain the octonion algebra. As a consequence, if a unital right alternative algebra contains the octonion algebra with the same unity then it is alternative.

1.3 Representation type of Jordan algebras

The (bi)representation type of an algebra A in a variety $\mathcal{M}$ is defined similarly to associative case [53]:

- A is of *finite representation type* if A has a finite number of non-isomorphic indecomposible $\mathcal{M}$-bimodules;
- A is of *tame representation type* if for any natural n, A has a finite number of 1-parametric series of non-isomorphic indecomposible $\mathcal{M}$-bimodules of dimension n;
- A is of *wild representation type* if A is not of tame type.

It is well known that, similarly to the associative case, semisimple finite dimensional Jordan algebras are of finite type. The study of Jordan bimodules over non-semisimple finite dimensional Jordan algebras was initiated by I. Kashuba and I. Shestakov in [119], where they considered Jordan algebras $\mathcal{J}$ whose semisimple component is a sum of fields (these algebras are called *basic*). The authors describe all 3-dimensional Jordan algebras and obtained a criterion for determining the representation type for a nilpotent Jordan algebra. In this way, they have found a general criterion for deciding whether a Jordan algebra with a sum of fields as semisimple part has finite or tame type.

The systematic study of Jordan bimodules over non-semisimple finite dimensional Jordan algebras over an algebraically closed field of characteristic not 2 and 3 was initiated by I. Kashuba, S. Ovsienko and I. Shestakov in [116]. The notions of diagram of a Jordan algebra and of Jordan tensor algebra of a bimodule are introduced, as well as the construction of a mapping (Qui) which associates the diagram of a Jordan algebra to the quiver of its universal associative enveloping algebra. The authors consider algebras whose semisimple component is a direct sum of Jordan matrix algebras, obtaining a criterion of finiteness and tameness for one-sided representations, in terms of diagram and mapping Qui, for Jordan tensor algebras and for algebras having null radical square.

In [117], the same authors study basic Jordan algebras over an algebraically closed field of characteristic zero. They describe all basic Jordan algebras with null radical square which are of finite or tame type. They considered the quiver of the multiplicative envelope $\mathcal{U}(\mathcal{J})$ which is a basic algebra. The classification is obtained in terms of the dimensions of the Peirce components of $\mathrm{Rad}(\mathcal{J})$, the radical of the algebra $\mathcal{J}$, relative to a complete set of primitive orthogonal idempotents.

The relationship between representations of a Jordan algebra and the Lie algebra obtained from it by the Tits-Kantor-Koecher construction was explored by I. Kashuba and V. Serganova in [118]. The authors use the equivalence between the category of unital representations of a Jordan algebra $\mathcal{J}$ and the category of representations of the Lie algebra $\mathrm{TKK}(\mathcal{J})$ which have a $\mathbb{Z}_3$-grading to classify Jordan algebras $\mathcal{J}$ whose semisimple part is a sum of Jordan algebras of quadratic forms and the category of unital bimodules for $\mathcal{J}$ is tame.

1.4 Lie algebras and superalgebras

1.4.1 Simple Lie algebras over a field of characteristic 2

The classification of simple Lie algebras is known for algebras over a field of characteristic not 2 or 3. Some contributions for the classification of simple Lie algebras in characteristic 2 was given by A. Grishkov and M. Guerreiro. For instance, in [75] they classify 7-dimensional simple Lie algebras over an algebraically closed field and in [76] they give a characterization of finite dimensional simple Lie algebras of type $B_{2l}, C_{2l}, D_{2l+1}, E_7$ and E_8 in terms of some gradations of these algebras.

1.4.2 Modules for Schur superalgebras

Schur algebras are examples of quasi-hereditary algebras, which play an important role in representation theory of Lie algebras. They also appear in the theory of algebraic groups and of quantum groups. Given positive integers n, r, the Schur algebra $S(n, r)$ can be defined in the following way: consider $F[x_{ij}]$ the commutative algebra of polynomials in n^2 variables x_{ij}, $i, j = 1, \dots, n$, over the field F with the canonical coalgebra structure ($\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}$, $\varepsilon(x_{ij}) = \delta_{ij}$). Let $A_F(n, r)$ the subspace of homogeneous polynomials of degree r. Then $A_F(n, r)$ is a subcoalgebra of $F[x_{ij}]$ and $S(n, r)$ is defined to be the dual algebra of $A_F(n, r)$, that is, $S(n, r) = \mathrm{Hom}_F(A_F(n, r), F)$, with multiplication satisfying $(fg)(x_{ij}) = \sum_l f(x_{il})g(x_{lj})$. The notions of standard and costandard modules are central in the theory of Schur algebras. These concepts are related to a ordering of the set of all irreducible modules of the given algebra. Costandard modules, for instance, are obtained considering certain maximal submodules of the injective hull of simple modules. Some generalizations of these algebras were considered, for instance, Schur superalgebras. The Schur superalgebra $S(m|n, r)$ is obtained taking $(m + n)^2$ variables x_{ij}, with parity of x_{ij} defined as $i + j (\mathrm{mod}\, 2)$. Schur superalgebras are not always quasi-hereditary. This property holds if and only if the superalgebra is semisimple. However, it is still relevant to describe their (co)standard modules.

The description of costandard modules for the Schur superalgebra $S(2|1, r)$ over a field of characteristic p is obtained by A. Grishkov, F. Marko and A. Zubkov in [79]. In particular, the classification of irreducible $S(2|1, r)$-modules in positive characteristic is obtained.

Irreducible modules for the Schur superalgebra $S(2|2, r)$ are classified by A. Grishkov and F. Marko in [77]. The authors consider two cases: characteristic zero and $p > 2$. As a continuation of this study, the same authors describe, in [78], the costandard modules $\nabla(\lambda)$ of restricted highest weight λ for the Schur superalgebra $S(2|2, r)$ over an algebraically closed field of odd characteristic.

1.4.3 Filtered multiplicative basis

Let A be an associative algebra over a field F. A F-basis B of A is called a *filtered multiplicative basis* if $B \cup \{0\}$ is closed under multiplication and $B \cap \mathrm{rad}\,(A)$ is a basis of $\mathrm{rad}\,(A)$, the Jacobson radical of A. The existence of filtered multiplicative basis in a given class of algebras implies that the number of isomorphism classes of algebras in this class is finite and the problem of classification of these algebras becames combinatorial. For instance, finite dimensional algebras with finite representation type have filtered multiplicative basis.

In [21], V. Bovdi, A. Grishkov and S. Siciliano study the problem of the existence of filtered multiplicative bases of a restricted enveloping algebra $u(L)$, for L a finite dimensional and p-nilpotent restricted Lie algebra over a field of positive characteristic p. They prove that the p-enveloping algebra of a nil abelian Lie algebra affords a filtered multiplicative basis if and only if L is a direct sum of nil-cyclic restricted Lie algebras.

In [22] the same authors deal with the existence problem of a filtered multiplicative basis for finite dimensional associative algebras and when this property is preserved by homomorphic images. This is used to determine when some restricted enveloping algebras in characteristic 2 have filtered multiplicative basis.

1.4.4 Normal enveloping algebras

An algebra A with involution $*$ over a field F is called *normal* if $xx^* = x^*x$, for all $x \in A$. In [80], A. Grishkov, M. Rasskazova and S. Siciliano investigate when ordinary and restricted enveloping algebras of Lie algebras are normal, when considered with their canonical involution, that is, the involution which satisfies $x^* = -x$, for every x in the Lie algebra.

They give necessary and sufficient conditions in order to the restricted enveloping algebra of a Lie algebra over a field of characterist $p > 0$ be normal and use this result to obtain the classification of Lie algebras over arbitrary fields for which their ordinary enveloping algebra are normal. In both cases, or the Lie algebra L is abelian or $p = 2$ and L is nilpotent of class 2.

1.4.5 Automorphism groups

In [81] A. Grishkov and M. Rasskazova describe the group of automorphisms of $\mathbb{Z}_p$ -diagonal forms of the Lie algebra $sl_2(\mathbb{Q}_p)$.

1.5 More general classes of algebras

1.5.1 Structurable superalgebras

Introduced by I. Kantor [109] and B. Allison [2], structurable algebras are unital algebras with involution defined by an identity of degree 4. They include associative, alternative and Jordan algebras and admit a generalization of the famous Tits-Kantor-Koecher construction, which associates a Lie algebra with $(-2, 2)$-graduation to each structurable algebra. Using this construction, one can obtain all finite dimensional simple Lie algebras over a field of characteristic zero in a unified way.

A. Pozhidaev and I. Shestakov contributed to the classification of structurable superalgebras. They obtained the description of simple structurable superalgebras of Cartan type in [162, 163]. With this classification and the results obtained by J. Faulkner [55] a complete classification of finite dimensional simple structurable superalgebra over an algebraically closed field of characteristic zero is obtained.

1.5.2 Noncommutative Jordan superalgebras

The class of noncommutative Jordan algebras form a wide generalization of Jordan algebras, which includes alternative, quasi-associative, quadratic flexible, and anticommutative algebras. They are defined by the identities

$$(x, y, x) = 0 \quad \text{and} \quad (x^2, y, x) = 0.$$

The classification of central simple finite dimensional noncommutative Jordan superalgebras in any characteristic was obtained by I. Shestakov and A. Pozhidaev in [160, 161, 164, 165], including a Coordinatization Theorem (see Sect. 1.2.2).

1.5.3 Right alternative algebras and superalgebras

An algebra is called *right alternative* if it satisfies the identity

$$(x, y, y) = 0.$$

Therefore, this class of algebras contains alternative algebras. Although finite dimensional simple right alternative algebras are alternative, these classes of algebras seem to be not so close. One of the inkling for this belief is the behaviour of bimodules over matrix algebras in both classes of algebras.

Right alternative bimodules over $M_2(F)$, where F is a field of characteristic not 2, were considered in [144] by L. Murakami and I. Shestakov. The authors have found an infinite family of non isomorphic 4-dimensional irreducible bimodules, as well as a family of $4n$-dimensional irreducible bimodules. On the other side, there are only two non isomorphic irreducible alternative bimodules for $M_2(F)$.

In [159], J. Picanço, L. Murakami and I. Shestakov use the 4-dimensional family of irreducible right alternative bimodules to prove the existence of an infinite family of 8-dimensional simple right alternative superalgebras which have $M_2(F)$ as even part.

In [143], L. Murakami, S. Pchelintsev and O. Shashkov investigate the structure of finite dimensional unital right alternative superalgebras with semisimple alternative even part over an algebraically closed field. The notion of weak annihilator is introduced and the classification of superalgebras with zero weak annihilator in this class of superalgebras is obtained. Also, it is proved that every superalgebra in this class splits into the direct sum of its weak annihilator and a superalgebra with zero weak annihilator.

1.5.4 Lie-Jordan algebras

An algebra L with a bilinear operation $[a, b]$ and a trilinear operation $\{a, b, c\}$ is called a *Lie-Jordan algebra* if it is a Lie algebra with respect to $[a, b]$, it is a triple Jordan system with respect to operation $\{a, b, c\}$, and these operations are related by the identities

$$\begin{aligned} \{a, b, c\} - \{b, a, c\} &= [[a, b], c], \\ [a, \{b, c, d\}] &= \{[a, b], c, d\} + \{b, [a, c], d\} + \{b, c, [a, d]\}. \end{aligned}$$

Any associative algebra A becomes a Lie-Jordan algebra $A^{\pm}$ under the operations $[a, b] = ab - ba$, $\{a, b, c\} = abc + bca$. A Lie-Jordan algebra L is called special if it can be embedded into a Lie-Jordan algebra $A^{\pm}$ for some associative algebra A. A. Grishkov and I. Shestakov in [82] prove that any Lie-Jordan algebra is special.

