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Twisting functors and Gelfand–Tsetlin modules over semisimple Lie algebras

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We associate to an arbitrary positive root α of a complex semisimple finite-dimensional Lie algebra $\mathfrak{g}$ a twisting endofunctor T_α of the category of $\mathfrak{g}$-modules. We apply this functor to generalized Verma modules in the category $\mathcal{O}(\mathfrak{g})$ and construct a family of α-Gelfand–Tsetlin modules with finite Γ_α-multiplicities, where Γ_α is a commutative $\mathbb{C}$-subalgebra of the universal enveloping algebra of $\mathfrak{g}$ generated by a Cartan subalgebra of $\mathfrak{g}$ and by the Casimir element of the $\mathfrak{sl}(2)$-subalgebra corresponding to the root α. This covers classical results of Andersen and Stroppel when α is a simple root and previous results of the authors in the case when $\mathfrak{g}$ is a complex simple Lie algebra and α is the maximal root of $\mathfrak{g}$. The significance of constructed modules is that they are Gelfand–Tsetlin modules with respect to any commutative $\mathbb{C}$-subalgebra of the universal enveloping algebra of $\mathfrak{g}$ containing Γ_α. Using the Beilinson–Bernstein correspondence we give a geometric realization of these modules together with their explicit description. We also identify a tensor subcategory of the category of α-Gelfand–Tsetlin modules which contains constructed modules as well as the category $\mathcal{O}(\mathfrak{g})$.

Keywords: Twisting functor; generalized Verma module; Gelfand–Tsetlin module; Weyl algebra; $\mathcal{D}$-module.

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0. Introduction

The theory of *Gelfand–Tsetlin modules* has been developed extensively in recent years following a number of important results obtained in [8, 10–12, 14–16, 20, 23, 31, 33–36]. These modules are connected with the study of Gelfand–Tsetlin

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integrable systems (e.g. [21, 22, 29]), BGG differential operators (e.g. [8, 15]), tensor product categorifications and KLRW-algebras in [24].

Gelfand–Tsetlin modules were originally defined and studied in [7] and [6] aiming to parameterize simple modules for the universal enveloping algebra of a complex simple Lie algebra by characters of a certain maximal commutative $\mathbb{C}$-subalgebra — the *Gelfand–Tsetlin subalgebra*. Up to now the theory is mainly evolved for simple Lie algebras of type A, that is $U(\mathfrak{sl}(n))$ and its generalizations. In [16] a new approach was developed which allows us to construct certain Gelfand–Tsetlin-like modules for an arbitrary simple finite-dimensional Lie algebra $\mathfrak{g}$.

Let θ be the maximal root of $\mathfrak{g}$ and let $\mathfrak{s}_\theta$ be the Lie subalgebra of $\mathfrak{g}$ based on the root θ, and hence isomorphic to $\mathfrak{sl}(2)$. Let us denote by Γ_θ the commutative $\mathbb{C}$-subalgebra of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ generated by a Cartan subalgebra of $\mathfrak{g}$ and by the center of $U(\mathfrak{s}_\theta)$. The *θ-Gelfand–Tsetlin modules*, modules admitting a locally finite action of Γ_θ, were studied in [16]. In the case of $\mathfrak{sl}(n)$ such modules give examples of the so-called *partial Gelfand–Tsetlin modules* studied in [19]. In particular, a family of θ-Gelfand–Tsetlin modules $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\theta)$ with finite Γ_θ-multiplicities were constructed explicitly using a free field realization. Here $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ with the corresponding Cartan subalgebra $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$. Using the Beilinson–Bernstein correspondence one can view $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\theta)$ as the vector space of "δ-functions" on the flag variety G/B supported at the 1-dimensional subvariety being an orbit of a unipotent subgroup of G and going through the point eB.

Let Γ be a commutative $\mathbb{C}$-subalgebra of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Then we denote by $\mathcal{H}(\mathfrak{g},\Gamma)$ the category of Γ-weight $\mathfrak{g}$-modules and by $\mathcal{H}_{\rm fin}(\mathfrak{g},\Gamma)$ the full subcategory of $\mathcal{H}(\mathfrak{g},\Gamma)$ consisting of Γ-weight $\mathfrak{g}$-modules with finite-dimensional Γ-weight subspaces. The modules $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\theta)$ belong to the category $\mathcal{H}_{\rm fin}(\mathfrak{g},\Gamma_\theta)$ and as an easy consequence also to the category $\mathcal{H}_{\rm fin}(\mathfrak{g},\Gamma)$ for any commutative $\mathbb{C}$-subalgebra Γ of $U(\mathfrak{g})$ containing Γ_θ.

The purpose of this paper is to extend the construction of $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\theta)$ to any positive root α for an arbitrary complex semisimple finite-dimensional Lie algebra. We show that such modules can be obtained from the corresponding Verma modules by applying the twisting functor. This functor was used successfully by Mathieu [30] to classify simple torsion free weight modules with finite-dimensional weight spaces over complex simple Lie algebras of type A and C. If α is a simple root, then the twisting functor is well understood, it is related to the Arkhipov's twisting functor [4] on the category $\mathcal{O}(\mathfrak{g})$. We are mostly interested in the case when α is not simple root. In addition, we replace the Borel subalgebra $\mathfrak{b}$ by any standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. The corresponding *α-Gelfand–Tsetlin* module $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda,\alpha)$ is obtained by applying the twisting functor to the generalized Verma module $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)$ induced from the finite-dimensional simple $\mathfrak{p}$-modules with highest weight λ. We show that all modules $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda,\alpha)$ are cyclic weight modules with respect to the Cartan subalgebra $\mathfrak{h}$. Moreover, every element of Γ_α has a Jordan decomposition on $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda,\alpha)$ with Jordan cells of size at most 2. The same holds for the elements of any commutative $\mathbb{C}$-subalgebra Γ of $U(\mathfrak{g})$ containing Γ_α. In the case $\mathfrak{g} = \mathfrak{sl}(3)$ the subalgebra Γ_θ is generically diagonalizable on $W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda, \theta)$ and also the latter module is generically simple, θ is the maximal root of $\mathfrak{g}$, see [16]. We expect the same properties of modules $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ in general. This will be addressed in a subsequent paper.

In addition, we provide the $\mathcal{D}$-module realization of $\mathfrak{g}$-modules $W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda, \alpha)$ using the Beilinson–Bernstein correspondence (Theorem 4.5). We also give explicit formulas for the Lie algebra action in this realization (Theorem 4.10).

One of the inspirations and motivations for our original quest was the question of Kleshchev [28] about the existence of tensor categories of Γ-Gelfand–Tsetlin modules beyond the category $\mathcal{O}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}(n)$ and $\mathfrak{g} = \mathfrak{sl}(n)$, where Γ is the Gelfand–Tsetlin subalgebra of $\mathfrak{g}$. We observe that Γ_α-Gelfand–Tsetlin modules form a tensor category which contains the category $\mathcal{O}(\mathfrak{g})$. In particular, if $\mathfrak{g} = \mathfrak{gl}(n)$ then Γ-Gelfand–Tsetlin modules which are Γ_α-Gelfand–Tsetlin modules form a tensor category if Γ contains Γ_α. We provide a recipe how to construct more general tensor categories of partial Gelfand–Tsetlin modules. The method of twisting functors was also successfully used to construct positive energy representations of affine vertex algebras, see [17].

Let us briefly summarize the content of our paper. In Sec. 1, we recall a general notion of twisting functors for rings and introduce Γ-Gelfand–Tsetlin modules for a commutative $\mathbb{C}$-subalgebra Γ of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Section 2 is devoted to study of α-Gelfand–Tsetlin modules over complex semisimple Lie algebras. We also introduce various tensor categories of α-Gelfand–Tsetlin modules. In Sec. 3, we define the twisting functor T_α for any positive root α of a complex semisimple Lie algebra $\mathfrak{g}$ as an endofunctor of the category $\mathcal{M}(\mathfrak{g})$ of $\mathfrak{g}$-modules and describe main properties of this functor. By applying the twisting functor T_α on generalized Verma modules $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ we construct α-Gelfand–Tsetlin modules $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ with finite Γ_α-multiplicities (Theorem 4.3). Basic characteristics of these modules together with their geometric realization are described in Sec. 4.

1. Preliminaries

We denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}_0$ and $\mathbb{N}$ the set of complex numbers, real numbers, integers, non-negative integers and positive integers, respectively. All algebras and modules are considered over the field of complex numbers.

1.1. *Twisting functors*

We recall the definition and basis properties of twisting functors for rings.

Definition 1.1. Let R be a ring. A multiplicative set S in R is called a left denominator set if it satisfies the following two conditions:

(i) $Sr \cap Rs \neq \emptyset$ for all $r \in R$ and $s \in S$ (the set S is left permutable);

(ii) if $rs = 0$ for $r \in R$ and $s \in S$, then there exists $s' \in S$ such that $s'r = 0$ (the set S is left reversible).

The above two conditions are usually called Ore's conditions.

If S is a left denominator set in a ring R, then we may construct a left ring of fractions for R with respect to S. Let us recall that a left ring of fractions for R with respect to S is a ring homomorphism $f : R \to T$ such that

(i) $f(s)$ is a unit in T for all $s \in S$;
(ii) any element of T can be written in the form $f(s)^{-1}f(r)$ for some $s \in S$ and $r \in R$;
(iii) $\ker f = \{r \in R;\, sr = 0 \text{ for some } s \in S\}$.

Let us note that a left ring of fractions is unique up to isomorphism, we will denote it by $S^{-1}R$.

Further, if M is a left R-module, then a module of fractions for M with respect to S is an R-module homomorphism $\varphi : M \to N$, where N is a left $S^{-1}R$-module, such that

(i) any element of N can be written in the form $s^{-1}\varphi(a)$ for some $s \in S$ and $a \in M$;
(ii) $\ker \varphi = \{a \in M;\, sa = 0 \text{ for some } s \in S\}$.

Let us note that a module of fractions is unique up to isomorphism, we will denote it by $S^{-1}M$.

Theorem 1.2. *Let S be a left denominator set in a ring R and let M be a left R-module. Then we have*

$$S^{-1}R \otimes_R M \simeq S^{-1}M, \tag{1.1}$$

where the mapping is given by $s^{-1}r \otimes a \mapsto s^{-1}ra$ for $s \in S, r \in R$ and $a \in M$.

Lemma 1.3. *Let R be a ring and let $t \in R$ be a locally* ad-*nilpotent regular element. Then the multiplicative set $S_t = \{t^n;\, n \in \mathbb{N}_0\}$ in R is a left denominator set.*

Proof. To check Ore's conditions for S_t, it is sufficient to verify them for the generator t of S_t. As t is a locally ad-nilpotent element, for any $r \in R$ we have $\mathrm{ad}(t)^n(r) = 0$ for some $n \in \mathbb{N}$. Then the identity $\mathrm{ad}(t)^n(r) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^{n-k} r t^k = 0$ can be written as $t^n r = r' t$ for some $r' \in R$. Moreover, since t is a regular element, Ore's conditions are satisfied for t. □

Let R be a ring and let $t \in R$ be a locally ad-nilpotent regular element. A left ring of fractions for R with respect to S_t we will denote by $R_{(t)}$, and similarly a module of fractions for M with respect to S_t we will denote by $M_{(t)}$. Since t is a regular element of R, the ring homomorphism $R \to R_{(t)}$ is injective. Hence, we may regard R as a subring of $R_{(t)}$. We also have (R,R)-bimodule $R_{(t)}/R$, which enables us to define the twisting functor

$$T_t : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$$

by

$$T_t(M) = (R_{(t)}/R) \otimes_R M \tag{1.2}$$

for $M \in \mathrm{Mod}(R)$. Besides, we introduce the 1-parameter family of automorphisms $\theta_t^\nu : R_{(t)} \to R_{(t)}$ by

$$\theta_t^\nu(r) = t^\nu r t^{-\nu} = \sum_{k=0}^{\infty} \binom{\nu+k-1}{k} t^{-k}\,\mathrm{ad}(t)^k(r) \tag{1.3}$$

for $\nu \in \mathbb{C}$ and $r \in R_{(t)}$. Let us note that the sum on the right-hand side is well defined since t is a locally ad-nilpotent element of R. The following lemma is straightforward.

Lemma 1.4. *Let R be a ring and $t \in R$ a locally* ad-*nilpotent regular element. Then we have $\theta_t^\mu \circ \theta_t^\nu = \theta_t^{\mu+\nu}$ for $\mu, \nu \in \mathbb{C}$.*

Proposition 1.5. *Let R be a ring and $t \in R$ a locally* ad-*nilpotent regular element. Then the twisting functor T_t is right exact.*

In this paper, we will mostly interested in two special cases when the ring R is the universal enveloping algebra of a complex semisimple Lie algebra or the Weyl algebra of a complex vector space.

1.2. Γ-*Gelfand–Tsetlin modules*

Let $\mathfrak{g}$ be a complex semisimple finite-dimensional Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, by Δ^+ a positive root system in Δ and by $\Pi \subset \Delta^+$ the set of simple roots. For $\alpha \in \Delta^+$, let $h_\alpha \in \mathfrak{h}$ be the corresponding coroot and let e_α and f_α be basis of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, respectively, defined by the requirement $[e_\alpha, f_\alpha] = h_\alpha$. We also set

$$Q = \sum_{\alpha\in\Delta^+} \mathbb{Z}\alpha \quad \text{and} \quad Q_+ = \sum_{\alpha\in\Delta^+} \mathbb{N}_0\alpha$$

together with

$$\Lambda = \{\lambda \in \mathfrak{h}^*; (\forall \alpha \in \Pi)\lambda(h_\alpha) \in \mathbb{Z}\}, \quad \Lambda^+ = \{\lambda \in \mathfrak{h}^*; (\forall \alpha \in \Pi)\lambda(h_\alpha) \in \mathbb{N}_0\}.$$

We call Q the root lattice and Λ the weight lattice. Further, we define the Weyl vector $\rho \in \mathfrak{h}^*$ by

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha.$$

The standard Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is defined through $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ with the nilradical $\mathfrak{n}$ and the opposite nilradical $\bar{\mathfrak{n}}$ given by

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \quad \text{and} \quad \bar{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}.$$

Besides, we have the corresponding triangular decomposition

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$$

of the Lie algebra $\mathfrak{g}$. Further, let $(\cdot\,,\cdot)_\mathfrak{g} : \mathfrak{g} \otimes_\mathbb{C} \mathfrak{g} \to \mathbb{C}$ be the Cartan–Killing form on $\mathfrak{g}$. Whenever $\alpha \in \mathfrak{h}^*$ satisfies $(\alpha, \alpha)_\mathfrak{g} \neq 0$, we define $s_\alpha \in \mathrm{GL}(\mathfrak{h}^*)$ by

$$s_\alpha(\gamma) = \gamma - \frac{2(\alpha, \gamma)_\mathfrak{g}}{(\alpha, \alpha)_\mathfrak{g}}\, \alpha$$

for $\gamma \in \mathfrak{h}^*$. The subgroup W of $\mathrm{GL}(\mathfrak{h}^*)$ defined through

$$W = \langle s_\alpha;\ \alpha \in \Pi \rangle$$

is called the Weyl group of $\mathfrak{g}$. Let us note that W is a finite Coxeter group. Let $\mathfrak{z}_\mathfrak{g}$ be the center of $U(\mathfrak{g})$ and let $\gamma : \mathfrak{z}_\mathfrak{g} \to U(\mathfrak{h})$ be the Harish-Chandra homomorphism (its image coincides with the set of W-invariant elements of $U(\mathfrak{h})$). For each $\lambda \in \mathfrak{h}^*$ we define a central character $\chi_\lambda : \mathfrak{z}_\mathfrak{g} \to \mathbb{C}$ by

$$\chi_\lambda(z) = (\gamma(z))(\lambda)$$

for $z \in \mathfrak{z}_\mathfrak{g}$, where we identify $U(\mathfrak{h})$ with the $\mathbb{C}$-algebra of polynomial functions on $\mathfrak{h}^*$.

For a commutative $\mathbb{C}$-algebra Γ we denote by $\mathrm{Hom}(\Gamma, \mathbb{C})$ the set of all characters of Γ, i.e. $\mathbb{C}$-algebra homomorphisms from Γ to $\mathbb{C}$. Let us note that if Γ is finitely generated, then there is a natural identification between the set $\mathrm{Hom}(\Gamma, \mathbb{C})$ of all characters of Γ and the set $\mathrm{Specm}\,\Gamma$ of all maximal ideals of Γ, which corresponds to a complex algebraic variety. Let M be a Γ-module. For each $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ we set

$$M_\chi = \{v \in M;\ (\exists k \in \mathbb{N})\ (\forall\, a \in \Gamma)\ (a - \chi(a))^k v = 0\}. \tag{1.4}$$

When $M_\chi \neq \{0\}$, we say that χ is a Γ-weight of M, the vector space M_χ is called the Γ-weight subspace of M with weight χ and the elements of M_χ are Γ-weight vectors with weight χ. If a Γ-module M satisfies

$$M = \bigoplus_{\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})} M_\chi, \tag{1.5}$$

then we call M a Γ-*weight module*. The dimension of the vector space M_χ will be called the Γ-*multiplicity* of χ in M. Let us note that any submodule or factor-module of a Γ-weight module is also a Γ-weight module.

Let Γ be a commutative $\mathbb{C}$-subalgebra of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Then we denote by $\mathcal{H}(\mathfrak{g}, \Gamma)$ the category of all Γ-weight $\mathfrak{g}$-modules and by $\mathcal{H}_{\mathrm{fin}}(\mathfrak{g}, \Gamma)$ the full subcategory of $\mathcal{H}(\mathfrak{g}, \Gamma)$ consisting of Γ-weight $\mathfrak{g}$-modules with finite-dimensional Γ-weight subspaces. If Γ and Γ' are commutative $\mathbb{C}$-subalgebras of $U(\mathfrak{g})$ satisfying $\Gamma \subset \Gamma'$, then we have the following obvious inclusions

$$\mathcal{H}_{fin}(\mathfrak{g}, \Gamma) \subset \mathcal{H}_{fin}(\mathfrak{g}, \Gamma') \subset \mathcal{H}(\mathfrak{g}, \Gamma') \subset \mathcal{H}(\mathfrak{g}, \Gamma), \tag{1.6}$$

which are strict in general. If Γ contains the Cartan subalgebra $\mathfrak{h}$, then a Γ-weight $\mathfrak{g}$-module M is called a Γ-*Gelfand–Tsetlin* module. In particular, a usual *weight* $\mathfrak{g}$-module M, i.e. a $\mathfrak{g}$-module M satisfying

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda,$$

where $M_\lambda = \{v \in M;\ (\forall h \in \mathfrak{h})\, hv = \lambda(h)v\}$, is a $U(\mathfrak{h})$-Gelfand–Tsetlin module.

Let $\{e, h, f\}$ denote the standard basis of the Lie algebra $\mathfrak{sl}(2)$. The vector subspace $\mathfrak{h} = \mathbb{C}h$ is a Cartan subalgebra of $\mathfrak{sl}(2)$. If we define $\alpha \in \mathfrak{h}^*$ by $\alpha(h) = 2$, then the root system of $\mathfrak{sl}(2)$ with respect to $\mathfrak{h}$ is $\Delta = \{\pm\alpha\}$ with $\Delta^+ = \{\alpha\}$. The standard Borel subalgebra $\mathfrak{b}$ of $\mathfrak{sl}(2)$ is defined as $\mathfrak{b} = \mathbb{C}h \oplus \mathbb{C}e$ with the nilradical $\mathfrak{n} = \mathbb{C}e$ and the opposite nilradical $\bar{\mathfrak{n}} = \mathbb{C}f$. Besides, the center $\mathfrak{z}_{\mathfrak{sl}(2)}$ of $U(\mathfrak{sl}(2))$ is freely generated by the quadratic Casimir element Cas given by

$$\mathrm{Cas} = ef + fe + \frac{1}{2}h^2.$$

If M is an $\mathfrak{sl}(2)$-module then M is also a $\mathfrak{z}_{\mathfrak{sl}(2)}$-module. The following proposition is standard but we include the details for convenience of the reader.

Proposition 1.6. (i) *Let M be an $\mathfrak{sl}(2)$-module. If for a weight vector $v \in M$ with weight $\lambda \in \mathfrak{h}^*$ there exists $n \in \mathbb{N}$ such that $f^n v = 0$, then we have*

$$\prod_{k=0}^{n-1} (z - \chi_{\lambda-\rho-k\alpha}(z))v = 0 \tag{1.7}$$

for all $z \in \mathfrak{z}_{\mathfrak{sl}(2)}$.

(ii) *Let M be a weight $\mathfrak{sl}(2)$-module which is locally $\bar{\mathfrak{n}}$-finite. Then M is a $\mathfrak{z}_{\mathfrak{sl}(2)}$-weight module. Moreover, the $\mathfrak{z}_{\mathfrak{sl}(2)}$-weights of M may be only of the form $\chi_{\lambda-\rho}$ for $\lambda \in \mathfrak{h}^*$ and Jordan blocks corresponding to the $\mathfrak{z}_{\mathfrak{sl}(2)}$-weight $\chi_{\lambda-\rho}$ are of size at most 2 if $w(\lambda - \rho) - \rho \in \Lambda^+$ for some $w \in W$ and of size 1 otherwise.*

Proof. (i) It is sufficient to prove the statement for the generator Cas of $\mathfrak{z}_{\mathfrak{sl}(2)}$. If $n = 1$ then $fv = 0$ and we have

$$\mathrm{Cas}\, v = \left(2ef + \frac{1}{2}h(h-2)\right) v = \chi_{\lambda-\rho}(\mathrm{Cas})v.$$

The rest of the proof is by induction on n. Let us assume that the statement holds for some $n \in \mathbb{N}$. If for a weight vector $v \in M$ with weight $\lambda \in \mathfrak{h}^*$ holds $f^{n+1}v = 0$, then $w = fv$ is a weight vector with weight $\lambda - \alpha \in \mathfrak{h}^*$ and $f^n w = 0$, which by the induction assumption gives us

$$\begin{aligned} 0 &= \prod_{k=1}^{n}(\mathrm{Cas} - \chi_{\lambda-\rho-k\alpha}(\mathrm{Cas}))w = \prod_{k=1}^{n}(\mathrm{Cas} - \chi_{\lambda-\rho-k\alpha}(\mathrm{Cas}))fv \\ &= f\prod_{k=1}^{n}(\mathrm{Cas} - \chi_{\lambda-\rho-k\alpha}(\mathrm{Cas}))v. \end{aligned}$$

Together with the fact $\frac{1}{2}h(h-2)v = \chi_{\lambda-\rho}(\mathrm{Cas})v$, we immediately obtain

$$\begin{aligned} 0 &= \left(2ef + \frac{1}{2}h(h-2) - \chi_{\lambda-\rho}(\mathrm{Cas})\right)\prod_{k=1}^{n}(\mathrm{Cas} - \chi_{\lambda-\rho-k\alpha}(\mathrm{Cas}))v \\ &= \prod_{k=0}^{n}(\mathrm{Cas} - \chi_{\lambda-\rho-k\alpha}(\mathrm{Cas}))v. \end{aligned}$$

This implies the first statement.