In [83] they prove that a Lie algebra L admits a triple product $\{a, b, c\}$ under which it becomes a Lie-Jordan algebra if and only if it is a Lie algebra of skew-symmetric elements in an associative algebra with involution.

2 Nonassociative Lie Theory: Malcev enveloping algebras, Sabinin algebras and non associative Hopf algebras

Malcev algebras are a generalization of Lie algebras. They were introduced in 1955 by A.I. Malcev as tangent algebras of analytic Moufang loops (see the definition of Moufang loop in Sect. 5.1). A Malcev algebra is an anticommutative algebra which satisfies the identity

$$J(x, y, xz) = J(x, y, z)x,$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$. If A is an alternative algebra then the algebra A^-, which is obtained from A considering the new product given by the comutator $[x, y] = xy - yx$, is a Malcev algebra.

The universal enveloping algebras of Malcev algebras are, in general, nonassociative. However, they have a structure close to Hopf algebras. They admit a cocommutative and coassociative coalgebra structure and the Malcev algebra is recovered as the space of primitive elements. The multiplication and comultiplication are related by a Hopf version of the Moufang identities.

The connection between Malcev algebras and local analytic Moufang loops is an extension of the relationship between Lie algebras and local Lie groups. This connection can be more extended to Bol algebras and local analytic Bol loops (see Sect. 4.3.14 for the definition of Bol algebras and Sect. 5.1 for the definition of Bol loops). In order to have a Lie correspondence between these structures, it was needed to consider two operations on the tangent space of local analytic loops. The corresponding algebra is called *Akivis algebra* (see Sect. 2.3.1). Moreover, in order to have a Lie correspondence for arbitrary analytic loop, one needs an infinite sequence of multilinear operations on the tangent space. The algebraic structure obtained is now called *Sabinin algebra*.

2.1 Enveloping algebras for Malcev algebras

In the development of the theory of universal envelopes for Lie algebras, the study of their centers and the structure of the enveloping algebra as a module over its center play an importante role. For instance, it is known that the center of the universal envelope $U(L)$, for a finite dimensional simple Lie algebra L over a field of characteristic zero, is a polynomial ring with n variables, where n is the dimension of Cartan subalgebra of L. Moreover, the classical Kostant Theorem states that $U(L)$ is a free module over its center $Z\big(U(L)\big)$.

J.M. Pérez Izquierdo and I. Shestakov prove in [153] that, for every Malcev algebra M, there is an algebra $U(M)$ and a monomorphism from M to the commutator algebra $U(M)^-$ such that the image of M lies into the alternative center of $U(M)$, and

$U(M)$ is a universal object with respect to such homomorphisms. The algebra $U(M)$, in general, is not alternative, but it has a nonassociative Hopf algebra structure and has some similar properties to the universal enveloping algebras of Lie algebras.

The similarity between properties of enveloping algebras of Lie algebras and enveloping algebras of Malcev algebras indicated above was reinforced in [198]. V. Zhelyabin and I. Shestakov prove that if M is finite dimensional semisimple Malcev algebra over a field of characteristic zero and $U = U(M)$ is its universal envelope then the center $Z = Z(U)$ is a polynomial ring in n variables, where n is the dimension of the Cartan subalgebra of M and U is a free module over its center Z.

There is only one 4-dimensional non Lie Malcev algebra. Its enveloping algebra U is not alternative and has infinite dimension. In [23], M. Bremner, I. Hentzel, L. Peresi and H. Usefi determine some structural constants of U and the quotient algebra of U by its alternator ideal, obtaining its universal alternative envelope.

The center of the universal nonassociative enveloping algebra of the 7-dimensional simple Malcev algebra over a field of characteristic $p > 3$ was studied by J.M. Pérez Izquierdo and I. Shestakov in [154]: the center is an extension of the polynomial ring in 7 variables by an element which is analogous to the Casimir element.

In [155], the same authors prove that, over fields of characteristic zero, any bialgebra deformation of the universal enveloping algebra of the algebra of traceless octonions satisfying the dual of the left and right Moufang identities must be coassociative and cocommutative. This result shows that Malcev algebras of traceless octonions are rigid in the category of Hopf-Moufang algebras.

2.2 Free Malcev algebras and superalgebras

One of the most important questions for any class of algebras is the structure of free algebras.

It is known that the free Malcev algebra of rank more than 4 is not semiprime, that is, it contains nilpotent ideals. In [179], A. Kornev and I. Shestakov prove that the prime radical of a free Malcev algebra $\mathcal{M}$ of rank more than 2 over a field of characteristic not 2 coincides with the set of all engelian elements of $\mathcal{M}$. This result is an analogue of the similar characterizations obtained by I. Shestakov and E. Zelmanov for radicals of free alternative and free Jordan algebras. It is still an open question whether the free Malcev algebras of rank 3 and 4 are semiprime.

Contrary to Lie algebras, effective bases of free Malcev algebra are unknown. I. Shestakov in [173] constructs a basis of the free Malcev superalgebra M on one odd generator, which produces a basis of the subspace of multilinear skew-symmetric elements in the free Malcev algebra of countable rank. As a corollary, an infinite series of central skew-symmetric elements in the free alternative algebra of countable rank is constructed. The first element in this series was found earlier by I. Hentzel and L.Peresi [101] with computer algebra.

The free Malcev superalgebra $\mathcal{M}$ on one odd generator and some related superalgebras were studied later by I. Shestakov and N. Zhukavets in [190, 191]. It was proved that $\mathcal{M}$ is special, that is, it can be embedded in a superalgebra A^{-}, for some alternative superalgebra A. In these papers, in particular, a basis for the Malcev Poisson superalgebra $\tilde{S}(\mathcal{M})$ is constructed, as well as a pre-basis for the free alternative superalgebra $\mathcal{A}$ on one odd generator, that is, a set which spans $\mathcal{A}$ as a vector space. In [192, 199], the authors prove that the elements of that pre-basis are linearly independent. As a consequence, some results on the structure of the radical and the center of the free alternative algebra of infinite rank are obtained. A basis for the Grassmann alternative algebra is also constructed.

2.3 Akivis and Sabinin algebras

2.3.1 Akivis algebras

Akivis algebras are generalization of Malcev algebras. An *Akivis algebra* is a vector space V endowed with a skew-symmetric bilinear product $[x, y]$ and a trilinear product $\mathcal{A}(x, y, z)$ that satisfy the identity

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = \mathcal{A}(x,y,z) + \mathcal{A}(y,z,x) + \mathcal{A}(z,x,y) - \mathcal{A}(y,x,z) - \mathcal{A}(x,z,y) - \mathcal{A}(z,y,x).$$

These algebras were introduced in 1976 by M. A. Akivis [1] as local algebras of three-webs. For any (non-associative) algebra B one may obtain an Akivis algebra $\mathrm{Ak}(B)$ by considering in B the usual commutator $[x, y] = xy - yx$ and associator $\mathcal{A}(x, y, z) = (xy)z - x(yz)$. M. A. Akivis posed the problem whether every Akivis algebra is isomorphic to a subalgebra of $\mathrm{Ak}(B)$ for a certain algebra B. This problem was positively solved by I. Shestakov in [172].

2.3.2 Primitive elements in free nonassociative algebra and Sabinin algebras

Let $F\{X\}$ be a free nonassociative algebra on a set of generators X. It follows from [172] that the subalgebra $\mathrm{Ak}\{X\}$ of the Akivis algebra $\mathrm{Ak}(F\{X\})$ generated by X is a free Akivis algebra on a set of generators X. It was conjected by K. Hofmann and K. Strambach in [106] that the algebra $\mathrm{Ak}\{X\}$ coincides with the subspace of primitive elements $\mathrm{Prim}\{X\} = \{f \in F\{X\} \mid \Delta(f) = f \otimes 1 + 1 \otimes f\}$, where $\Delta : F\{X\} \to F\{X\} \otimes F\{X\}$ is a comultiplication homomorfism defined by the condition $\Delta(x) = x \otimes 1 + 1 \otimes x,\ x \in X$. The conjecture has been proved not to be true: I. Shestakov and U. Umirbaev in [183] prove that the space $\mathrm{Prim}\{X\}$ is generated by an infinite number of multilinear operations and forms with respect to these operations, a so called *Sabinin algebra* which was introduced earlier by P. Mikheev and L. Sabinin [168] in their study of local analytic loops, as a tangent algebra, generalizing Lie, Malcev, and Akivis algebras.

Sabinin algebras play a role of Lie algebras in the Nonassociative Lie Theory, which establishes an equivalence of categories of Sabinin algebras, nonassociative Hopf algebras, and formal loops. A survey on this theory is given in [140].

2.3.3 Nilpotent Sabinin algebras

Properties of nilpotent Sabinin algebras are presented by J. Mostovoy, J.M. Pérez Izquierdo and I. Shestakov in [141]. It is shown that these algebras can be integrated to produce nilpotent loops, satisfy an analogue of the Ado Theorem (which states the existence of finite dimensional associative enveloping algebra for finite dimensional Lie algebras) and have nilpotent Lie envelopes.

2.3.4 Nonassociative Baker–Campbell–Hausdorff formula

An analogue to the famous Baker–Campbell–Hausdorff formula for associative algebra of formal power series in two non commutative variables over a field of characteristic zero was proved to arbitrary nonassociative algebras by J. Mostovoy, J.M. Pérez Izquierdo and I. Shestakov in [142]. The authors use the fact that the unital algebra of formal power series in a set of free non associative and non commutative variables has a structure of a nonassociative Hopf algebra.

2.3.5 A base of free Sabinin algebra

In [32] E. Chibrikov, under the supervision of I. Shestakov, constructs an efective base for a free Sabinin algebra. As a consequence, it is obtained that every subalgebra of a free Sabinin algebra is free. In other words, the variety of Sabinin algebras is a Schreier variety.

2.3.6 Loops of diffeomorphisms and their Hopf algebras

A. Frabetti and I. Shestakov [59] investigate the functor of diffeomorphisms which associate to a nonassociative and noncommutative algebra a loop (an analogue of Nottingham groups). It is proved that this functor is representable in a nonassociative Hopf algebra which generalize the Faà di Bruno Hopf algebra.

2.3.7 PBW-pairs of linear algebras

A.A. Mikhalev and I. Shestakov in [139] introduce the notion of a Poincaré–Birkhoff–Witt (PBW)-pair of varieties of linear algebras. Examples of PBW-pairs are given. It is proved that if (V, W) is a PBW-pair and the variety W is homogeneous and Schreier, then so is V; the results similar to the Schreier property for PBW-pairs are also true for the Freiheitssatz and the Word Problem. In particular, it follows that the Freiheitssatz is true for the varieties of Akivis and Sabinin algebras. Moreover, examples of varieties that do not satisfy the Freiheitssatz are also given. It is shown that an element u of a free algebra $W[X]$ in a homogeneous Schreier variety of algebras W satisfying the Freiheitssatz is a primitive element (a coordinate polynomial) if and only if the factor algebra of $W[X]$ by the ideal generated by the element u is a free algebra in W.