(ii) Let $v \in M$ be a weight vector with weight $\lambda \in \mathfrak{h}^*$. Since M is locally $\bar{\mathfrak{n}}$-finite, there exists $n \in \mathbb{N}$ such that $f^n v = 0$. We have

$$\prod_{k=0}^{n-1}(z - \chi_{\lambda-\rho-k\alpha}(z))v = 0$$

for all $z \in \mathfrak{z}_{\mathfrak{sl}(2)}$, which implies that $v \in \sum_{k=0}^{n-1} M_{\chi_{\lambda-\rho-k\alpha}}$. By using the fact that M is a weight $\mathfrak{sl}(2)$-module we immediately obtain that M is a $\mathfrak{z}_{\mathfrak{sl}(2)}$-weight module and the $\mathfrak{z}_{\mathfrak{sl}(2)}$-weights of M may be only of the form $\chi_{\mu-\rho}$ for $\mu \in \mathfrak{h}^*$.

Since $\chi_{\mu_1-\rho} = \chi_{\mu_2-\rho}$ for $\mu_1, \mu_2 \in \mathfrak{h}^*$ and $\mu_1 \neq \mu_2$ if and only if $\mu_2 - \rho = s(\mu_1 - \rho) = -\mu_1 + \rho$, where s is the nontrivial element of the Weyl group W of $\mathfrak{sl}(2)$, we get that $\chi_{\lambda-\rho-k_1\alpha} = \chi_{\lambda-\rho-k_2\alpha}$ for $0 \leq k_1 < k_2$ provided $\lambda - \rho = \frac{1}{2}(k_1+k_2)\alpha$. Moreover, we have $\lambda - \rho - k_1\alpha = \frac{1}{2}(k_2 - k_1)\alpha$ and $\lambda - \rho - k_2\alpha = -\frac{1}{2}(k_2 - k_1)\alpha$. Therefore, the multiplicity of a $\mathfrak{z}_{\mathfrak{sl}(2)}$-character $\chi_{\mu-\rho}$ for $\mu \in \mathfrak{h}^*$ occurring in the decomposition $\prod_{k=0}^{n-1}(z - \chi_{\lambda-\rho-k\alpha}(z))$ is at most 2 and it happens only if $w(\mu-\rho) - \rho \in \Lambda^+$ for some $w \in W$, otherwise is 1. This gives us the second statement. □

2. α-Gelfand–Tsetlin Modules

2.1. *Definitions and examples*

Let us denote by $\mathfrak{s}_\alpha$ for a positive root $\alpha \in \Delta^+$ the Lie subalgebra of $\mathfrak{g}$ determined through the $\mathfrak{sl}(2)$-triple $(e_\alpha, h_\alpha, f_\alpha)$. Then the quadratic Casimir element Cas_α given by

$$\mathrm{Cas}_\alpha = e_\alpha f_\alpha + f_\alpha e_\alpha + \frac{1}{2}h_\alpha^2 \tag{2.1}$$

is a free generator of the center $\mathfrak{z}_{\mathfrak{s}_\alpha}$ of $U(\mathfrak{s}_\alpha)$.

Definition 2.1. Let us denote by Γ_α for $\alpha \in \Delta^+$ the commutative $\mathbb{C}$-subalgebra of $U(\mathfrak{g})$ generated by the Cartan subalgebra $\mathfrak{h}$ and by the center $\mathfrak{z}_{\mathfrak{s}_\alpha}$ of $U(\mathfrak{s}_\alpha)$. The $\mathbb{C}$-algebra Γ_α is freely generated by the coroots h_γ for $\gamma \in \Pi$ and by the quadratic Casimir element Cas_α.

The Γ_α-Gelfand–Tsetlin modules will be simply called *α-Gelfand–Tsetlin modules.* When α is the maximal root of a simple Lie algebra $\mathfrak{g}$ such modules were considered in [16].

The following proposition is an immediate consequence of Proposition 1.6.

Proposition 2.2. *Let $\alpha \in \Delta^+$ and let M be a weight $\mathfrak{g}$-module which is locally $\mathfrak{s}_\alpha^-$-finite. Then M is an α-Gelfand–Tsetlin module with Jordan blocks of size at most* 2.

Let us consider a weight $\mathfrak{g}$-module M. Since $[\mathfrak{h}, \mathfrak{s}_\alpha^-] = \mathfrak{s}_\alpha^-$ for $\alpha \in \Delta^+$, the Lie algebra cohomology groups $H^n(\mathfrak{s}_\alpha^-; M)$ are weight $\mathfrak{h}$-modules for all $n \in \mathbb{N}_0$. Let us note that $H^n(\mathfrak{s}_\alpha^-; M) = 0$ for all $n > 1$ and $H^0(\mathfrak{s}_\alpha^-; M) \simeq M^{\mathfrak{s}_\alpha^-}$, $H^1(\mathfrak{s}_\alpha^-; M) \simeq M/\mathfrak{s}_\alpha^- M$.

Theorem 2.3. *Let $\alpha \in \Delta^+$ and let M be a weight $\mathfrak{g}$-module which is locally $\mathfrak{s}_\alpha^-$-finite. Then M is an α-Gelfand–Tsetlin module with finite Γ_α-multiplicities if and only if the zeroth cohomology group $H^0(\mathfrak{s}_\alpha^-; M)$ is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces.*

Proof. As f_α acts locally nilpotently on M, by Proposition 1.6 we obtain that M is a $\mathfrak{z}_{\mathfrak{s}_\alpha}$-weight module and hence Γ_α-weight module. Further, since M_μ for $\mu \in \mathfrak{h}^*$ is a $\mathfrak{z}_{\mathfrak{s}_\alpha}$-module, we only need to show that M_μ has finite $\mathfrak{z}_{\mathfrak{s}_\alpha}$-multiplicities if and only if $H^0(\mathfrak{s}_\alpha^-; M)$ is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces.

Let us define an increasing filtration $\{F_n M_\mu\}_{n \in \mathbb{N}_0}$ of M_μ by $F_n M_\mu = \{v \in M_\mu;\ f_\alpha^n v = 0\}$. Since $F_n M_\mu$ is a $\mathfrak{z}_{\mathfrak{s}_\alpha}$-module for $n \in \mathbb{N}_0$, we get that $F^n M_\mu = F_{n+1} M_\mu / F_n M_\mu$ is a $\mathfrak{z}_{\mathfrak{s}_\alpha}$-module for $n \in \mathbb{N}_0$. Therefore, for $z \in \mathfrak{z}_{\mathfrak{s}_\alpha}$ we have the induced linear mapping

$$\mathrm{gr}_n^F\ z : F^n\ M_\mu \to F^n\ M_\mu.$$

Moreover, by Proposition 1.6 we have

$$(z - \chi_{\mu_\alpha - \rho_\alpha - n\alpha}(z)) f_\alpha^n v = 0$$

for $v \in F_{n+1} M_\mu$ and $z \in \mathfrak{z}_{\mathfrak{s}_\alpha}$, where $\rho_\alpha = \rho_{\mathfrak{s}_\alpha} = \frac{1}{2}\alpha$ and $\mu_\alpha = \mu_{|\mathfrak{s}_\alpha \cap \mathfrak{h}}$, which immediately implies that $(z - \chi_{\mu_\alpha - \rho_\alpha - n\alpha}(z)) v \in F_n M_\mu$. Hence, we have $\mathrm{gr}_n^F z_{|F^n M_\mu} = \chi_{\mu_\alpha - \rho_\alpha - n\alpha}(z)\, \mathrm{id}_{F^n M_\mu}$, which means that $F^n M_\mu = (F^n M_\mu)_{\chi_{\mu_\alpha - \rho_\alpha - n\alpha}}$. Using the fact that

$$(F_{n+1} M_\mu)_\chi / (F_n M_\mu)_\chi \simeq (F^n\ M_\mu)_\chi$$

for $\chi \in \mathrm{Hom}(\mathfrak{z}_{\mathfrak{s}_\alpha}, \mathbb{C})$, we obtain

$$\dim(M_\mu)_{\chi_{\mu_\alpha-\rho_\alpha-n\alpha}} = \begin{cases} \dim F^n\ M_\mu & \text{if } \mu(h_\alpha)-n-1 \notin \mathbb{N}_0\backslash\{n\}, \\ \dim F^n\ M_\mu + \dim F^{\mu(h_\alpha)-n-1}M_\mu & \text{if } \mu(h_\alpha)-n-1 \in \mathbb{N}_0\backslash\{n\} \end{cases}$$

for $n \in \mathbb{N}_0$, where we used that $\chi_{\mu_\alpha-\rho_\alpha-k\alpha} = \chi_{\mu_\alpha-\rho_\alpha-n\alpha}$ if and only if $k = n$ or $k = \mu(h_\alpha) - n - 1$ and $k \in \mathbb{N}_0$.

Now, let us assume that $H^0(\mathfrak{s}_\alpha^-; M) \simeq F_1M$ is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces. We show by induction on n that $\dim F_nM_\mu < \infty$ for $n \in \mathbb{N}_0$ and $\mu \in \mathfrak{h}^*$. For $n = 0$ it is trivial and for $n = 1$ it follows from the requirement on $H^0(\mathfrak{s}_\alpha^-; M)$. Let us assume that it holds for some $n \in \mathbb{N}$ and let us consider the linear mapping $f^n_{\alpha\,|F_{n+1}M} : F_{n+1}M \to F_1M$. Since $\operatorname{Ker} f_\alpha^n = F_nM$, we obtain that the induced linear mapping $f^n_{\alpha\,|F_{n+1}M} : F_{n+1}M/F_nM \to F_1M$ is injective. As we have $f_\alpha^n F_{n+1}M_\mu \subset F_1M_{\mu-n\alpha}$, $\dim F_1M_{\mu-n\alpha} < \infty$ and by the induction assumption also $\dim F_nM_\mu < \infty$, we get $\dim F_{n+1}M_\mu < \infty$. Hence, we have $\dim\,(M_\mu)_\chi < \infty$ for $\chi \in \mathrm{Hom}(\mathfrak{z}_{\mathfrak{s}_\alpha}, \mathbb{C})$.

On the other hand, if M_μ has finite $\mathfrak{z}_{\mathfrak{s}_\alpha}$-multiplicities, then we have $\dim\,(F_1M)_\mu = \dim F_1M_\mu = \dim F^1M_\mu \leq \dim\,(M_\mu)_{\chi_{\mu_\alpha-\rho_\alpha-\alpha}} < \infty$. Hence, we have that $H^0(\mathfrak{s}_\alpha^-; M) \simeq F_1M$ is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces. □

By duality we can analogously prove a similar statement if we replace f_α by e_α and $H^0(\mathfrak{s}_\alpha^-; M)$ by $H^0(\mathfrak{s}_\alpha^+; M)$.

2.2. *Categories of α-Gelfand–Tsetlin modules*

We denote by $\mathcal{M}(\mathfrak{g})$ the category of $\mathfrak{g}$-modules and by $\mathcal{E}(\mathfrak{g})$ the category of finite-dimensional $\mathfrak{g}$-modules. For a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ we introduce the full subcategory $\mathcal{I}(\mathfrak{g}, \mathfrak{a})$ of $\mathcal{M}(\mathfrak{g})$ consisting of locally $\mathfrak{a}$-finite weight $\mathfrak{g}$-modules and the full subcategory $\mathcal{I}_f(\mathfrak{g}, \mathfrak{a})$ of $\mathcal{M}(\mathfrak{g})$ of finitely generated locally $\mathfrak{a}$-finite weight $\mathfrak{g}$-modules. Therefore, for $\alpha \in \Delta^+$ we have the full subcategories $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$ and $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$ of $\mathcal{M}(\mathfrak{g})$ assigned to the Lie subalgebras $\mathfrak{s}_\alpha^+ = \mathfrak{s}_\alpha \cap \mathfrak{n}$ and $\mathfrak{s}_\alpha^- = \mathfrak{s}_\alpha \cap \bar{\mathfrak{n}}$ of $\mathfrak{g}$.

The following statement is obvious, see Proposition 1.3 in [16] for details.

Proposition 2.4. (i) *The categories $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$ and $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$ are full subcategories of $\mathcal{H}(\mathfrak{g}, \Gamma_\alpha)$ for $\alpha \in \Delta^+$.*

(ii) *Let $M \in \mathcal{M}(\mathfrak{g})$ be a simple weight $\mathfrak{g}$-module and let $\alpha \in \Delta^+$. If there exists a positive integer $n \in \mathbb{N}$ and a nonzero vector $v \in M$ such that $e_\alpha^n v = 0$(or $f_\alpha^n v = 0$), then $M \in \mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$ (or $M \in \mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$).*

We introduce $\mathcal{I}_{fin}(\mathfrak{g}, \mathfrak{s}_\alpha^+) = \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^+) \cap \mathcal{H}_{\mathrm{f}in}(\mathfrak{g}, \Gamma_\alpha)$ and $\mathcal{I}_{fin}(\mathfrak{g}, \mathfrak{s}_\alpha^-) = \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^-) \cap \mathcal{H}_{\mathrm{f}in}(\mathfrak{g}, \Gamma_\alpha)$ for $\alpha \in \Delta^+$.

Remark.

- We have $\mathcal{E}(\mathfrak{g}) \subset \mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^+) \cap \mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$. Moreover, it holds $\mathcal{E}(\mathfrak{g}) = \mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^+) \cap \mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$ provided $\mathfrak{g} = \mathfrak{sl}(2)$;
- The category $\mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$ contains the category $\mathcal{O}(\mathfrak{g})$.
- Examples of simple $\mathfrak{g}$-modules with infinite-dimensional weight spaces but with finite Γ_α-multiplicities were constructed in [16] for the maximal root α of a simple Lie algebra $\mathfrak{g}$.

Proposition 2.5. *Let $\mathfrak{a}$ be a nilpotent Lie subalgebra of $\mathfrak{g}$. Then the category $\mathcal{I}(\mathfrak{g}, \mathfrak{a})$ is a tensor category.*

Proof. Since $\mathfrak{a}$ is a nilpotent Lie algebra, the condition that a $\mathfrak{g}$-module M is locally $\mathfrak{a}$-finite is equivalent to saying that each element $a \in \mathfrak{a}$ acts locally nilpotently on M. The rest of the proof follows immediately from the definition of the tensor product of $\mathfrak{g}$-modules. □

Let us note that the categories $\mathcal{H}(\mathfrak{g}, \Gamma_\alpha)$, $\mathcal{H}_{\text{fin}}(\mathfrak{g}, \Gamma_\alpha)$, $\mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^+)$, $\mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^-)$, $\mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$, $\mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$ and $\mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^+) \cap \mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$ are not tensor categories (they are not closed with respect to the tensor product of $\mathfrak{g}$-modules) for $\alpha \in \Delta^+$, except the case $\mathfrak{g} = \mathfrak{sl}(2)$ when $\mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^+) \cap \mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^-) = \mathcal{E}(\mathfrak{g})$. On the other hand, by Proposition 2.5 we have that $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$ and $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$ are tensor categories.

Let Γ be a commutative $\mathbb{C}$-subalgebra of $U(\mathfrak{g})$ containing Γ_α. Then we have the following chain of embeddings

$$\mathcal{H}_{fin}(\mathfrak{g}, \Gamma_\alpha) \subset \mathcal{H}_{fin}(\mathfrak{g}, \Gamma) \subset \mathcal{H}(\mathfrak{g}, \Gamma) \subset \mathcal{H}(\mathfrak{g}, \Gamma_\alpha)$$

of categories. In particular, we see that $\mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^\pm)$ is a full subcategory of $\mathcal{H}_{\text{fin}}(\mathfrak{g}, \Gamma)$. However, $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^\pm)$ and $\mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^\pm)$ are not subcategories of $\mathcal{H}(\mathfrak{g}, \Gamma)$ in general. Also let us note that neither $\mathcal{H}(\mathfrak{g}, \Gamma)$ nor its subcategories $\mathcal{H}(\mathfrak{g}, \Gamma) \cap \mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$ and $\mathcal{H}(\mathfrak{g}, \Gamma) \cap \mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$ are tensor categories if $\mathfrak{g} \neq \mathfrak{sl}(2)$. Nevertheless, we have the following result for $\mathfrak{sl}(3)$.

Theorem 2.6. *Let $\mathfrak{g} = \mathfrak{sl}(3)$ and $\alpha \in \Delta^+$. Let V and W be simple $\mathfrak{g}$-modules in $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$(or in $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$). Then the modules V, W and every simple subquotient of $V \otimes_{\mathbb{C}} W$ belong to the category $\mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$ (or to the category $\mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$) and hence to the category $\mathcal{H}_{\text{fin}}(\mathfrak{g}, \Gamma)$ for any commutative $\mathbb{C}$-subalgebra Γ of $U(\mathfrak{g})$ containing Γ_α.*

Proof. Indeed, as Γ is a commutative $\mathbb{C}$-subalgebra containing Γ_α, then either $\Gamma = \Gamma_\alpha$ or it is generated over Γ_α by central elements of $U(\mathfrak{g})$. Then the simplicity of modules and Proposition 2.4(i) imply that V and W belong to the category $\mathcal{H}(\mathfrak{g}, \Gamma)$ for any commutative $\mathbb{C}$-subalgebra Γ of $U(\mathfrak{g})$ containing Γ_α. In particular, we have $V, W \in \mathcal{H}(\mathfrak{g}, \Gamma)$, where Γ is a Gelfand–Tsetlin subalgebra generated by Γ_α and by the center of $U(\mathfrak{g})$. The structure of simple Γ-Gelfand–Tsetlin modules for $\mathfrak{sl}(3)$ was described in [13], see also [9]. It follows that $V, W \in \mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$ (or that $V, W \in \mathcal{I}_{\text{fin}}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$). Every simple subquotient U of $V \otimes_{\mathbb{C}} W$ is clearly a weight module with a locally finite action of $\mathfrak{s}_\alpha^+$ (or $\mathfrak{s}_\alpha^-$). Again, the results of [13] imply that Γ_α-multiplicities in U are finite, and the statement follows. □

Theorem 2.6 suggests a possible generalization for $\mathfrak{sl}(n)$. Indeed, let us consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$ with a triangular decomposition $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ and with the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ such that the corresponding Cartan matrix $A = (a_{ij})_{1\leq i,j\leq n-1}$ is given by $a_{ii} = 2$, $a_{ij} = -1$ if $|i-j| = 1$ and $a_{ij} = 0$ if $|i-j| \geq 2$. Further, let us denote by $\mathfrak{g}_k$ for $k = 1, 2, \dots, n$ the Lie subalgebra of $\mathfrak{g}$ generated by the root subspaces $\mathfrak{g}_{\alpha_1}, \dots, \mathfrak{g}_{\alpha_{k-1}}$ and $\mathfrak{g}_{-\alpha_1}, \dots, \mathfrak{g}_{-\alpha_{k-1}}$. Then we obtain a finite sequence

$$0 = \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \dots \subset \mathfrak{g}_{n-1} = \mathfrak{g}$$

of Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g}_k \simeq \mathfrak{sl}(k)$ for $k = 2, 3, \dots, n$. We have also the induced triangular decomposition

$$\mathfrak{g}_k = \bar{\mathfrak{n}}_k \oplus \mathfrak{h}_k \oplus \mathfrak{n}_k$$

of the Lie algebra $\mathfrak{g}_k$ for $k = 2, 3, \dots, n$, where $\bar{\mathfrak{n}}_k = \bar{\mathfrak{n}} \cap \mathfrak{g}_k$, $\mathfrak{n}_k = \mathfrak{n} \cap \mathfrak{g}_k$ and $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{g}_k$. Besides, we have a sequence $U(\mathfrak{g}_2) \subset U(\mathfrak{g}_3) \subset \dots \subset U(\mathfrak{g}_n)$ of $\mathbb{C}$-subalgebras of the universal enveloping algebra $U(\mathfrak{g})$. Let us denote by $\mathfrak{z}_{\mathfrak{g}_k}$ the center of $U(\mathfrak{g}_k)$ for $k = 2, 3, \dots, n$. Then the Gelfand–Tsetlin subalgebra Γ of $U(\mathfrak{g})$ is generated by $\mathfrak{z}_{\mathfrak{g}_k}$ for $k = 2, 3, \dots, n$ and by the Cartan subalgebra $\mathfrak{h}$, cf. [6]. It is a maximal commutative $\mathbb{C}$-subalgebra of $U(\mathfrak{g})$.

The following theorem provides a generalization of Theorem 2.6 to the Lie algebra $\mathfrak{sl}(n)$.

Theorem 2.7. *Let* $\mathfrak{g} = \mathfrak{sl}(n)$.

(i) *For* $k = 2, 3, \dots, n$, *the categories* $\mathcal{I}(\mathfrak{g}, \mathfrak{n}_k)$ *and* $\mathcal{I}(\mathfrak{g}, \bar{\mathfrak{n}}_k)$ *are tensor categories.*
(ii) *Let* V *and* W *be simple* $\mathfrak{g}$*-modules in* $\mathcal{I}(\mathfrak{g}, \mathfrak{n}_{n-1})$ *or in* $\mathcal{I}(\mathfrak{g}, \bar{\mathfrak{n}}_{n-1})$. *Then the* $\mathfrak{g}$*-modules* V, W *and every simple subquotient of* $V \otimes_{\mathbb{C}} W$ *belongs to* $\mathcal{H}(\mathfrak{g}, \Gamma)$.

Proof. The first statement follows from Proposition 2.5. The proof of the second statement is analogous to the proof of Theorem 2.6. □

3. Twisting Functors and Generalized Verma Modules

We define the twisting functor T_α assigned to a positive root $\alpha \in \Delta^+$ of a semisimple Lie algebra $\mathfrak{g}$ and describe main properties of this functor. By applying of the twisting functor T_α to generalized Verma modules we construct α-Gelfand–Tsetlin $\mathfrak{g}$-modules with finite Γ_α-multiplicities. If α is a simple root, then these modules are twisted Verma modules up to conjugation of the action of $\mathfrak{g}$, see [1], [26], [32].