3 Geometric aspects

3.1 Geometric structure of Jordan and alternative algebras

For 3-dimensional Jordan algebras, the description and the number of irreducible components is given by I. Kashuba and I. Shestakov in [111, 119]. Invariants of corresponding components, such as radicals and automorphism groups is also described.

The estimate of the dimension of irreducible components in the variety of alternative and Jordan algebras was investigated by the same authors in [120]. The authors prove that if Ω is an arbitrary family of non isomorphic n-dimensional alternative algebras over an algebraically closed field which depends continuously on parameters $p_1, \dots, p_N$ then the asymptotic dimension of Ω is $\frac{4}{27} n^3 + O(n^{8/3})$. They also formulate a conjecture for the asymptotic of parameters which determine a family of irreducible n-dimensional Jordan algebras: $N = \frac{1}{6\sqrt{3}} n^3 + O(n^{8/3})$. This conjecture is confirmed for a certain closed subvariety of Jordan algebras.

The classification of 4-dimensional Jordan algebras is given by I. Kashuba and M.E. Martin in [112]. The authors give a list of 73 non isomorphic algebras, describe all deformation between these algebras and determine the rigid ones. The same was done for 3-dimensional real Jordan algebras, by the same authors in [113] and, in [114], they obtain the geometric classification of 5-dimensional nilpotent Jordan algebras. Rigid algebras and families of semi-rigid algebras are described.

3.2 Algebraic geometry over Lie algebras

Recently, a newsworthy area has been intensively developed in algebra: the algebraic geometry over algebraic systems. For instance, a successful achievement in this area is the solution of the famous Tarski Problem obtained by O. Kharlampovich and A. Myasnikov, by using algebraic geometry over groups.

N. Romanovsky and I. Shestakov contributed to the theory of algebraic geometry over Lie algebras. Algebraic geometry over free metabelian Lie algebras and free Lie algebras are studied in [167]. An interesting problem in this field is to determine whether a finitely generated free solvable Lie algebra is *equationally Noetherian*, that is, whether for each n, every system of equations in n variables over A is equivalent to a finite subsystem of the given system. The authors prove that the finitely generated free solvable Lie algebra of index 2 is equationally Noetherian with respect to the equations of its universal enveloping algebra.

3.3 Automorphic equivalence of the representations of Lie algebras

The basic notions of the algebraic geometry of representations of Lie algebras are defined in a similar way to the basic notions of the algebraic geometry of representations of groups. I. Shestakov and A. Tsurkov in [182] prove that if a field F has no nontrivial automorphisms then the automorphic equivalence of representations of Lie algebras coincides with the geometric equivalence.

4 Combinatorial aspects

4.1 Free algebras, their automorphisms and derivations

4.1.1 The Nagata Problem

Let A be a free algebra in the variables $x_1, \dots, x_n$ on a class of algebras over a field F of characteristic zero. An automorphism f of A is called *elementary* if it fixes all but one of the variables x_i and, for this x_i, $f(x_i) = \alpha x_i + g$, with $\alpha \in F$ is nonzero and g is a polynomial which does not contain the variable x_i. An automorphism of A is *tame* if it is a composition of elementary automorphisms; in other case, it is called *wild*. The classical Jung Theorem states that, in the case $A = F[x, y]$, the polynomial algebra, all automorphisms of A are tame. I. Shestakov and U. Umirbaev [184–186] presented an algorithm for deciding if an automorphism of $F[x, y, z]$ is tame or wide, using methods of Poisson brackets and estimates for the degree of elements of subalgebras of $F[x_1, \dots, x_n]$ generated by two elements. In particular, they proved that the famous Nagata automorphism is wild. Therefore, tame automorphisms of $\mathrm{Aut}(F[x, y, z])$ form a proper subgroup of $\mathrm{Aut}(F[x, y, z])$.

In the proof of the Nagata Conjecture, the free Poisson algebras were effectively used. The free Poisson algebras are closely related to algebras of polynomials, free Lie algebras, and free associative algebras.

A *Poisson algebra* is a vector space B over a field F with two bilinear operations $x \cdot y$ (multiplication) and $[x, y]$ (a *Poisson bracket*) which satisfies the following conditions

- B is a commutative associative algebra with $x \cdot y$;
- B is a Lie algebra with $[x, y]$;
- $[x \cdot y, z] = [x, y] \cdot y + x \cdot [y, z]$ (*Leibniz identity*).

If L is a Lie algebra, $\{l_1, \dots, l_k, \dots\}$ is a F-basis and $P(L)$ denotes the algebra of polynomials on the variables $l_1, \dots, l_k, \dots$, then the operation $[x, y]$ of the Lie algebra L can be uniquely extended to a Poisson bracket on $P(L)$ which makes $P(L)$ into a Poisson algebra. If L is a free Lie algebra then $P(L)$ is a free Poisson algebra on the same set of generators.

4.1.2 Algebraic dependence in free Poisson algebras

Let P be a free Poisson algebra over a field of characteristic zero and let Q be its field of quotients. The structure of Poisson algebra of P can be naturally extended from P to Q. In [136], L. Makar-Limanov and I. Shestakov prove that a pair of elements $f, g \in Q$ are Poisson dependent (which means that the subalgebra generated by f and g is not a free Poisson algebra) if, and only if, f and g are algebraically dependent. This result is used to give a new proof of the tameness of automorphisms for free Poisson algebras of rank two.

4.1.3 Free generic Poisson fields and algebras

The concept of *generic Poisson algebra* was introduced by I. Shestakov in 2000. The difference between classic and generic Poisson algebras is that in the last case, the Poisson bracket does not need to satisfy the Jacobi identity. Free generic Poisson algebras (GP-algebras) over a field of characteristic zero are considered by I. Kaygorodov, I. Shestakov and U. Umirbaev in [121]. It is proved that some properties of free Poisson algebras are true for free GP-algebras. For instance, the left dependence of a finite system of elements is algorithmically recognizable in the universal multiplicative enveloping algebra of the free GP-algebra $U(GP\langle X\rangle)$; the universal multiplicative enveloping algebra of a free GP-field is a free ideal ring; also, the Poisson and polynomial dependence of two elements are equivalent in the free GP-field $GP(x_1, \ldots, x_n)$. As a consequence, all automorphisms of the free GP-algebra $GP(x, y)$ are tame.

4.1.4 The Freiheitssatz for generic Poisson algebras

The Freiheitssatz for generic Poisson algebras over a field F of characteristic zero was proved by P. Kolesnikov, L. Makar-Limanov and I. Shestakov in [124]: let $P = P[x_1, \ldots, x_n]$ be the free generic Poisson algebra and let $f \in P$ be an element which contains x_n. Then, the intersection of the ideal $\langle f\rangle$ generated by f and the subalgebra $P[x_1, \ldots, x_{n-1}]$ is trivial, and the image of the subalgebra $P[x_1, \ldots, x_{n-1}]$ in the quotient algebra $P/\langle f\rangle$ is a free algebra. As a consequence, it was proved that automorphisms of the free algebra $P[x, y]$ are tame. Moreover, there exists an isomorphism between $\mathrm{Aut}\,(P[x, y])$ and $\mathrm{Aut}\,(F[x, y])$, where $F[x, y]$ is the polynomial algebra over the field F.

4.1.5 Locally nilpotent derivations

A derivation D of an algebra A is called *locally nilpotent* if for any $a \in A$ there exists $n = n(a)$ such that $D^n(a) = 0$. If D is a locally nilpotent derivation of an algebra A over a field of characteristic 0 then $\exp(D) = 1 + D + D^2/2! + \cdots$ is an automorphism of A. An example of a locally nilpotent derivation of a free algebra of some variety with generators $x_1, \ldots, x_n$ is given by $D(x_1) = 0$, $D(x_2) = f_1(x_1), \ldots, D(x_i) = f_i(x_1, \ldots, x_{i-1}), \ldots$. A derivation D is called *triangulable* if has the above form under certain choice of generators $x_1, \ldots, x_n$. The classical Rentchler theorem says that in a ring of polynomials

$F[x, y]$ over a field F of characteristic 0 every locally nilpotent derivation is triangulable. I. Shestakov and his PhD student S.D. Crode generalize this theorem to the free associative algebra $F\langle x, y\rangle$ in [52].

4.1.6 Constants of partial derivations and primitive operations

The description of the algebras of constants of the set of all partial derivations in free algebras of unitarily closed varieties over a field of characteristic zero is given by S. Pchelintsev and I. Shestakov in [147]. It is proved that a subalgebra of proper polynomials coincides with the subalgebra generated by values of commutators and Umirbaev-Shestakov primitive elements on a set of generators for a free algebra. Varieties of all algebras, all commutative algebras, right alternative algebras and Jordan algebras were considered.

4.1.7 Rings of constants of linear derivations on Fermat rings

The characterization of all linear $\mathbb{C}$-derivations of the Fermat ring is presented in [195]. M. Veloso and I. Shestakov prove that the Fermat ring has linear $\mathbb{C}$-derivations with trivial ring of constants and some examples are constructed.

4.1.8 Free Jordan algebras

Let $J(D)$ be the free D-generated Jordan algebra (without unit) over a field F of characteristic zero. Then $J(D) = \oplus_{n\geq 1} J_n(D)$, where $J_n(D)$ is the homogenous component of degree n of $J(D)$. I. Kashuba and O. Mathieu [115] make the following conjecture about the dimension a_n of each homogenous component $J_n(D)$ of $J(D)$:

The sequence a_n is the unique solution of the following equation:

$$Res_{t=0}\,\psi \prod_{n}^{\infty}(1 - z^n(t + t^{-1}) + z^{2n})^{a_n} dt = 0,$$

where $\psi = Dzt^{-1} + (1 - Dz) - t$.

It is easy to see the equation provides a recurrence relation to uniquely determine the integers a_n, but a closed formula is not known.

Let $sl_2J(D)$ be the Tits-Allison-Gao construction of $J(D)$ [3]. The authors state two natural conjectures for the homology of Lie algebra $sl_2J(D)$, and each of them implies the previous conjecture.

The cyclicity of the Jordan structures, namely that the symmetric group σ_{D+1} acts on the multilinear part of $J(D)$, plays an essential role to connect the Lie algebra homology of $sl_2J(D)$ and the dimension of $J_n(D)$.

4.2 Lie algebras and related algebras

4.2.1 Self-iterated associative, Lie and Jordan algebras

First examples of infinite dimensional nil affine algebras were constructed in 1964 by E.S. Golod and I.R. Shafarevich [62]. These algebras have strictly exponential growth. In 2000, L. Bartholdi and R.I. Grigorchuk [14] proved that the Lie algebra associated to the Grigorchuk self iterated group is graded, nil and has Gelfand-Kirillov dimension equal to 1. Using this result, L. Bartholdi [13] constructed, in 2006, an infinite dimensional affine associative algebra over a field of characteristic 2 which is graded, nil and has Gelfand-Kirillov dimension equal to 2. In 2007, T. Lenagan and A. Smoktunowicz [128] constructed a family of infinite dimensional affine nil algebras with finite Gelfand-Kirillov dimension over an arbitrary countable field. In 2006, V. Petrogradsky [156] constructed a self iterated Lie algebra L with two generators over an arbitrary field of characteristic 2 which is nil and satisfies $1 < GK \dim L < 2$.