3.1. *Twisting functor T_α*

We use the notation introduced in previous sections. Let $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta^+$ be nonzero elements of $\mathfrak{g}$ satisfying $[e_\alpha, f_\alpha] = h_\alpha$. The multiplicative set $\{f_\alpha^n;\ n \in \mathbb{N}_0\}$ in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is a left (right) denominator set, since f_α is a locally ad-nilpotent regular element. Therefore, based on the previous general construction we have defined the *twisting functor*

$$T_\alpha = T_{f_\alpha} : \mathcal{M}(\mathfrak{g}) \to \mathcal{M}(\mathfrak{g})$$

for $\alpha \in \Delta^+$.

Next we prove some basic characteristics of the twisting functor T_α for $\alpha \in \Delta^+$. Let us note that the twisting functor T_α is usually considered only for a simple root $\alpha \in \Pi$ in the literature (see e.g. [2]), which is caused by the fact that T_α preserves the category $\mathcal{O}(\mathfrak{g})$ up to conjugation of the action of $\mathfrak{g}$.

For $\alpha \in \Delta^+$ let us consider the subset

$$\Phi_\alpha = \{\gamma \in \Delta \backslash \{\pm\alpha\};\ \mathfrak{g}_{\gamma,\alpha} \subset \mathfrak{n}\}$$

of Δ, where $\mathfrak{g}_{\gamma,\alpha}$ for $\gamma \in \Delta$ is the simple finite-dimensional $\mathfrak{s}_\alpha$-module given by

$$\mathfrak{g}_{\gamma,\alpha} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{\gamma + j\alpha}.$$

It easily follows that Φ_α is closed, i.e. if $\gamma_1, \gamma_2 \in \Phi_\alpha$ and $\gamma_1 + \gamma_2 \in \Delta$, then we have $\gamma_1 + \gamma_2 \in \Phi_\alpha$, and that $\Phi_\alpha \subset \Delta^+$. Therefore, the subset Φ_α of Δ^+ gives rise to the Lie subalgebras $\mathfrak{t}_\alpha^+$, $\mathfrak{t}_\alpha^-$ and $\mathfrak{t}_\alpha$ of $\mathfrak{g}$ defined by

$$\mathfrak{t}_\alpha^+ = \mathfrak{g}_\alpha \oplus \bigoplus_{\gamma \in \Phi_\alpha} \mathfrak{g}_\gamma, \quad \mathfrak{t}_\alpha = \bigoplus_{\gamma \in \Phi_\alpha} \mathfrak{g}_\gamma, \quad \mathfrak{t}_\alpha^- = \mathfrak{g}_{-\alpha} \oplus \bigoplus_{\gamma \in \Phi_\alpha} \mathfrak{g}_\gamma.$$

As we have $\mathfrak{s}_\alpha^\pm \subset \mathfrak{t}_\alpha^\pm$ for $\alpha \in \Delta^+$, we get immediately $\mathcal{I}(\mathfrak{g}, \mathfrak{t}_\alpha^\pm) \subset \mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^\pm)$. Besides, we have

$$\mathrm{Ad}(\dot{s}_\alpha)(\mathfrak{s}_\alpha^\pm) = \mathfrak{s}_\alpha^\mp \quad \text{and} \quad \mathrm{Ad}(\dot{s}_\alpha)(\mathfrak{t}_\alpha^\pm) = \mathfrak{t}_\alpha^\mp$$

for $\alpha \in \Delta^+$, where $\dot{s}_\alpha \in N_G(H)$ is a representative of the element $s_\alpha \in W \simeq N_G(H)/H$, where G and H are connected algebraic groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. The inclusions $\mathfrak{s}_\alpha^+ \subset \mathfrak{t}_\alpha^+ \subset \mathfrak{n}$ of Lie algebras give rise to the embeddings

$$\mathcal{O}(\mathfrak{g}) \subset \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^+) \subset \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^+) \tag{3.1}$$

of categories for $\alpha \in \Delta^+$.

Proposition 3.1. *Let M be a $\mathfrak{g}$-module. Then $T_\alpha(M)$ is a locally $\mathfrak{s}_\alpha^-$-finite $\mathfrak{g}$-module for any $\alpha \in \Delta^+$. Moreover, if M is a locally $\mathfrak{s}_\alpha^-$-finite $\mathfrak{g}$-module, then $T_\alpha(M) = 0$.*

Proof. Let M be a $\mathfrak{g}$-module. Then by definition we have

$$T_\alpha(M) = \left(\frac{U(\mathfrak{g})_{(f_\alpha)}}{U(\mathfrak{g})}\right) \otimes_{U(\mathfrak{g})} M \simeq M_{(f_\alpha)}/M$$

for $\alpha \in \Delta^+$. Since every element of $M_{(f_\alpha)}$ can be written in the form $f_\alpha^{-n}v$ for $n \in \mathbb{N}_0$ and $v \in M$, we obtain immediately that $T_\alpha(M)$ is locally $\mathfrak{s}_\alpha^-$-finite. Further, let us assume that M is a locally $\mathfrak{s}_\alpha^-$-finite $\mathfrak{g}$-module. Since for each $v \in M$ there exists $n_v \in \mathbb{N}_0$ such that $f_\alpha^{n_v}v = 0$, we may write $f_\alpha^{-n}v = f_\alpha^{-n-n_v}f_\alpha^{n_v}v = 0$ for $n \in \mathbb{N}_0$. This implies the required statement. □

Lemma 3.2. *Let $\alpha \in \Delta^+$. Then we have*

$$\begin{aligned} e_\alpha f_\alpha^{-n} &= f_\alpha^{-n}e_\alpha - nf_\alpha^{-n-1}h_\alpha - n(n+1)f_\alpha^{-n-1}, \\ hf_\alpha^{-n} &= f_\alpha^{-n}h + n\alpha(h)f_\alpha^{-n}, \\ f_\alpha f_\alpha^{-n} &= f_\alpha^{-n}f_\alpha \end{aligned} \tag{3.2}$$

in $U(\mathfrak{g})_{(f_\alpha)}$ for $n \in \mathbb{Z}$ and $h \in \mathfrak{h}$.

Proof. It follows immediately from the formula

$$af_\alpha^{-n} = f_\alpha^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}f_\alpha^{-k}\,\mathrm{ad}(f_\alpha)^k(a)$$

in $U(\mathfrak{g})_{(f_\alpha)}$ for all $a \in U(\mathfrak{g})$ and $n \in \mathbb{N}_0$. □

Theorem 3.3. *We have*

(i) *if $M \in \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^+)$, then $T_\alpha(M) \in \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^-)$;*
(ii) *if $M \in \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^+)$, then $T_\alpha(M) \in \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^-)$*

for $\alpha \in \Delta^+$. Therefore, we have the restricted functors

$$T_\alpha : \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^+) \to \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^-) \quad \text{and} \quad T_\alpha : \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^+) \to \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^-)$$

for $\alpha \in \Delta^+$.

Proof. (i) If M is a weight $\mathfrak{g}$-module, then it easily follows from definition that $T_\alpha(M)$ is also a weight $\mathfrak{g}$-module. Moreover, by Proposition 3.1 we have that $T_\alpha(M)$ is locally $\mathfrak{s}_\alpha^-$-finite. Hence, the rest of the proof is to show that $T_\alpha(M)$ is finitely generated provided M is finitely generated and locally $\mathfrak{s}_\alpha^+$-finite.

Let $R \subset M$ be a finite set of generators of M. Then the vector subspace $V = U(\mathfrak{s}_\alpha^+)\langle R\rangle$ of M is finite dimensional. Further, let us introduce a filtration $\{F_kV\}_{k\in\mathbb{N}_0}$ on V by

$$F_kV = \{v \in V;\ e_\alpha^k v = 0\}$$

for $k \in \mathbb{N}_0$ and let $n_0 \in \mathbb{N}$ be the smallest positive integer satisfying

$$n_0 \geq \max\{-\mu(h_\alpha);\ \mu \in \mathfrak{h}^*,\ V_\mu \neq \{0\},\ \mu(h_\alpha) \in \mathbb{R}\}.$$

Let us consider a vector $v \in F_kV \cap V_\mu$ for $\mu \in \mathfrak{h}^*$ and $k \in \mathbb{N}_0$. Then by Lemma 3.2 we obtain

$$e_\alpha f_\alpha^{-(n_0+n)}v = f_\alpha^{-(n_0+n)}e_\alpha v - (n_0+n)(\mu(h_\alpha)+n_0+n+1)f_\alpha^{-(n_0+n+1)}v$$

for $n \in \mathbb{N}_0$, which together with $f_\alpha f_\alpha^{-n-1} v = f_\alpha^{-n} v$ for $n \in \mathbb{N}_0$ gives us

$$U(\mathfrak{s}_\alpha) f_\alpha^{-n_0} F_k V / \mathbb{C}[f_\alpha^{-1}] F_{k-1} V = \mathbb{C}[f_\alpha^{-1}] F_k V / \mathbb{C}[f_\alpha^{-1}] F_{k-1} V$$

for all $k \in \mathbb{N}$. As $F_0 V = \{0\}$ and V is a finite-dimensional vector space, we immediately get $U(\mathfrak{s}_\alpha) f_\alpha^{-n_0} V = \mathbb{C}[f_\alpha^{-1}] V$.

Further, since for any $a \in U(\mathfrak{g})$ and $n \in \mathbb{N}_0$ there exist $b \in U(\mathfrak{g})$ and $m \in \mathbb{N}_0$ satisfying $f_\alpha^{-n} a = b f_\alpha^{-m}$, we obtain $f_\alpha^{-n} U(\mathfrak{g}) \subset U(\mathfrak{g}) \mathbb{C}[f_\alpha^{-1}] \subset U(\mathfrak{g})_{(f_\alpha)}$ for $n \in \mathbb{N}_0$. Hence, we may write

$$f_\alpha^{-n} M = f_\alpha^{-n} U(\mathfrak{g}) V \subset U(\mathfrak{g}) \mathbb{C}[f_\alpha^{-1}] V,$$

which implies $M_{(f_\alpha)} = U(\mathfrak{g}) \mathbb{C}[f_\alpha^{-1}] V = U(\mathfrak{g}) f_\alpha^{-n_0} V$. In other words, this means that $T_\alpha(M) \simeq M_{(f_\alpha)}/M$ is finitely generated and the number of generators is bounded by $\dim V$.

(ii) Since $\mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^+)$ is a full subcategory of $\mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^+)$, by item (i) we only need to show that $T_\alpha(M)$ is locally $\mathfrak{t}_\alpha^-$-finite if M is locally $\mathfrak{t}_\alpha^+$-finite. As $\mathfrak{t}_\alpha^-$ is a nilpotent Lie algebra and $T_\alpha(M)$ is locally $\mathfrak{s}_\alpha^-$-finite the condition that $T_\alpha(M)$ is locally $\mathfrak{t}_\alpha^-$-finite which is equivalent to say that the element e_γ, where $\mathfrak{g}_\gamma = \mathbb{C} e_\gamma$, acts locally nilpotently on $T_\alpha(M)$ for $\gamma \in \Phi_\alpha$.

Let $\mathrm{ht} : \mathbb{Z}\Delta \to \mathbb{Z}$ be the $\mathbb{Z}$-linear height function with $\mathrm{ht}(\alpha) = 1$ if α is a simple root. Then we get an $\mathbb{N}_0$-grading $U(\mathfrak{t}_\alpha) = \bigoplus_{n \in \mathbb{N}_0} U(\mathfrak{t}_\alpha)_n$, where

$$U(\mathfrak{t}_\alpha)_n = \bigoplus_{\mu \in \mathfrak{h}^*,\ \mathrm{ht}(\mu) = n} U(\mathfrak{t}_\alpha)_\mu.$$

Let $\gamma \in \Phi_\alpha$ and let $r \in \mathbb{N}_0$ be the smallest nonnegative integer such that $\gamma - (r+1)\alpha \notin \Phi_\alpha$. Let us recall that if $\gamma - k\alpha \in \Delta$ for some $k \in \mathbb{Z}$ then $\gamma - k\alpha \in \Phi_\alpha$. Let us consider a vector $v \in M$. Then from the formula

$$a f_\alpha^{-n} = f_\alpha^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} f_\alpha^{-k} \operatorname{ad}(f_\alpha)^k(a)$$

in $U(\mathfrak{g})_{(f_\alpha)}$ for $a \in U(\mathfrak{g})$ and $n \in \mathbb{N}_0$, we obtain

$$e_\gamma^t f_\alpha^{-n} v = f_\alpha^{-n} \sum_{k=0}^{tr} \binom{n+k-1}{k} f_\alpha^{-k} \operatorname{ad}(f_\alpha)^k(e_\gamma^t) v$$

for $t \in \mathbb{N}$ and $n \in \mathbb{N}_0$, where we used the fact that $\operatorname{ad}(f_\alpha)^k(e_\gamma) \neq 0$ only for $k = 0, 1, \ldots, r$. Moreover, we have $\operatorname{ad}(f_\alpha)^k(e_\gamma^t) \in U(\mathfrak{t}_\alpha)_{t\gamma - k\alpha} \subset U(\mathfrak{t}_\alpha)_{\mathrm{ht}(t\gamma - k\alpha)}$ for $k = 0, 1, \ldots, tr$. Since M is locally $\mathfrak{t}_\alpha^+$-finite, there exists an integer $n_v \in \mathbb{N}_0$ such that $U(\mathfrak{t}_\alpha)_n v = \{0\}$ for $n > n_v$. Therefore, it is enough to show that $\mathrm{ht}(t\gamma - k\alpha) > n_v$ for $k = 0, 1, \ldots, tr$.

As we may write

$$\mathrm{ht}(t\gamma - k\alpha) \geq \mathrm{ht}(t\gamma - tr\alpha) = t\, \mathrm{ht}(\gamma - r\alpha) \geq t$$

for $k = 0, 1, \ldots, tr$, since $\gamma - r\alpha \in \Phi_\alpha$ and hence $\mathrm{ht}(\gamma - r\alpha) \geq 1$, we obtain that $e_\gamma^t f_\alpha^{-n} v = 0$ for $n \in \mathbb{N}_0$ provided $t > n_v$. Hence, the element e_γ acts locally nilpotently on $T_\alpha(M)$ for $\gamma \in \Phi_\alpha$. ◻

The following shows how the twisting functor T_α can be used to construct α-Gelfand–Tsetlin modules with finite Γ_α-multiplicities.

Theorem 3.4. *Let M be a weight $\mathfrak{g}$-module and let $\alpha \in \Delta^+$.*

(i) *The $\mathfrak{g}$-module $T_\alpha(M)$ is an α-Gelfand–Tsetlin module with finite Γ_α-multiplicities if and only if the first cohomology group $H^1(\mathfrak{s}_\alpha^-; M)$ is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces.*

(ii) *If M is a highest weight $\mathfrak{g}$-module then $T_\alpha(M)$ is a locally $\mathfrak{t}_\alpha^-$-finite cyclic weight $\mathfrak{g}$-module with finite Γ_α-multiplicities.*

Proof. (i) Let M be a weight $\mathfrak{g}$-module. Then by Proposition 3.1 we have that $T_\alpha(M)$ is a locally $\mathfrak{s}_\alpha^-$-finite weight $\mathfrak{g}$-module for $\alpha \in \Delta^+$. Hence, we may apply Theorem 2.3 on $T_\alpha(M)$ and we obtain that $T_\alpha(M)$ is an α-Gelfand–Tsetlin module with finite Γ_α-multiplicities if and only if $H^0(s_\alpha^-; T_\alpha(M))$ is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces. Further, the $\mathbb{C}$-linear mapping $\varphi_\alpha : M \to T_\alpha(M)$ defined by

$$\varphi_\alpha(v) = f_\alpha^{-1} v$$

for $v \in M$ gives rise to the $\mathbb{C}$-linear mapping

$$\widetilde{\varphi}_\alpha : H^1(\mathfrak{s}_\alpha^-; M) \to H^0(\mathfrak{s}_\alpha^-; T_\alpha(M))$$

for $\alpha \in \Delta^+$, which is in fact an isomorphism. Therefore, we obtain an isomorphism $H^1(\mathfrak{s}_\alpha^-; M) \simeq H^0(\mathfrak{s}_\alpha^-; T_\alpha(M)) \otimes_{\mathbb{C}} \mathbb{C}_{-\alpha}$ of $\mathfrak{h}$-modules, where $\mathbb{C}_{-\alpha}$ is the 1-dimensional $\mathfrak{h}$-module determined by the character $-\alpha$ of $\mathfrak{h}$, which implies the first statement.

(ii) If M is a highest weight $\mathfrak{g}$-module, then it belongs to the category $\mathcal{O}(\mathfrak{g})$. Hence, using Theorem 3.3 we obtain that $T_\alpha(M)$ is a locally $\mathfrak{t}_\alpha^-$-finite weight $\mathfrak{g}$-module. In fact, from the proof of Theorem 3.3(i) it follows that $T_\alpha(M)$ is not only finitely generated by also cyclic. To finish the proof, we need to show by Theorem 3.4 that $H^1(\mathfrak{s}_\alpha^-; M)$ is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces. Since $H^1(\mathfrak{s}_\alpha^-; M) \simeq M/\mathfrak{s}_\alpha^- M$ and M is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces, we immediately obtain that also $M/\mathfrak{s}_\alpha^- M$ is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces. This gives us the required statement. □

If we denote by $\Theta_\alpha : \mathcal{M}(\mathfrak{g}) \to \mathcal{M}(\mathfrak{g})$ the functor sending $\mathfrak{g}$-module to the same $\mathfrak{g}$-module with the action twisted by the automorphism $\mathrm{Ad}(\dot{s}_\alpha) : \mathfrak{g} \to \mathfrak{g}$, then we obtain the endofunctor

$$\Theta_\alpha \circ T_\alpha : \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^+) \to \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^+).$$

Moreover, for $\alpha \in \Pi$ we have $\mathfrak{t}_\alpha^+ = \mathfrak{n}$ which implies $\mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^+) = \mathcal{O}(\mathfrak{g})$ and in this case the functor coincides with the Arkhipov's twisting functor, see [3], [4]. On the other hand, we have $\mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^+) = \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^+)$ if α is the maximal root of a simple Lie algebra $\mathfrak{g}$.

3.2. *Left derived functor of T_α*

Let us recall that we have the following obvious inclusions

$$\mathcal{O}(\mathfrak{g}) \subset \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^+) \subset \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^+)$$

of categories for $\alpha \in \Delta^+$. For that reason, we may restrict the twisting functor T_α for $\alpha \in \Delta^+$ on the category $\mathcal{O}(\mathfrak{g})$. Since $T_\alpha : \mathcal{O}(\mathfrak{g}) \to \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^-)$ is right exact and the category $\mathcal{O}(\mathfrak{g})$ has enough projective objects, we may consider the left derived functor

$$LT_\alpha : D^b(\mathcal{O}(\mathfrak{g})) \to D^b(\mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^-))$$

of T_α for $\alpha \in \Delta^+$. The following theorem is similar to [2, Theorem 2.2].

Theorem 3.5. *Let $\alpha \in \Delta^+$. Then we have $L_iT_\alpha = 0$ for all $i > 1$. Besides, if M is a $U(\mathfrak{s}_\alpha^-)$-free $\mathfrak{g}$-module, then we also have $L_1T_\alpha(M) = 0$.*

Proof. Let $\mathcal{A}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$ be the full subcategory of $\mathcal{O}(\mathfrak{g})$ consisting of $U(\mathfrak{s}_\alpha^-)$-free $\mathfrak{g}$-modules. Clearly, it contains Verma $\mathfrak{g}$-modules and $\mathfrak{g}$-modules with a Verma flag, thus all projective objects from $\mathcal{O}(\mathfrak{g})$. For that reason, the category $\mathcal{A}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$ has enough projective objects. Besides, $U(\mathfrak{g})_{(f_\alpha)}/U(\mathfrak{g}) \simeq (U(\mathfrak{s}_\alpha^-)_{(f_\alpha)}/U(\mathfrak{s}_\alpha^-)) \otimes_{U(\mathfrak{s}_\alpha^-)} U(\mathfrak{g})$ as left $U(\mathfrak{g})$-modules, which gives us that the functor T_α is exact on $\mathcal{A}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$. Since every projective object in $\mathcal{O}(\mathfrak{g})$ is a projective object in $\mathcal{A}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$, we get $L_iT_\alpha(M) = 0$ for $i > 0$ if M is a $U(\mathfrak{s}_\alpha^-)$-free $\mathfrak{g}$-module.

Further, let us consider a $\mathfrak{g}$-module $M \in \mathcal{O}(\mathfrak{g})$ and its projective cover $P \to M$. Hence, the short exact sequence

$$0 \to N \to P \to M \to 0$$

of $\mathfrak{g}$-module and the fact that $L_iT_\alpha(P) = 0$ for $i > 0$ gives us $L_iT_\alpha(M) \simeq L_{i-1}T_\alpha(N)$ for $i > 1$. Further, since $U(\mathfrak{s}_\alpha^-)$ is a principal ideal domain and N is a $\mathfrak{g}$-submodule of a projective object in $\mathcal{O}(\mathfrak{g})$, i.e. N is a $U(\mathfrak{s}_\alpha^-)$-free $\mathfrak{g}$-module, we have $L_{i-1}T_\alpha(N) = 0$ for $i > 1$, which implies $L_iT_\alpha(M) = 0$ for $i > 1$. □

3.3. *Tensoring with finite-dimensional $\mathfrak{g}$-modules*

In this subsection we show that the twisting functors behave well with respect to tensoring with finite-dimensional $\mathfrak{g}$-modules, analogously to [2], where this is considered only for simple roots.

Let us recall that $U(\mathfrak{g})$ is a Hopf algebra with the comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$, the counit $\varepsilon : U(\mathfrak{g}) \to \mathbb{C}$ and the antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ given by

$$\Delta(a) = a \otimes 1 + 1 \otimes a, \quad \varepsilon(a) = 0, \quad S(a) = -a$$

for $a \in \mathfrak{g}$. Since $U(\mathfrak{g})_{(f_\alpha)}$ has the structure of a left $\mathbb{C}[f_\alpha^{-1}]$-module, hence also $U(\mathfrak{g})_{(f_\alpha)} \otimes_{\mathbb{C}} U(\mathfrak{g})_{(f_\alpha)}$ is a left $\mathbb{C}[f_\alpha^{-1}]$-module and we denote by $U(\mathfrak{g})_{(f_\alpha)} \mathbin{\widehat{\otimes}}_{\mathbb{C}} U(\mathfrak{g})_{(f_\alpha)}$ its extension to a left $\mathbb{C}[[f_\alpha^{-1}]]$-module, i.e. we set

$$U(\mathfrak{g})_{(f_\alpha)} \mathbin{\widehat{\otimes}}_{\mathbb{C}} U(\mathfrak{g})_{(f_\alpha)} = \mathbb{C}[[f_\alpha^{-1}]] \otimes_{\mathbb{C}[f_\alpha^{-1}]} U(\mathfrak{g})_{(f_\alpha)} \otimes_{\mathbb{C}} U(\mathfrak{g})_{(f_\alpha)}.$$

There is an obvious extension of the $\mathbb{C}$-algebra structure on $U(\mathfrak{g})_{(f_\alpha)} \otimes_{\mathbb{C}} U(\mathfrak{g})_{(f_\alpha)}$ to the completion $U(\mathfrak{g})_{(f_\alpha)} \mathbin{\widehat{\otimes}_{\mathbb{C}}} U(\mathfrak{g})_{(f_\alpha)}$.