More properties were obtained for this algebra L and its associative envelope A, by V. Petrogradsky and I. Shestakov:

(i) $GK \dim L = \ln 2/\ln((1+\sqrt{5})/2)$,
(ii) $GK \dim A = 2GK \dim L$,
(iii) L has a nil 2-application,
(iv) L and A are direct sum (as vector spaces) of two locally nilpotent subalgebras.

These properties are analogous to Grigorchuk and Gupta-Sidki groups. Moreover, similar results were obtained for arbitrary fields of characteristic $p > 0$, by V. Petrogradsky, I. Shestakov and E. Zelmanov. See [157, 188] and [158].

4.2.2 Binary Lie algebras and assocyclic algebras

An anticommutative algebra A is called *binary Lie algebra* if every pair of elements of A generates a Lie algebra. Every Malcev algebra is a binary Lie algebra. An algebra is *assocyclic* if the identity $(a, b, c) = (b, c, a)$ is satisfied, where $(a, b, c) = (ab)c = a(bc)$ is the associator of a, b and c. The class of assocyclic algebras contains alternative algebras and it is well known that the algebra $A^{(-)}$, for A an alternative algebra, is Malcev. However, it is an open problem if every Malcev algebra is special, that is, is isomorphic to a subalgebra of $A^{(-)}$, for an alternative algebra A.

V. Filippov noticed that for each assocyclic algebra A, the algebra $A^{(-)}$ is a binary Lie algebra and formulated the question if every binary Lie algebra is special in this sense, that is, is isomorphic to a subalgebra of $A^{(-)}$, for some assocyclic algebra A. This problem was solved by M. Arenas and I. Shestakov in [6]. They proved that the answer is negative, constructing a binary Lie algebra which does not belong to the variety generated by special binary Lie algebras. The construction is based on an example of a finite dimensional simple binary Lie superalgebra which is not a Malcev superalgebra.

4.2.3 Derivations of the Lie algebra of infinite strictly upper triangular matrices over a commutative ring

Let R be a commutative ring with unity. Then, the set $T(\infty, R)$ of infinite $\mathbb{N} \times \mathbb{N}$ -upper triangular matrices over F is an associative ring with the usual operations and $T(\infty, R)^{-}$ is a Lie algebra with respect to the Lie product $[A, B] = AB - BA$. The Lie subalgebra of $T(\infty, R)^{-}$ of strictly upper triangular matrices is denoted by $\mathcal{N}(\infty, R)$.

In [108], W. Holubowski, I. Kashuba and S. Zurek describe the Lie algebra $\mathcal{N}(\infty, R)$. They show that every derivation of $\mathcal{N}(\infty, R)$ is a sum of diagonal and inner derivations.

4.3 Identities

4.3.1 Gradings of simple algebras and superalgebras

The description of all group gradings on a finite dimensional simple associative superalgebra over and algebraically closed field by a finite abelian group is given by Y. Bahturin and I. Shestakov in [9].

In the case of gradings on simple Jordan algebras by abelian groups, some results were obtained. For instance, all group gradings of the simple Jordan algebra of a non-degenerate symmetric form over a field of characteristic not 2 were described in [8] by Y. Bahturin and I. Shestakov and all group gradings by a finite abelian group on some types of simple Jordan (and Lie) algebras over an algebraically closed field of characteristic zero was presented in [10] by the same authors in collaboration with M. Zaicev.

In [7], Y. Bahturin, M. Bresar and I. Shestakov apply the method of so called *functional identities* to the study of group gradings by an abelian group on simple Jordan algebras, under mild restrictions on the grading group or the base field of coefficients. The method allows one to reduce the question of describing the gradings by abelian groups on simple Jordan algebras to the same problem on associative algebras.

4.3.2 Skew-symmetric identities of octonions

The question concerning identities for octonions has been investigated for a long time. An explicit basis of octonions identities is known only when the field is finite. When the field has characteristic zero, the algebra of octonions possesses a finite basis of identities, however, such a basis is still unknown. Using computer algebra, I. Hentzel and L. Peresi [100] found a new identity of degree 6.

Using the basis of a free alternative superalgebras on one odd generator constructed by I. Shestakov and N. Zhukavets in [192] and introducing the concept of quadratic alternative superalgebra, the same authors describe in [193] all skew-symmetric identities of octonions and prove that all of them are consequence of octonions being quadratic.

4.3.3 Associative identities of octonions

As in Sect. 4.3.2, another direction in order to understand octonions identities is developed by I. Shestakov in [176]. The aim here is to describe a basis of identities modulo an associator ideal of a free alternative algebra. More specifically, let $I(O)$ the T-ideal of polynomial identities for the octonions in the free alternative algebra. The homomorphic image $I_{as}(O)$ of $I(O)$ in the free associative algebra is studied and a basis of generators of the T-ideal $I_{as}(O)$ is obtained.

4.3.4 Graded identities of octonions

F. Henry, a PhD student of I. Shestakov, described in [97] the graded identities of octonions under the $\mathbb{Z}_2^3$ and $\mathbb{Z}_2^2$ gradings related with the Cayley-Dickson process.

4.3.5 Polynomial identities for ternary algebras

In [24], M. Bremner and L. Peresi use representation theory of the symmetric group S_3 to classify, up to equivalence, all ternary operations over the rational field.

The classic triple products are: Lie, Jordan and anti-Jordan. Each one of these products, together with their polynomial identities, defines the corresponding variety of triple systems.

For a representative of each equivalence class, the polynomial identities of degree less than or equal to to 5 satisfied by the ternary operation in every totally associative ternary algebra are obtained in [25]. These identities are obtained as a null space of a big matrix with integer coefficients. Lenstra-Lenstra-Lovász algorithm is used for reducing the number of terms and the size of the coefficients. Using the same techniques, polynomial identities of degree less than or equal to 9 in two and three variables which are satisfied by the ternary cyclic sum $[a, b, c] = abc + bca + cab$ in totally associative ternary algebras are obtained in [27].

4.3.6 Identities of finitely generated Malcev algebras

I. Shestakov in [178] proves that for any n there exists $N(n)$ such that for any Malcev algebra with n generators all multilinear skew-symmetric polynomials of degree $\geq N(n)$ are trivial ($= 0$).

4.3.7 Codimension growth in finite dimensional nonassociative algebras

Let A be an algebra over a field F of characteristic zero. A quantitative estimate of the polynomial identities satisfied by A is achieved through the study of the asymptotics of the sequence of codimensions of A. This sequence is defined as follows. Let $F\{X\}$ be the free non-associative algebra over F on $X = \{x_1, \dots, x_n, \dots\}$. Let $\mathrm{Id}(A)$ be the T-ideal of polynomial identities satisfied by the algebra A and let P_n be the space of multilinear polynomials in $x_1, \dots, x_n$. The *sequence of codimensions* of A is the sequence $c_n(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)}$, $n = 1, 2, \dots$.

It was proved by Regev that this sequence is exponentially bounded for associative PI-algebras. He also showed, in this case, that $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is an integer. The *exponent* of A is defined to be $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$, if this limit exists.

However, in general, the sequence of codimensions of an arbitrary algebra can have overexponential growth. For that ones which have exponentially bounded sequence of codimensions, one can ask if their exponents exist and if the exponent is a integer number.

The codimension growth for finite dimensional algebras in several classes of algebras were investigated by A. Giambruno, I. Shestakov and M. Zaicev in [61]. They proved that in a wide class of simple algebras including noncommutative Jordan algebras, $\exp(A)$ exists and is equal to the dimension of A. Also, they determine finite dimensional algebras, including Jordan and alternative algebras, for which $\exp(A)$ exists and is a non negative integer.

4.3.8 Polynomial identities for finite dimensional simple algebras

It is well known that two non isomorphic algebras may have the same identities. For instance, over the field $\mathbb{R}$ of real numbers, the quaternion algebra $\mathbb{H}$ and the matrix algebra $M_2(\mathbb{R})$ are not isomorphic. However they have the same identities, because $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ and $M_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ are isomorphic. In [187], I. Shestakov and M. Zaicev prove that over an algebraically closed field, two finite dimensional simple algebras are isomorphic if, and only if, they satisfy the same polynomial identities.

4.3.9 Prime degenerate algebras

Identities of prime degenerate algebras in the variety of Jordan, alternative and $(-1, 1)$ type were investigated. This kind of algebras was called *monsters* since the first examples of such algebras, obtained by S. Pchelintsev, were very complicated. After that, Yu. Medvedev, E. Zelmanov and I. Shestakov have constructed other monsters using the theory of superalgebras. In [146] S. Pchelintsev and I. Shestakov prove that the variety of algebras generated by the Pchelintsev's prime degenerate $(-1, 1)$-algebra is the same as the generated by the Grassmann $(-1, 1)$-algebra and the same variety is generated by the Grassmann envelope of any simple nonassociative $(-1, 1)$-superalgebra. Moreover, this variety is the smallest variety of $(-1, 1)$ -algebra that contains monsters. Similar results were obtained for Jordan algebras.

4.3.10 Associative nil algebras over finite fields

Let F be a field. Denote by $C^F_{n,d}$ the nilpotency index of the relatively free algebra with d generators in the variety of algebras defined by the identity $x^n = 0$. In [130], A. Lopatin and I. Shestakov prove that if F is a finite field then $C^F_{3,d}$ is determined for any F and, if $\#F > n$ then $C^F_{n,d}$ does not depend on the cardinality of F.

4.3.11 Nuclear elements of the free alternative algebra

In 1953 E. Kleinfeld proved that, for every pair of elements x, y in an alternative algebra, $[x, y]^4$ is a nuclear element. This fact was used to prove that every simple alternative algebra is associative of a Cayley-Dickson algebra. Since then, many authors have found nuclear elements in an alternative algebra.

In [101], I. Hentzel and L. Peresi present another element of degree 7 in the center of free alternative algebras. I. Shestakov and N. Zhukavets, in [190], showed that this element is the first one in a infinite set of central elements.

For free alternative algebras over $\mathbb{Z}_{103}$, I. Hentzel and L. Peresi prove that the least degree of a nuclear element is 5 and all nuclear elements of degree 5 and 6 are obtained. These results can be found in [102–104].

4.3.12 Identities of nonhomogeneous subalgebras of Lie and special Jordan superalgebras

Polyomial identities satisfied by nonhomogeneous subalgebras of Lie and special Jordan superalgebras are considered in [26] by M. Bremner and L. Peresi. The authors ignore the grading and regard the superalgebra as an ordinary algebra. The Lie case has been studied earlier by I. Volichenko and A. Baranov: they have found identities in degrees 3, 4 and 5 which imply all the identities in degrees less than or equal to 6. Their identities of degree 5 were simplified and it was shown that there are no new identities in degree 7. For the Jordan case, identities of degree less than or equal to 6 are also found. The method used is the same as described in Sect. 4.3.5: representation theory of symmetric groups and Lenstra–Lenstra–Lovász algorithm.