Let $\alpha \in \Delta^+$. Then the $\mathbb{C}$-linear mapping

$$\widetilde{\Delta} : U(\mathfrak{g})_{(f_\alpha)} \to U(\mathfrak{g})_{(f_\alpha)} \mathbin{\widehat{\otimes}_{\mathbb{C}}} U(\mathfrak{g})_{(f_\alpha)}$$

given through

$$\widetilde{\Delta}(f_\alpha^{-n} u) = \left(\sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} f_\alpha^{-n-k} \otimes f_\alpha^k \right) \Delta(u),$$

where $n \in \mathbb{N}_0$ and $u \in U(\mathfrak{g})$, defines a $\mathbb{C}$-algebra homomorphism, see [2]. The following theorem is analogous to [2, Theorem 3.2].

Theorem 3.6. *Let $\alpha \in \Delta^+$. Then there is a family $\{\eta_E\}_{E \in \mathcal{E}(\mathfrak{g})}$ of natural isomorphisms*

$$\eta_E : T_\alpha \circ (\bullet \otimes_{\mathbb{C}} E) \to (\bullet \otimes_{\mathbb{C}} E) \circ T_\alpha$$

of functors such that the following diagrams commute.

(i) *For $E, F \in \mathcal{E}(\mathfrak{g})$ we have*

$$\begin{array}{ccc} T_\alpha(M \otimes_{\mathbb{C}} E \otimes_{\mathbb{C}} F) & \xrightarrow{\eta_F(M \otimes_{\mathbb{C}} E)} & T_\alpha(M \otimes_{\mathbb{C}} E) \otimes_{\mathbb{C}} F \\ & \searrow^{\eta_{E \otimes_{\mathbb{C}} F}(M)} & \downarrow{\scriptstyle \eta_E(M) \otimes \mathrm{id}_F} \\ & & T_\alpha(M) \otimes_{\mathbb{C}} E \otimes_{\mathbb{C}} F. \end{array}$$

(ii) *For $E \in \mathcal{E}(\mathfrak{g})$ we have*

$$\begin{array}{ccc} T_\alpha(M \otimes_{\mathbb{C}} E \otimes_{\mathbb{C}} E^*) & \xrightarrow{\eta_{E \otimes_{\mathbb{C}} E^*}(M)} & T_\alpha(M) \otimes_{\mathbb{C}} E \otimes_{\mathbb{C}} E^* \\ {\scriptstyle T_\alpha(\mathrm{id}_M \otimes i_E)} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{T_\alpha(M)} \otimes i_E} \\ T_\alpha(M \otimes_{\mathbb{C}} \mathbb{C}) & \xrightarrow[\eta_{\mathbb{C}}(M)]{} & T_\alpha(M) \otimes_{\mathbb{C}} \mathbb{C}, \end{array}$$

where $i_E : \mathbb{C} \to E \otimes_{\mathbb{C}} E^*$ is given by $1 \mapsto \sum_{i=1}^d e_i \otimes e_i^*$ for a fixed basis $\{e_i\}_{1 \le i \le d}$ of E with the dual basis $\{e_i^*\}_{1 \le i \le d}$ of E^*.

Remark. Theorem 3.6 implies that for any $\alpha \in \Delta^+$ the twisting functor T_α commutes with the translation functors.

3.4. *Twisting of generalized Verma modules*

In Sec. 3.1, we defined the twisting functor T_α for $\alpha \in \Delta^+$ as an endofunctor of the category $\mathcal{M}(\mathfrak{g})$. Besides, we introduce the *partial Zuckerman functor*

$$S_\alpha : \mathcal{M}(\mathfrak{g}) \to \mathcal{M}(\mathfrak{g})$$

defined by

$$S_\alpha(M) = \{v \in M;\ (\exists n \in \mathbb{N})\ f_\alpha^n v = 0\}$$

for $M \in \mathcal{M}(\mathfrak{g})$. To construct new simple weight $\mathfrak{g}$-modules with infinite-dimensional weight spaces and finite Γ_α-multiplicities we restrict both functors to the subcategory $\mathcal{O}(\mathfrak{g})$. Therefore, we have

$$T_\alpha : \mathcal{O}(\mathfrak{g}) \to \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^-) \quad \text{and} \quad S_\alpha : \mathcal{O}(\mathfrak{g}) \to \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^-),$$

which follows by Theorem 3.3.

In addition to the standard Borel subalgebra of $\mathfrak{g}$ we also consider the standard parabolic subalgebras of $\mathfrak{g}$. For a subset Σ of Π we denote by Δ_Σ the root subsystem in $\mathfrak{h}^*$ generated by Σ. Then the standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ associated to Σ is defined as $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ with the nilradical $\mathfrak{u}$ and the opposite nilradical $\bar{\mathfrak{u}}$ given by

$$\mathfrak{u} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_\Sigma} \mathfrak{g}_\alpha \quad \text{and} \quad \bar{\mathfrak{u}} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_\Sigma} \mathfrak{g}_{-\alpha}$$

and with the Levi subalgebra $\mathfrak{l}$ defined by

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_\Sigma} \mathfrak{g}_\alpha.$$

Moreover, we have the corresponding triangular decomposition

$$\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u}$$

of the Lie algebra $\mathfrak{g}$. Note that if $\Sigma = \emptyset$ then $\mathfrak{p} = \mathfrak{b}$ and if $\Sigma = \Pi$ then $\mathfrak{p} = \mathfrak{g}$.

Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be the standard parabolic subalgebra of $\mathfrak{g}$ associated to a subset Σ of Π. We denote

$$\Delta_{\mathfrak{u}}^+ = \{\alpha \in \Delta^+;\ \mathfrak{g}_\alpha \subset \mathfrak{u}\}, \quad \Delta_{\mathfrak{l}}^+ = \{\alpha \in \Delta^+;\ \mathfrak{g}_\alpha \subset \mathfrak{l}\}$$

and set

$$\Lambda^+(\mathfrak{p}) = \{\lambda \in \mathfrak{h}^*;\ (\forall \alpha \in \Sigma)\ \lambda(h_\alpha) \in \mathbb{N}_0\}.$$

The elements of $\Lambda^+(\mathfrak{p})$ are called $\mathfrak{p}$-dominant and $\mathfrak{p}$-algebraically integral weights.

Besides, we denote by $\sigma_\lambda : \mathfrak{p} \to \operatorname{End} \mathbb{F}_\lambda$ the simple finite-dimensional $\mathfrak{p}$-module with highest weight $\lambda \in \Lambda^+(\mathfrak{p})$. Let us note that the nilradical $\mathfrak{u}$ of $\mathfrak{p}$ acts trivially on $\mathbb{F}_\lambda$.

Definition 3.7. Let $\lambda \in \Lambda^+(\mathfrak{p})$. The generalized Verma $\mathfrak{g}$-module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is the induced module

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathbb{F}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{F}_\lambda \simeq U(\bar{\mathfrak{u}}) \otimes_{\mathbb{C}} \mathbb{F}_\lambda, \tag{3.3}$$

where the last isomorphism of $U(\bar{\mathfrak{u}})$-modules follows from the Poincaré–Birkhoff–Witt theorem.

Now, we are in the position to define our main objects — $\mathfrak{g}$-modules which are twistings of generalized Verma modules. Later we will give their explicit realization.

Definition 3.8. Let $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^+_{\mathfrak{u}}$. Then the α-Gelfand–Tsetlin $\mathfrak{g}$-module $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$ is defined by

$$W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha) = T_\alpha(M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)), \tag{3.4}$$

where $T_\alpha : \mathcal{O}(\mathfrak{g}) \to \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}^-_\alpha)$ is the twisting functor.

Let us note that for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^+_{\mathfrak{l}}$ we have $T_\alpha(M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)) = 0$ by Proposition 3.1. This explains the restriction $\alpha \in \Delta^+_{\mathfrak{u}}$ in the definition above.

By using the triangular decomposition $\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u}$ and the Poincaré–Birkhoff–Witt theorem we get an isomorphism

$$U(\mathfrak{g})_{(f_\alpha)} \simeq U(\bar{\mathfrak{u}})_{(f_\alpha)} \otimes_{\mathbb{C}} U(\mathfrak{p})$$

of left $U(\bar{\mathfrak{u}})$-modules for $\alpha \in \Delta^+$. Hence, we may write

$$U(\mathfrak{g})_{(f_\alpha)} \otimes_{U(\mathfrak{g})} M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda) \simeq U(\mathfrak{g})_{(f_\alpha)} \otimes_{U(\mathfrak{g})} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{F}_\lambda \simeq U(\bar{\mathfrak{u}})_{(f_\alpha)} \otimes_{\mathbb{C}} \mathbb{F}_\lambda$$

for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^+_{\mathfrak{u}}$, which gives us an isomorphism

$$W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha) \simeq \big(U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}})\big) \otimes_{\mathbb{C}} \mathbb{F}_\lambda$$

of $U(\bar{\mathfrak{u}})$-modules. Besides, we have that

$$M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda) \simeq U(\bar{\mathfrak{u}}) \otimes_{\mathbb{C}} \mathbb{F}_\lambda \quad \text{and} \quad W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha) \simeq \big(U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}})\big) \otimes_{\mathbb{C}} \mathbb{F}_\lambda \tag{3.5}$$

are also isomorphisms of $U(\mathfrak{l})$-modules for the adjoint action of $U(\mathfrak{l})$ on $U(\bar{\mathfrak{u}})$ and $U(\bar{\mathfrak{u}})_{(f_\alpha)}$.

For later use, we also want to clarify the relation between $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda, \alpha)$ and $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^+_{\mathfrak{u}}$. Since we have a canonical surjective homomorphism

$$M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda) \to M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)$$

of generalized Verma modules, by applying the twisting functor T_α, which is right exact, we get a surjective homomorphism

$$W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda, \alpha) \to W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$$

of α-Gelfand–Tsetlin modules. Therefore, we obtain that $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$ is a quotient of $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda, \alpha)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^+_{\mathfrak{u}}$.

4. α-Gelfand–Tsetlin Modules $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$

In this section we discuss the properties of α-Gelfand–Tsetlin module $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^+_{\mathfrak{u}}$ and show that they belong to the category $\mathcal{H}_{\text{fin}}(\mathfrak{g}, \Gamma_\alpha)$. We also give their explicit realization.