4.3.13 Special identities for quasi-Jordan algebras

Algebras which satisfy identities $a(bc) = a(cb)$, $(ba)a^2 = (ba^2)a$ and $(b, a^2, c) = 2(b, a, c)a$, are called *quasi-Jordan algebras*. An *associative dialgebra* is a vector space with two bilinear operations $a \dashv b$ and $b \vdash a$ satisfying

$$\begin{aligned} (a \vdash b) \vdash c =& (a \dashv b) \vdash c, \quad a \dashv (b \dashv c) = a \dashv (b \vdash c), \\ (a \dashv b) \dashv c =& a \dashv (b \dashv c), \ (a \vdash b) \vdash c = a \vdash (b \vdash c), \ (a \vdash b) \dashv c = a \vdash (b \dashv c). \end{aligned}$$

If $(D, \dashv, \vdash)$ is an associative dialgebra, the algebra D^+ with multiplication $a \lhd b = a \dashv b + b \vdash a$ defined on the vector space D is a quasi-Jordan algebra. A quasi-Jordan algebra is *special* if it is isomorphic to a subalgebra of D^+, for some associative dialgebra D. An identity is called *special* if it is satisfied by special quasi-Jordan algebras, but not for every quasi-Jordan algebra. In [28] M. Bremner and L. Peresi show that the least degree of a special identity is 8 and examples of such identities are presented. Some of these identities are noncommutative preimages of the Glennie identity for Jordan algebras.

4.3.14 Special identities for Bol algebras

The role of Bol algebras in the generalization of Lie algebras as tangent algebras of Lie groups was mentioned in Sect. 2. A (left) Bol algebra is a vector space equipped with a binary operation $[a, b]$ and a ternary operation $\{a, b, c\}$ which satisfy

$$[a,b]+[b,a]=0, \quad \{a,b,c\}+\{b,a,c\}=0, \quad \{a,b,c\}+\{b,c,a\}+\{c,a,b\}=0,$$
$$[\{a,b,c\},d]-[\{a,b,d\},c]+\{c,d,[a,b]\}-\{a,b,[c,d]\}+[[a,b],[c,d]]=0,$$
$$\{a,b,\{c,d,e\}\}-\{\{a,b,c\},d,e\}-\{c,\{a,b,d\},e\}-\{c,d,\{a,b,e\}\}=0.$$

If A is a left or right alternative algebra then A^b is a Bol algebra, where $[a,b]=ab-ba$ is the commutator of a and b, and $\{a,b,c\}=\langle b,c,a\rangle$ is the Jordan associator $(b\circ c)\circ a-b\circ(c\circ a)$. A *special identity* is an identity satisfied by A^b, for all right alternative algebra A, but not satisfied by the free Bol algebra. In [105] I. Hentzel and L. Peresi prove that there are no special identities of degree less than or equal to 7 and obtain all the special identities of degree 8 in partition six-two.

4.3.15 Malcev dialgebras

The notion of a dialgebra in any variety of algebras was given by Kolesnikov, in [123]. Using this approach, M. Bremner, L. Peresi and J. Sanchez considered Malcev dialgebras in [30]. They define the *dicommutator* $\{a,b\}=a\dashv b-b\vdash a$ and show that if D is an alternative dialgebra then $D^-=(D,+,\{,\})$ is a Malcev dialgebra. A *special identity* for Malcev dialgebra is an identity which holds for D^-, for any alternative dialgebra D but do not hold for every Malcev dialgebra. They prove that any special identity for Malcev dialgebras must have degree at least 7. They also introduce a trilinear operation which makes any Malcev dialgebra into a Leibniz triple system.

4.3.16 Identities for the ternary commutator

Polynomial identities for the trilinear operation

$$[a,b,c]=abc-acb-bac+bca+cab-cba$$

in the free associative algebra are studied by M. Bremner and L. Peresi in [29]. The authors use representation theory of the symmetric group to prove the existence of new identities in degree 11.

5 Applications and generalization

5.1 Loops

A *loop* is a set L with a binary operation which admits a neutral element 1 and, for every $a,b\in L$, the equations $ax=b$ and $ya=b$ have unique solutions in L.

Therefore, for each $a \in L$, the left and right translations $L_a, R_a : L \to L$ are bijections of L. The *multiplication group* of L, Mult(L), is the permutation group generated by all left and right translations. The stabilizer of $1 \in L$ in Mult(L) is the inner group Inn(L) of L. A loop L is called *automorphic loop* if every element of Inn(L) is an automorphism of L.

A loop L is a *Moufang loop* is the identities

$$((zx)y)x = z((xy)x), \quad x(y(xz)) = (x(yx))z$$

hold in L. If a loop L satisfies the first identity above, it is called a *left Bol loop*. The second identity define the class of *right Bol loops*.

A loop L is an *algebraic loop* if L is an algebraic variety over an algebraically closed field F with regular morphisms $m, \phi, \psi : L \times L \to L$ and an element $e \in L$ such that the identities

$$x = m(e, x) = m(x, e) = m(y, \phi(x, y)) = m(\psi(x, y), y)$$

hold for $x, y \in L$. In this case, m is the multiplication of L and ϕ and ψ are, respectively, the left and right divisions. If $m, \phi, \psi : L \times L \to L$ are well-defined rational maps for which the identities above hold on a Zariski open subset of $L \times L$ then L is called a *local algebraic loop*.

5.1.1 Moufang loops and alternative algebras

In [174] I. Shestakov constructed an example of a Moufang loop which is not embedded into a loop of invertible elements of a unital alternative algebra.

5.1.2 Commutative Moufang loops and alternative algebras

In [84] A. Grishkov and I. Shestakov prove that a free commutative Moufang loop L of *exponent* 3 (that is, $x^3 = 1$, for all $x \in L$) with at most 7 generators can be embedded in the free commutative alternative algebra over a field of characteristic 3. As a consequence, the order of this loop is determined.

5.1.3 Self-dual gauge fields in seven dimension

In [129] A. Grishkov and E. Loginov apply the theory of Moufang loops to the construction of a gauge-invariant Lagrangian and to find a solution of modified Yang-Mills equations in seven dimension.

5.1.4 Algebraic Bol loops

In [70] A. Grishkov and G. Nagy study the category of algebraic Bol loops over an algebraically closed field. They apply techniques from the theory of algebraic groups and obtain structural theorems for this category: if B is an algebraic right Bol loop then the set $\text{Mult}_r(B)$ of all right translations of B is an algebraic group. It is shown that these loops lack some nice properties of algebraic groups. For instance, they construct local algebraic Bol loops which are not birationally equivalent to global algebraic loops.

5.1.5 Kimmerle conjecture for the Held and O'Nam sporadic simple groups

The Kimmerle conjecture for the Held and O'Nam sporadic simple groups is proved by V. Bovdi, A. Grishkov and A. Konovalov in [20]. For a finite group G, this conjecture states that $\pi(G)$ and $\pi\big(V(\mathbb{Z}G)\big)$ coincide, where $\pi(G)$ is the Gruenberg–Kegel graph, which has the prime divisors of the group order as its vertices and two vertices p and q are connected if and only if G possesses an element of order pq, and $V(\mathbb{Z}G)$ is the normalised unit group of the integral group ring $\mathbb{Z}G$.

5.1.6 Moufang loops

Moufang loops have been studied by A. Grishkov and A. Zavarnitsine since 2005. For instance, an analog of Lagrange's Theorem for finite Moufang loops was obtained in [85]. Groups with triality are naturally connected with Moufang loops and the description of groups with triality associated with a given Moufang loop was given in [86]. An analog of the first Sylow Theorem giving a criterion for the existence of a p-Sylow subloop and the maximal order of p-subloops in the Moufang loops that do not possess p-Sylow subloops are presented in [87]. Using groups with triality they construct a series of nonassociative Moufang loops in [88]. Certain members of this series contain an abelian normal subloop with the corresponding quotient being a cyclic group. In particular, they give a new series of examples of finite abelian-by-cyclic Moufang loops. Some of the examples are shown to be embeddable into a Cayley algebra. And, also using groups with triality, they obtain in [89] some general multiplication formulas in Moufang loops, construct Moufang extensions of abelian groups, and describe the structure of minimal extensions for finite simple Moufang loops over abelian groups.

In [60], S. Gagola and A. Grishkov worked on the principal problem in the theory of finite Moufang loops: to describe loops without abelian normal subloops. They proved that any Moufang loop M which contains an associative simple normal subloop N such that M/N is a cyclic group of order $p \neq 3$ (p prime) is a group.

5.1.7 Commutative automorphic loops

In [11] D. Barros, A. Grishkov and P. Vojtechovsky classify loops in the class of all 2-generated commutative automorphic loops Q possessing a central subloop $Z \cong \mathbb{Z}_p$ such that $Q/Z \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

In [12] the same authors construct the free commutative 2-generated loop of nilpotency class 3. It turns out that it has dimension 8 over the integers.

Solvability of commutative automorphic loops was studied by A. Grishkov, M. Knyon and P. Nagy in [69]. It is proved that every finite commutative automorphic loop is solvable and that every automorphic 2-loop is solvable.

5.1.8 Automorphic loops

A general construction of automorphic loops is given by A. Grishkov, M. Rasskazova and P. Vojtechovsky in [73]. If R is a commutative ring, V is a R-module, $E = \mathrm{End}_R(V)$ is the ring of R-endomorphisms of V and W is a subgroup of $(E, +)$ such that $ab = ba$, for every $a, b \in W$, and $1 + a$ is invertible for every $a \in W$, then the space $W \times V$ with multiplication defined by $(a, u)(b, v) = \big(a + b, u(1 + b) + v(1 - a)\big)$, for $a, b \in W$ and $u, v \in V$, is an automorphic loop. In the same paper, it is shown an example of an infinite 2-generated abelian-by-cyclic automorphic loop of prime exponent.

5.1.9 Steiner loops

A *Steiner triple system* is an incidence structure consisting of points and blocks such that every two distinct points are contained in precisely one block and any block has exactly 3 points. There is a loop (*Steiner loop*) associated to each Steiner triple system, defining xy as the third element in the block containing x and y and adjoining an element e such that $x \cdot e = e \cdot x = x$ and $x \cdot x = e$. Reciprocally, a Steiner loop determines a Steiner triple system.

The paper [71] is devoted to the study of free objects in the variety of Steiner loops and of the combinatorial structures behind them, focusing on their automorphism groups. A. Grishkov, D. Rasskazova, M. Rasskazova, and I. Stuhl prove that all automorphisms are tame and the automorphism group is not finitely generated if the loop is more than 3-generated. For the free Steiner loop with three generators they describe the generator elements of the automorphism group and some relations between them.

In [72] the same authors describe Steiner Loops of nilpotency class 2 and establish the classification of finite 3-generated nilpotent Steiner loops of nilpotency class 2.

5.2 Commutative power associative nilalgebras

Power associative algebras are a natural generalization of associative, alternative, and Jordan algebras. An algebra A is *power associative* if, for each $a \in A$, the subalgebra of A generated by a is associative. The class of commutative power associative algebras generalizes Jordan algebras and some results connecting these two varieties of algebras are known. For instance, every simple commutative power associative algebra of degree greater than 2 is Jordan and a commutative power associative algebra of degree 2 is Jordan if and only if it is stable. If the characteristic of the ground field is zero, then every semisimple commutative power associative algebra is Jordan.

For Jordan algebras it is true that every finite dimensional nilalgebra must be nilpotent. In 1948, A. Albert asked if the same occurs in the variety of commutative power associative algebras. This question was answered by D. Suttles, who constructed a 5-dimensional algebra which is commutative power associative, nil of index 4, but it is not nilpotent. However, this algebra is solvable. Since then, the following question has remained open.

Albert Problem: Is every finite dimensional commutative power associative nilalgebra solvable?

In [91], J.C. Gutiérrez proves that every commutative power associative nilalgebra of dimension n and nilindex greater than or equal to $n-2$ is solvable, for algebras over a field of characteristic zero or of sufficiently high characteristic compared to the nilindex. It is also shown that every commutative nilalgebra of dimension less than or equal to 6 over a field of characteristic not 2, 3 or 5 is solvable. In [95] J.C. Gutiérrez and A. Suazo improve this result, proving that commutative power associative nilalgebras of dimension less than or equal to 8 over a field of characteristic not 2, 3 or 5 are solvable. The fact that every power associative nilalgebra of dimension n and niindex greater than n is either nilpotent of the same index or isomorphic to the Suttles' example was obtained by L. Elgueta, A. Suazo and J.C. Gutiérrez in [54].

Modules over a trivial algebra of dimension two in the variety of commutative and power associative algebras were studied by J.C. Gutiérrez, A. Grishkov, M. Montoya and L. Murakami in [94] and irreducible modules were classified. These results were used to understand the structure of finite dimensional power associative algebras of nilindex 4.

The description of commutative, power associative nilalgebra of dimension n and nilindex n over a field of characteristic not 2, 3 or 5 was given by J.C. Gutiérrez, C. Garcia and M. Montoya in [93] and special cases of algebras of dimension $n \geq 6$ and nilindex $n-1$ are treated in [92] by J.C. Gutiérrez, C. Garcia, J. Martínez and M. Montoya.

6 Baric algebras

As mentioned before, the study of nonassociative algebras was initiated at IME-USP by R. Costa in 1980. Inicially it was concentrated in the study of algebras describing genetic inheritance; see A. Wörz-Busekros [196] and M. Reed [166]. Let F be a field. If the otherwise is not stated, in this section all algebras are over F and char $(F) \neq 2$.

Let A be a K-algebra, where K is a commutative ring with a unit element. The *duplicate* $D(A)$ of A consists of the tensor product K-module $A \otimes A$ with multiplication $(a \otimes b)(c \otimes d) = ab \otimes cd$. Now, suppose that A is commutative. In general $D(A)$ is not commutative. But the quotient algebra $A \otimes A/I$, where I is the ideal of A generated as a submodule by $\{a \otimes b - b \otimes a \mid a, b \in A\}$, is commutative. The K-algebra $A \otimes A/I$ is called the *commutative duplicate* of A.

A *baric algebra* is a pair (A, ω) where A is an algebra and $\omega : A \to F$ is a nonzero homomorphism of algebras. The homomorphism ω is called the *weight function* and, for $x \in A$, $\omega(x)$ is the *weight* of x. We denote by N the kernel of ω.

6.1 Gametic and zygotic algebras of a 2m-ploid population

A *genetic algebra* is a real nonassociative algebra for which there exists a basis $c_0, c_1, \dots, c_n$, with a multiplication table satisfying the following conditions: if $c_i c_j = \sum_{k=0}^{n} \lambda_{ijk} c_k$ then $\lambda_{000} = 1$, $\lambda_{0jk} = \lambda_{j0k} = 0$ for $k < j$ and $\lambda_{ijk} = 0$ for $i,j > 0$ and $k \leq max(i,j)$. Such a basis is called a *canonical basis*. Any genetic algebra is baric since the linear map ω defined by $w(c_0) = 1$, $\omega(c_i) = 0$ $(1 \leq i \leq n)$ is a homomorphism. The $t_j = \lambda_{0jj}$ are independent of the particular canonical basis since they are eigenvalues of any linear transformation $y \to xy$, x an element of weight 1. They are called the *left train roots*.

Let $G = G(n+1, 2m)$ be the gametic algebra of a $2m$-ploid population with $n+1$ alleles $D_0, D_1, \dots, D_n$. Each monomial in the D's of degree m represents one of the gametic types of the population, and so the algebra has dimension $\begin{pmatrix} m+n \\ m \end{pmatrix}$. H. Gonshor [65] proved that G is genetic, the elements of a certain canonical basis can be represented formally by all monomials $X_0^{m-p} X_{i_1} \dots X_{i_p}$ of degree m in the variables $X_0, X_1, \dots, X_n$. The nonzero products are

$$(X_0^{m-p} X_{i_1} \dots X_{i_p})(X_0^{m-q} X_{j_1} \dots X_{j_q}) = c(p,q) X_0^{m-(p+q)} X_{i_1} \dots X_{i_p} X_{j_1} \dots X_{i_q}$$

where $p+q \leq m$ and $c(p,q) = \begin{pmatrix} 2m \\ p+q \end{pmatrix}^{-1} \begin{pmatrix} m \\ p+q \end{pmatrix}$. The monomial X_0^m is an idempotent and $G^2 = G$. The $c(p, q)$ are completely determined by the left train roots of G. The corresponding zygotic algebra $Z = Z(n+1, 2m)$ is the commutative duplicate of G. Using the technique of duplication, Gonshor gave a canonical basis of Z.

The Lie algebra $\mathbb{R}^n \oplus gl(n, \mathbb{R})$ is the derivation algebra of the affine group $A(n)$ of $\mathbb{R}^n$. R. Costa [38, 39] studied the derivations of G and Z. The main results are that for G and Z the derivation algebra is isomorphic to $\mathbb{R}^n \oplus gl(n, \mathbb{R})$.

L. Peresi [148] proved that the automorphism group of G is isomorphic to $A(n)$ and constructed a more comprehensive class $\Omega(m, n)$ of algebras of dimension $\begin{pmatrix} m+n \\ m \end{pmatrix}$, determined by their left train roots. For any algebra A in $\Omega(m, n)$, $A(n)$ is isomorphic to a subgroup of the automorphism group of A. For $1 \leq m \leq 5$, n arbitry, all algebras in $\Omega(m, n)$ having the automorphism group isomorphic to $A(n)$ were obtained. Hence it is clear that the derivation algebra of any algebra in $\Omega(m, n)$ contains a subalgebra isomorphic to $\mathbb{R}^n \oplus gl(n, \mathbb{R})$. In [150], for $1 \leq m \leq 5$, n arbitry, L. Peresi determined all algebras in $\Omega(m, n)$ having derivation algebra isomorphic to $\mathbb{R}^n \oplus gl(n, \mathbb{R})$.

Let A be a K-algebra. Assume that $A^2 = A$ and A has an idempotent. L. Peresi [149] proved that A, $D(A)$ and the commutative duplicate of A have isomorphic automorphism groups and isomorphic derivation algebras. The condition $A^2 = A$ is necessary. It was proved later that the other condition is not. As a consequence, the automorphism group of Z is isomorphic to $A(n)$.

The gametic algebra $G(n+1, 2)$ satisfies $x^2 = \omega(x)x$. For this algebra, R. Costa [40] proved that all identities of minimal degree are consequences of commutativity and $2((ab)c)d - ((bc)a)d - ((bc)d)a + 2((cd)b)a - 2(ab)(cd) = 0$. Using computer algebra, M. Bremner, Y. Piao and S. Richards [31] proved that every identity for $G(n+1,2)$ follows from commutativity and Costa's identity. These authors also proved that every identity for the zygotic algebra $Z(n+1,2)$ follows from commutativity and $2((ab)c)d-((ab)d)c-((ac)b)d-((bc)a)d+(ab)(cd)$.

6.2 Indecomposable baric algebras

R. Costa and H. Guzzo [42] introduced the notion of indecomposable baric algebra and proved an analogous to the Krull-Schmidt Theorem. A baric algebra (A,ω) with an idempotent of weight 1 is *decomposable* if there are non trivial ideals N_1 and N_2 of A, both contained in N, such that $N=N_1\oplus N_2$. Otherwise, it is *indecomposable*. For every $n\geq 1$ and $m\geq 2$, $G(n+1,2m)$ is indecomposable. The additive group $(N,+)$ can be endowed with a structure of an abelian M-group. The set M is formed by all right and left multiplications R_a and L_a, where a belongs to $A\cup F$. In this case, the M-subgroups of $(N,+)$ are the ideals of the algebra A, contained in N. If (A,ω) has an idempotent e of weight 1 then $A=Fe\oplus N$. The *join* of $(A_1,\omega_1)=Fe_1\oplus N_1$ and $(A_2,\omega_2)=Fe_2\oplus N_2$ is the baric algebra $(A_1\vee A_2,\omega_1\vee\omega_2)$ where $A_1\vee A_2=F(e_1+e_2)\oplus N_1\oplus N_2$, $\omega_1\vee\omega_2(e_1+e_2)=1$ and $\omega_1\vee\omega_2(N_1\oplus N_2)=0$. If $(A,\omega)=Fe\oplus N$ and the M-group N satisfies the d.c.c. then there exist m indecomposable baric subalgebras (A_i,ω_i) of (A,ω) such that $(A,\omega)=(A_1\vee\cdots\vee A_m,\omega_1\vee\cdots\vee\omega_m)$.

The Krull-Schmidt Theorem is as follows. Suppose that for $(A,\omega)=Fe\oplus N$ the M-group Nsatisfies both a.c.c. and d.c.c.. Let (A_i,ω_i) and (B_j,ω_j) be indecomposable baric algebras such that

$$(A,\omega)=(A_1\vee\cdots\vee A_m,\omega_1\vee\cdots\vee\omega_m,\quad (A,\omega)=(B_1\vee\cdots\vee B_n,\gamma_1\vee\cdots\vee\gamma_n),$$

Then $m=n$ and for some permutation $i\to j$ of indices, $(A_i,\omega_i)\cong(B_j,\gamma_j)$. R. Costa and H. Guzzo [43] obtained some classes of baric algebras for which the Krull-Schmidt Theorem is valid. They prove also that the commutative duplicate of an indecomposable commutative baric algebra A such that $A^2=A$ is indecomposable.

6.3 Bar-radical and Wedderburn decomposition

In [96], H. Guzzo introduced the bar-radical, and obtained results on Noetherian and Artinian baric algebras and the join of baric algebras. Let $A=(A,\omega)$ be a baric algebra and denote $\mathrm{bar}(A)=\ker(\omega)$. A baric subalgebra of A is a subalgebra B of A such that $B\nsubseteq \mathrm{bar}(A)$. In this case, $(B,\omega_{|B})$ is a baric algebra. If, furthermore, $\mathrm{bar}(B)$ is a two-sided ideal of $\mathrm{bar}(A)$ then B is called normal. A baric algebra A is called baric simple if $\mathrm{bar}(B)=\{0\}$ or $\mathrm{bar}(B)=\mathrm{bar}(A)$ for all normal baric subalgebras B of A. The *bar-radical* $R_b(A)$ of A is defined as follows: $R_b(A)=\{0\}$ if A is baric simple; otherwise, $R_b(A)=\bigcap\mathrm{bar}(B)$, where B runs over the maximal normal baric subalgebras of A. When $R_b(A)=\{0\}$, A is called baric semisimple. The algebra A has a *Wedderburn baric decomposition* if A decomposes as a direct sum (of vector spaces)

$A = S \oplus T \oplus R_b(A)$, where S is a semisimple baric subalgebra of A and T is a vector subspace of bar(A) such that $T^2 \subset R_b(A)$.

Let $A = (A, \omega)$ be a finite dimensional baric algebra. Denote by Nil(A) the nilradical (maximal nilideal). If A is alternative, or unital Jordan, or of (γ, δ)-type (char$(F) \neq 2, 3, 5$ in this last case) then $R_b(A) = \mathrm{Nil}(A) \cap (\mathrm{bar}(A))^2$. See M. Couto and H. Guzzo [51], J. Ferreira and H. Guzzo [57, 58]. In [56], B. Ferreira, J. Ferreira and H. Guzzo proved that if A is of (γ, δ)-type over algebraically closed field F of characteristic $\neq 2, 3, 5$ then A has a Wedderburn baric decomposition.

6.4 Bernstein algebras

In 1923 S. Bernstein proposed the problem of describing all evolution operator that achieves equilibrium in the second generation. Let $a_1, \dots, a_n$ be n hereditary types. Denote by δ_{ijk} the probalility to obtain a_k from a_i and a_j $(\delta_{ijk} = \delta_{jik} \geq 0, \quad \sum_{k=1}^{n} \delta_{ijk} = 1)$. If $x = (x_1, \dots, x_n)$, $x_1, \dots, x_n \geq 0$ and $\sum_{i=1}^{n} x_i = 1$, represents the frequency distribution of hereditary types in a given generation, then the frequency distribution in the next generation is $x' = (x'_1, \dots, x'_n)$, where $x'_k = \sum_{i,j=1}^{n} x_i x_j \delta_{ijk}$ $(k = 1, \dots, n)$.

The *evolution operator* is the quadratic operator V defined by $V(x) = x'$. The population achieves equilibrium in the second generation if and only if $V^2 = V$; in this case, we say that V is *stationary*. Bernstein solved the problem for $n = 3$ and obtained some results for arbitrary n. Y. Lyubich (see [135]) solved the problem for the regular and exceptional cases, and J.C. Gutiérrez [90] for the non-regular case completing the solution of Bernstein problem.

Lyubich (and also P. Holgate [107]) introduced the notion of Bernstein algebra, given an algebraic formulation for the Bernstein problem. Let $V(x) = x'$ be the evolution operator. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathbb{R}^n$. Considering in $\mathbb{R}^n$ the multiplication given by $e_i e_j = \sum_{k=1}^{n} \delta_{ijk} e_k$, we obtain an algebra A_V. In terms of A_V, $V(x) = x^2$ and stability in the second generation is given by $x^2 x^2 = x^2$. It is easy to see that the linear form $\omega_V : A_V \to \mathbb{R}$ defined by $\omega_V(x) = \sum_{i=1}^{n} x_i$ is a homomorfism and $x^2 x^2 = \omega_V(x)^2 x^2$, for all $x \in A_V$. A *Bernstein algebra* is a baric algebra (A, ω), where A is a commutative algebra over F, such that $x^2 x^2 = \omega(x)^2 x^2 \quad (\forall x \in A)$. In particular, (A_V, ω_V) is a Bernstein algebra. For a Bernstein algebra, the weight function ω is unically determined.

Let (A, ω) be a Bernstein algebra. If $y \in A$ and $\omega(y) \neq 0$ then $(y/\omega(y))^2$ is an idempotent. Chosen an idempotent e, the Peirce decomposition is $A = Fe \oplus U \oplus V$, where $U = \{n \in N \mid en = \frac{1}{2} n\}$ and $V = \{n \in N \mid en = 0\}$. The subspaces U and V satisfy $U^2 \subset V$, $UV \subset U$, $V^2 \subset U$, $UV^2 = 0$. *Invariant* means that it does not depend on the choise of the idempotent. The condition $U^2 = 0$ is invariant and defines the subclass of *exceptional* Bernstein algebras. Also the conditions $UV = 0$ and $V^2 = 0$ are invariants and define the subclass of *normal* Bernstein algebras. When (A, ω) is exceptional it satisfies $(xy)(xy) = \omega(xy)xy$; and when is normal it satisfies $x^2 y = \omega(x)xy$.

A class of exceptional Bernstein algebras associated to finite simple graphs was constructed by R. Costa and H. Guzzo [44]. And it was proved by R. Costa and A. Grishkov [68] that an isomorphism between exceptional Bernstein algebras associated to two graphs implies an isomorphism of the graphs.

6.4.1 Nilpotency and solvability

Let (A, ω) be a Bernstein algebra and $N = \ker(\omega)$. As proved by A. Grishkov [74], the question about the nilpotency of N arises because Bernstein algebras are genetic algebras if and only if N is nilpotent. A. Grishkov proved that if $A^2 = A$ and A is finite dimensional then N is nilpotent. I. Hentzel and L. Peresi [99] have shown that N is not in general nilpotent but is solvable for any finitely generated Bernstein algebra. Grishkov conjectured that if $A^2 = A$ and A is finitely generated by n generators then there is $c = c(n)$ such that N is nilpotent with index $\leq c$. L. Peresi [151] proved that the conjecture is true without finding c. A. Krapivin [126] also proved that the conjecture is true and gave c. For arbitrary Bernstein algebras in characteristic $\neq 2, 3$, I. Hentzel, D. Jacobs L. Peresi and S. Sverchkov [98] proved that N^2 is nilpotent of index at most 9 and N is solvable of index at most 4. The key fact to obtain these results, verified by computer algebra, is that if J is a commutative nilring of nilindex 3 and characteristic $\neq 2, 3$ then $J^2J^2J^2J^2J = 0$. J. Bernad, S. González and C. Martínez [16] improved these results. Let (A, ω) be a Bernstein algebra. Then N is solvable of index at most 3 and N^2 is nilpotent of index at most 5.

6.4.2 Bernstein-Jordan algebras

Let $A = Fe \oplus U \oplus V$ be a Bernstein algebra. The set $L = \{u \in U \mid uU = 0\}$ is an ideal of A and A/L is a Bernstein-Jordan algebra. The ideal L is invariant. When $L = 0$ we say that A is *reduced*. Bernstein-Jordan algebras of dimension 5 over the real field were classified by I. Correa and L. Peresi [35].

T. Cortés and F. Montaner [37] defined direct products of Bernstein algebras, together with the related notions of decomposable and indecomposable algebras, and obtained some of their properties, including a Krull-Schmidt theorem of uniqueness of such decompositions. These authors define $J(A)$ as the smallest ideal of A such that $A/J(A)$ is a Bernstein-Jordan algebra. They show that the correspondence $A \to A/J(A)$ is a functor preserving the direct product.

T. Cortés and F. Montaner [37] sugested the following approach for the structure of the finite dimensional Bernstein algebras (A, ω): the first step is to calculate the reduced algebra A/L; the second step is to decompose this reduced algebra as a direct product of indecomposable algebras; the final step consists in classifying the indecompossable Bernstein-Jordan algebras. This approach works for low dimensions, since in dimension 7 there are an infinity number of indecomposable Bernstein-Jordan algebras that are not isomorphic.

T. Cortés and F. Montaner [36] used the above approach to classify Bernstein-Jordan algebras of dimension ≤ 5 when F is algebraically closed. First, they described all reduced Bernstein algebras of dimension ≤ 5 throught their indecomponsable factors. Next, they described the Bernstein-Jordan algebras of dimension ≤ 5 using the following argument: if (A, ω) is such an algebra then A/L is reduced, hence is one of the already listed algebras; finally, they obtained the algebra A from A/L.

6.4.3 Radical and complete Bernstein algebras

S. González, C. Martínez and A. Grishkov [66] introduced the concepts of radical and complete algebra for the finite dimensional Bernstein algebra A over an algebraically closed field of characteristic 0. Denote by $\mathrm{As}(A)$ the subalgebra of $\mathrm{End}(A^2)$ generated by the right multiplications R_a $(a \in A)$. The *radical* of A is the set $R(A) = \{x \in A \mid R_x \in J(\mathrm{As}(A))\}$, where $J(\mathrm{As}(A))$ is the Jacobson radical of $\mathrm{As}(A)$.

If (A, ω) is a Bernstein algebra and $t : A \to A$ is a linear map, we define the algebra A_1 and the homomorphism ω_1 in the following way: $A_1 = A \oplus Ft$ and the multiplication is given by

$$(a_1 + \alpha_1 t)(a_2 + \alpha_2 t) = a_1 a_2 + \alpha_2 t(a_1) + \alpha_1 t(a_2),$$
$$w_1(a) = w(a) \ (a \in A), \ w(t) = 0.$$

We say that t is an *A-map* if (A_1, ω_1) is a Bernstein algebra. If e is an idempotent of A, we say that the A-map t is an *e-map* if $t(e) = 0$. The A-map t is called *inner* when there exists $a \in A$ such that $t(x) = R_a(x)$ $(\forall x \in A^2)$.

If $A = Fe \oplus U \oplus V$, $u, u_1, u_2 \in U$, $v_1, v_2 \in V$, the maps t_1, t_2, t_3 are e-maps, where

$$\begin{aligned} t_1 =& R_{v_1} \circ R_{v_2}, \\ t_2(e) =& 0, \ t_2 = R_{uv_1} R_{v_2} + R_{uv_2} R_{v_1} \ \text{ in } \ U \oplus V, \\ t_3(e) =& 0, \ t_3 = R_{(u_1 u_2)v_1} R_{v_2} + R_{(u_1 u_2)v_2} R_{v_1} \ \text{ in } \ U \oplus V. \end{aligned}$$

The Bernstein algebra A is called *complete* when t_1, t_2, t_3 are inners for any idempotent e de A.

Suppose that the finite dimensional Bernstein algebra A is not complete. Then A can be embedded into a complete finite dimensional Bernstein algebra. The e-map $t = t_i$ for some i $(i = 1, 2, 3)$ is not inner. The algebra $A_1 = A \oplus Ft$ is constructed and A is embedded into A_1. If A_1 is complete the process ends. If A_1 is not complete, the process continues, obtaining an algebra A_n $(n \geq 2)$ and A is embedded into A_n. Since $A^2 = A_1^2 = \cdots = A_n^2$, A_n is complete for some n. Let A be a finite dimensional Bernstein algebra over an algebraically closed field F of characteristic 0. The radical $R(A)$ is a nilpotent ideal. If A is complete and e is an idempotent of A then $A = Fe \oplus R(A) \oplus T$, where $eT = 0$ and $J = \{R_x : A^2 \to A^2 \mid x \in T\}$ is a Jordan algebra with $\mathrm{Nil}(J) = 0$.

6.4.4 Basis of identities

(Polynomial) identities have an important role in the structure theory of algebras. The problem of classifying all algebras does not have a satisfactory answer. It turns out interesting to study classes of algebras satisfying a set of identities.

Another problem consists in finding identities for a class of algebras. The central question in this direction is Specht Problem: Given a class of algebras $\mathcal{C}$, is it true that any algebra A in $\mathcal{C}$ has a finite basis of identities (that is, the T-ideal of identities $T(A)$ is generated by a finite number of identities without dependency relations)? In

1950, W. Specht proposed this problem for associative algebras over a field of characteristic 0. A. Kemer [122] gave a complete solution: Any associative algebra over a field of characteristic 0 has a finite basis of identities. For positive characteristic, A. Belov [15], A. Grishin [67] and V. Shchigolev [169] independently constructed T-ideals of the free associative algebra that do not have a finite basis of identities. Results similar to Kemer's theorem are valid for finite dimensional alternative, Jordan and Lie algebras over a field of characteristic zero.

A weak form of Specht Problem is the following: Given a class of algebras $\mathcal{C}$, is it true that the T-ideal of identities $T(\mathcal{C})$ of $\mathcal{C}$ has a finite basis of identities?

The class of Bernstein algebras is not a variety of algebras, that is, cannot be defined by a set of identities. Over a field of characteristic $\neq 2, 3, 5$, I. Correa, I. Hentzel and L. Peresi [34] obtained by computer algebra the identities of minimal degree, which are not consequence of commutativity, for Bernstein algebras, normal Bernstein algebras and exceptional Bernstein algebras. For Bernstein algebras, all identities of minimal degree are consequences of

$$(x^2x^2, y, x) - 2(x^2, y, x)x^2 = 0,$$
$$(y, x^2x^2, x) - 2(yx^2, x, x^2) + 2y(x^2, x^2, x) + 2(x, (yx)x, x^2) = 0.$$

I. Correa [33] studied commutative algebras satisfying one of these identities. Let A be a commutative algebra over a field F of characteristic zero containing an idempotent and satisfying $(x^2x^2, y, x) - 2(x^2, y, x)x^2 = 0$. Then the Peirce decomposition is $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_0$ and we have: (i) A is a Bernstein algebra if and only if $\dim A_1 = 1$, $A_{\frac{1}{2}}^2 \subset A_0$ and $A_0^2 \subset A_{\frac{1}{2}}$. (ii) A is a Jordan algebra if and only if A_0 is a Jordan algebra and $j(xy) = (jx)y + (jy)x$ for all $x, y \in A_0$ and $j \in A_{\frac{1}{2}}$.

For normal Bernstein algebras, all identities of minimal degree are consequences of

$$3(yx^2)x = 2((yx)x)x + yx^3,$$
$$2(xy)(xy) = (x^2y)y + (y^2x)x,$$
$$(x, yz, t) + (z, yt, x) + (t, yx, z) = 0.$$

And, for exceptional Bernstein algebras, they are consequences of

$$(ae, bd, c) + (be, cd, a) + (ce, ad, b) + (a, bd, ce) + (b, cd, ae) + (c, ad, be)$$
$$+ (a, e, b)(cd) + (b, e, c)(ad) + (c, e, a)(bd) = 0.$$

J. Bernad, S. González, C. Martínez and A. Iltyakov [17] gave an example of an exceptional Bernstein algebra that does not have a basis of identities. Therefore, Specht Problem does not have a positive solution for the class of Bernstein algebras. For certain subclasses the problem has a positive solution. They proved that if (A, ω) is a Bernstein-Jordan algebra or a Bernstein algebra satisfying $A^2 = A$, over a field of caracteristic 0, then $T(A)$ has a finite basis of identities.

Denote by $\mathcal{N}$ the subclass of normal Bernstein algebras, and by $\mathcal{M}$ the subclass of exceptional Bernstein algebras satisfying $UV = 0$. J. Bernad, S. González, C. Martínez and A. Iltyakov [18] found 5 identities that generate $T(\mathcal{N})$ and 7 identities that generate

$T(\mathcal{M})$ in characteristic 0. L. Peresi [152] proved that there are dependence relations among the generators and obtained bases of identities. A basis of identities for $T(\mathcal{N})$ is

$$(a,b,c)d + (d,b,a)c + (c,b,d)a = 0, \quad (x,x,y^2) - 2(x,y,y)x = 0.$$

A basis of identities for $T(\mathcal{M})$ is

$$\begin{aligned}&(a,(bc)d,e) - 2(a,bc,e)d = 0,\\ &(a,b,c)(de) - 2\,((a,b,c)d)e = 0, \quad 3\,(x,x^2,x^2) - 2\,(x,x,x^2)x = 0,\\ &(a,c,bd) - (b,c,ad) - (a,d,bc) + (b,d,ac) - 3(a,c,b)d + 3(a,d,b)c = 0.\end{aligned}$$

For normal Bernstein algebras, the three identities of minimal degree mentioned above form also a basis of identities [152].

6.4.5 Multiplication algebra

Let $(A,\omega) = Fe \oplus U \oplus V$ be a finite dimensional Bernstein algebra. The dimensions $r = \dim U$ and $s = \dim V$ are invariant and $(1+r,s)$ is the *type* of (A,ω).

The *multiplication algebra* $M(A)$ is the subalgebra of $\mathrm{End}(A)$ generated by $L_x : A \to A$, where $x \in A$ and $L_x(a) = xa$. The following basic results on $M(A)$ were obtained by R. Costa and A. Suazo [50]: $M(A)$ is baric with weight function $\bar{\omega}$ defined by $\bar{\omega}(L_x) = \omega(x)$ and $2L_e^2 - L_e$ is an idempotent of weight 1;

$$\begin{aligned}M(A) &= F(2L_e^2 - L_e) \oplus \tilde{U} \oplus \tilde{V},\\ \tilde{U} &= \{\sigma \mid \sigma(A) \subset N,\ \sigma(2L_e^2 - L_e) = \sigma\},\\ \tilde{V} &= \{\sigma \mid \sigma(A) \subset N,\ \sigma(2L_e^2 - L_e) = 0\};\\ \tilde{U}^2 &= 0,\ \tilde{U}\tilde{V} = 0,\ \tilde{V}\tilde{U} \subset \tilde{U},\ \tilde{V}^2 \subset \tilde{V};\end{aligned}$$

$\tilde{U} \cong U \oplus U^2$ and an isomorphism is given by $x \in U \oplus U^2 \to \psi_x \in \tilde{U}$, where $\psi_x(e) = x$ and $\psi_x(N) = 0$; the idempotent $4L_e - 4L_e^2 \in \tilde{V}$ gives the usual Peirce decomposition $\tilde{V} = \tilde{V}_{11} \oplus \tilde{V}_{10} \oplus \tilde{V}_{01} \oplus \tilde{V}_{00}$ and the dimensions of V_{ij} are invariant.

Further results on $M(A)$ were obtained by R. Costa, L. Murakami and A. Suazo [45]. The subspace $\{L_u - 2L_eL_u \mid u \in U\}$ of $\tilde{V}_{01}$ has the same dimension as U/L, so $\dim \tilde{V}_{01} \geq \dim U - \dim L$. If $r \geq 1$ then the following hold: $\tilde{V}_{11} \neq 0$; A is normal if and only if $\tilde{V}_{10} = 0$; A is exceptional if and only if $\tilde{V}_{01} = 0$; $U(UV) = 0$ if and only if $\tilde{V}_{00} = 0$, and in this case $\dim \tilde{V}_{01} = \dim U - \dim L$; if $\tilde{V}_{01} = 0$ or $\tilde{V}_{10} = 0$ then $\tilde{V}_{00} = 0$; $\dim M(A) \geq r + 2$, and the equality occurs if and only if $N^2 = 0$. If $A^2 = A$ and $s \geq 1$ then $\dim M(A) \geq 3 + r + s = 2 + \dim A$ and, when $\dim A \geq 3$, there is up to isomorphism one Bernstein algebra where $A^2 = A$ and the equality occurs.

We summarize some more main results obtained in [45] and by R. Costa and L. Murakami [46]. Let (A,ω), (A_1,ω_1) and (A_2,ω_2) be finite dimensional Bernstein algebras. (i) (A,ω) is normal if and only if $\dim M(A) = 2 + 2\,\dim U + \dim U^2 - \dim L$. (ii) Assume that (A_1,ω_1) and (A_2,ω_2) have isomorphic multiplication algebras. Then A_1 is exceptional (normal) if and only if A_2 is exceptional (normal). Also A_1^2 and A_2^2 have the same type. (iii) The ideal N of A is nilpotent if and only the ideal $\tilde{N} = \tilde{U} \oplus \tilde{V}$ of $M(A)$ is nilpotent. (iv)

Assume that (A, ω) has type $(1 + r, s)$. If N is nilpotent then all the idempotents in $\{\sigma \in M(A) \mid \sigma(A) \subset N\}$ have rank r. The converse is true if A is non exceptional.

Two numerical invariants of Bernstein algebras derived from their multiplication algebras were studied by R. Costa and L. Murakami [47]. The first one is the maximum rank $\rho(A)$ of the elements of $M(A)$. They investigated the bounds of $\rho(A)$ and its relations with the dimension of the invariant subspace $U(UV)$. Assume that (A, ω) has type $(1 + r, s)$. Then $1 + r \leq \rho(A) \leq r + s$, and $\rho(A) = 1 + r$ if and only if $U(UV) = 0$. In particular, when A is exceptional or normal we have $U(UV) = 0$ and then $\rho(A) = 1 + r$. More generally, if $\dim U(UV) = t$ then $\rho(A) \leq 1 + r + t$. The other invariant is the maximum dimension of $M(A)$. It is proved that

$$\dim M(A) \leq 1 + r + r^2 + rs$$

for each one of the following classes of Bernstein algebras: exceptional; $r > 1$ and $U(UV) = 0$; $r > s$ and N is nilpotent; $\dim U(UV) = 1$. For every $r \geq 1$ and $s \geq 2$, there exists an exceptional algebra with $\dim M(A) = 1 + r + r^2 + rs$.

R. Costa and L. Murakami [48] proved that if $A^2 = A$ then the group of automorphisms of $M(A)$ has a proper subgroup isomorphic to the group of automorphims of A.

6.4.6 Other invariants

Let $(A, \omega) = Fe \oplus U_e \oplus V_e$ be a finite dimensional Bernstein algebra of type $(1 + r, s)$. The set of nonzero idempotents of (A, ω) is $\mathrm{Ip}(A) = \{e + u + u^2 : u \in U_e\}$. Let $p = p(U, V)$ be a polynomial on the commutative and nonassociative variables U and V without constant term and all the coefficients equal to 1. Let $p_e = p(U_e, V_e)$ be the corresponding subspace of (A, ω). The polynomial p is invariant if $p_e = p_f$ and has invariant dimension if p_e and p_f have the same dimension, for all nonzero idempotents e and f. Well-known examples of polynomials which have invariant dimension are $U, V, UV + V^2$ and U^2.

R. Costa and J. Picanço [49] proved that every polynomial p has invariant dimension in every Bernstein-Jordan algebra, and in every Bernstein algebra if $p_e \subset V_e$. Since $U_e(U_e V_e^{(k)}) \subset V_e$ the monomial $U(UV^{(k)})$ has invariant dimension for all $k \geq 0$. The Bernstein algebra is *n-exceptional* if n is the least integer for which $U_e(U_e V_e^{(n)}) = 0$. See N. Bezerra and R. Costa [41].

N. Bezerra, R. Costa and J. Picanço [19] obtained some results for p and for the monomial UV. Let $\alpha_p = \max\{\dim p_e : e \in \mathrm{Ip}(A)\}$, $\beta_p = \min\{\dim p_e : e \in Ip(A)\}$ and $\Omega(p) = \alpha_p - \beta_p$. The main results are: (i) $0 \leq \Omega(p) \leq \dim L \leq r$, and $\dim L \leq r - 1$ when (A, ω) is n-exceptional. (ii) In a n-exceptional Bernstein algebra (A, ω) the monomial UV is invariant if and only if $n = 0$ or $n = 1$ and $U_e^2 V_e \subset U_e V_e$ for every $e \in \mathrm{Ip}(A)$. (iii) Let (A, ω) be a Bernstein algebra with $r \geq 2$. Then $\Omega(UV) = r - 1$ if and only if there exists $e \in \mathrm{Ip}(A)$ such that $U_e V_e = 0$ and $\dim U_e^2 V_e = r - 1$.

Funding L. S. I. Murakami was partially supported by Grant FAPESP 2018/23690-6; I. P. Shestakov was partially supported by grants FAPESP 2018/23690-6 and CNPq 303916/2014-1; L. A. Peresi is retired from IME-USP.

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