4.1. *Basic properties*

We denote by $\mathbb{Z}^{\Delta^+}$ and $\mathbb{Z}^{\Delta^+_{\mathfrak{u}}}$ the set of all functions from Δ^+ to $\mathbb{Z}$ and from $\Delta^+_{\mathfrak{u}}$ to $\mathbb{Z}$, respectively. Since $\Delta^+_{\mathfrak{u}} \subset \Delta^+$, an element of $\mathbb{Z}^{\Delta^+_{\mathfrak{u}}}$ will be also regarded as an element of $\mathbb{Z}^{\Delta^+}$ extended by 0 on $\Delta^+ \backslash \Delta^+_{\mathfrak{u}}$. A similar notation is introduced for $\mathbb{N}_0^{\Delta^+}$ and $\mathbb{N}_0{}^{\Delta^+_{\mathfrak{u}}}$.

Since any positive root $\gamma \in \Delta^+$ can be expressed as

$$\gamma = \sum_{i=1}^{r} m_{\gamma,\alpha_i} \alpha_i, \tag{4.1}$$

where $m_{\gamma,\alpha_i} \in \mathbb{N}_0$ for $i = 1, 2, \ldots, r$ and $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, we define $t^{\alpha}_{\gamma} \in \mathbb{Z}^{\Delta^+}$ for $\gamma \in \Delta^+ \backslash \Pi$ and $\alpha \in \Delta^+$ by

$$t^{\alpha}_{\gamma}(\beta) = \begin{cases} -(-1)^{\delta_{\alpha,\alpha_i}} m_{\gamma,\alpha_i} & \text{for } \beta = \alpha_i, \\ (-1)^{\delta_{\alpha,\gamma}} & \text{for } \beta = \gamma, \\ 0 & \text{for } \beta \neq \gamma \text{ and } \beta \notin \Pi, \end{cases}$$

and the subset Λ^{α}_{+} of $\mathbb{Z}^{\Delta^+}$ by

$$\Lambda^{\alpha}_{+} = \left\{ \sum_{\gamma \in \Delta^+ \backslash \Pi} n_\gamma t^{\alpha}_{\gamma};\ n_\gamma \in \mathbb{N}_0 \text{ for all } \gamma \in \Delta^+ \backslash \Pi \right\}. \tag{4.2}$$

Besides, we introduce a weight $\mu_{a,\alpha} \in \mathfrak{h}^*$ by

$$\mu_{a,\alpha} = -\sum_{\gamma \in \Delta^+} (-1)^{\delta_{\alpha,\gamma}} a_\gamma \gamma + \alpha \tag{4.3}$$

for $\alpha \in \Delta^+$ and $a \in \mathbb{Z}^{\Delta^+}$.

Lemma 4.1. *Let $\alpha \in \Delta^+_{\mathfrak{u}}$ and $a, b \in \mathbb{Z}^{\Delta^+_{\mathfrak{u}}}$. Then $\mu_{a,\alpha} = \mu_{b,\alpha}$ if and only if*

$$b = a + \sum_{\gamma \in \Delta^+_{\mathfrak{u}} \backslash \Pi} n_\gamma t^{\alpha}_{\gamma}, \tag{4.4}$$

where $n_\gamma \in \mathbb{Z}$ for all $\gamma \in \Delta^+_{\mathfrak{u}} \backslash \Pi$.

Proof. Let $\alpha \in \Delta^+_{\mathfrak{u}}$ and $a, b \in \mathbb{Z}^{\Delta^+_{\mathfrak{u}}}$. If

$$b = a + \sum_{\gamma \in \Delta^+_{\mathfrak{u}} \backslash \Pi} n_\gamma t^{\alpha}_{\gamma},$$

where $n_\gamma \in \mathbb{Z}$ for $\gamma \in \Delta^+_{\mathfrak{u}} \backslash \Pi$, then we easily get $\mu_{a,\alpha} = \mu_{b,\alpha}$. On the other hand, let us assume that $\mu_{a,\alpha} = \mu_{b,\alpha}$. Then we set $n_\gamma = (-1)^{\delta_{\alpha,\gamma}}(b_\gamma - a_\gamma)$ for $\gamma \in \Delta^+_{\mathfrak{u}} \backslash \Pi$ and define $c = a + \sum_{\gamma \in \Delta^+_{\mathfrak{u}} \backslash \Pi} n_\gamma t^{\alpha}_{\gamma}$. Hence, we have $c_\gamma = a_\gamma + (-1)^{\delta_{\alpha,\gamma}} n_\gamma = b_\gamma$ for $\gamma \in \Delta^+_{\mathfrak{u}} \backslash \Pi$ and

$$c_{\alpha_i} = a_{\alpha_i} - \sum_{\gamma \in \Delta^+_{\mathfrak{u}} \backslash \Pi} (-1)^{\delta_{\alpha,\alpha_i}} n_\gamma m_{\gamma,\alpha_i}$$

for $i = 1, 2, \ldots, r$. Further, since $\mu_{a,\alpha} = \mu_{b,\alpha}$, we may write

$$\begin{aligned}\mu_{b,\alpha} - \mu_{a,\alpha} &= \sum_{\gamma \in \Delta_{\mathfrak{u}}^+} (-1)^{\delta_{\alpha,\gamma}} (a_\gamma - b_\gamma)\gamma \\ &= \sum_{i=1}^r (-1)^{\delta_{\alpha,\alpha_i}} (a_{\alpha_i} - b_{\alpha_i})\alpha_i + \sum_{i=1}^r \sum_{\gamma \in \Delta_{\mathfrak{u}}^+ \setminus \Pi} (-1)^{\delta_{\alpha,\gamma}} (a_\gamma - b_\gamma) m_{\gamma,\alpha_i} \alpha_i \\ &= \sum_{i=1}^r (-1)^{\delta_{\alpha,\alpha_i}} \left(a_{\alpha_i} - b_{\alpha_i} - \sum_{\gamma \in \Delta_{\mathfrak{u}}^+ \setminus \Pi} (-1)^{\delta_{\alpha,\alpha_i}} n_\gamma m_{\gamma,\alpha_i} \right) \alpha_i = 0,\end{aligned}$$

where we used (4.1). As the set $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ forms a basis of $\mathfrak{h}^*$, we get

$$a_{\alpha_i} - b_{\alpha_i} - \sum_{\gamma \in \Delta_{\mathfrak{u}}^+ \setminus \Pi} (-1)^{\delta_{\alpha,\alpha_i}} n_\gamma m_{\gamma,\alpha_i} = 0,$$

which implies that $c_{\alpha_i} = b_{\alpha_i}$ for $i = 1, 2, \ldots, r$. Hence, we have $c_\gamma = b_\gamma$ for all $\gamma \in \Delta_{\mathfrak{u}}^+$ and we are done. □

Let $\{f_\alpha, f_{\gamma_1}, \ldots, f_{\gamma_n}\}$ be a root basis of the opposite nilradical $\bar{\mathfrak{u}}$, where $n = \dim \bar{\mathfrak{u}} - 1$, $\gamma_i \in \Delta_{\mathfrak{u}}^+$ and $f_{\gamma_i} \in \mathfrak{g}_{-\gamma_i}$ for $i = 1, 2, \ldots, n$. Then as a consequence of the Poincaré–Birkhoff–Witt theorem we obtain that the subset $\{u_{a,\alpha};\ a \in \mathbb{N}_0^{\Delta_{\mathfrak{u}}^+}\}$ of $U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}})$, where

$$u_{a,\alpha} = f_\alpha^{-a_\alpha - 1} f_{\gamma_1}^{a_{\gamma_1}} \cdots f_{\gamma_n}^{a_{\gamma_n}}$$

for $a \in {\mathbb{N}_0}^{\Delta_{\mathfrak{u}}^+}$, forms a basis of $U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}})$ for $\alpha \in \Delta_{\mathfrak{u}}^+$. Moreover, if $v \in \mathbb{F}_\lambda$ is a weight vector with weight $\mu_v \in \mathfrak{h}^*$, then $u_{a,\alpha} \otimes v \in W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ is a weight vector with weight $\mu_v + \mu_{a,\alpha}$ by Lemma 3.2.

Proposition 4.2. *Let $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_{\mathfrak{u}}^+$. Then all weight spaces of $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ are finite-dimensional if $\alpha \in \Delta_{\mathfrak{u}}^+ \cap \Pi$ and infinite-dimensional provided that $\alpha \in \Delta_{\mathfrak{u}}^+ \setminus \Pi$ and $m_{\alpha,\gamma} = 0$ for $\gamma \in \Pi \setminus \Delta_{\mathfrak{u}}^+$, where $\alpha = \sum_{\gamma \in \Pi} m_{\alpha,\gamma}\gamma$.*

Proof. Since $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ is isomorphic to $(U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}})) \otimes_{\mathbb{C}} \mathbb{F}_\lambda$ as an $\mathfrak{l}$-module for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_{\mathfrak{u}}^+$, and the $\mathfrak{g}$-module $\mathbb{F}_\lambda$ is finite-dimensional, it is sufficient to show that all weight spaces of $U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}})$ are finite-dimensional if $\alpha \in \Delta_{\mathfrak{u}}^+ \cap \Pi$ and infinite dimensional if $\alpha \in \Delta_{\mathfrak{u}}^+ \setminus \Pi$ and $m_{\alpha,\gamma} = 0$ for $\gamma \in \Pi \setminus \Delta_{\mathfrak{u}}^+$.

From the discussion above it follows that all weights of $U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}})$ are of the form $\mu_{a,\alpha}$ for some $a \in \mathbb{Z}^\Pi$. If we assume that $u_{b,\alpha}$ for $b \in {\mathbb{N}_0}^{\Delta_{\mathfrak{u}}^+}$ is a weight vector with weight $\mu_{a,\alpha}$, then we obtain $\mu_{a,\alpha} = \mu_{b,\alpha}$, which together with Lemma 4.1 implies

$$b = a + \sum_{\gamma \in \Delta^+ \setminus \Pi} n_\gamma t_\gamma^\alpha,$$

where $n_\gamma \in \mathbb{Z}$ for all $\gamma \in \Delta^+\backslash\Pi$. As $\{u_{b,\alpha};\ b \in \mathbb{N}_0{}^{\Delta_{\mathfrak{u}}^+}\}$ is a basis of $U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}})$, we immediately get

$$\dim\ \big(U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}})\big)_{\mu_{a,\alpha}} = \#\{t \in \Lambda_+^\alpha;\ a + t \in \mathbb{N}_0{}^{\Delta_{\mathfrak{u}}^+}\}.$$

Let us assume that $t = \sum_{\gamma\in\Delta^+\backslash\Pi} n_\gamma t_\gamma^\alpha \in \Lambda_+^\alpha$ satisfies $a + t \in \mathbb{N}_0{}^{\Delta_{\mathfrak{u}}^+}$. Then for $b = a + t$ we have $b_\gamma = a_\gamma + (-1)^{\delta_{\alpha,\gamma}} n_\gamma = (-1)^{\delta_{\alpha,\gamma}} n_\gamma$ for $\gamma \in \Delta^+\backslash\Pi$ and $b_{\alpha_i} = a_{\alpha_i} - \sum_{\gamma\in\Delta^+\backslash\Pi} (-1)^{\delta_{\alpha,\alpha_i}} n_\gamma m_{\gamma,\alpha_i}$ for $i = 1, 2, \ldots, r$.

If $\alpha \in \Delta_{\mathfrak{u}}^+ \cap \Pi$, then the condition $b_\gamma \in \mathbb{N}_0$ for $\gamma \in \Delta^+$ implies $n_\gamma \in \mathbb{N}_0$ for $\gamma \in \Delta^+\backslash\Pi$. Further, for $i = 1, 2, \ldots, r$ satisfying $\alpha_i \neq \alpha$ we may write

$$0 \leq b_{\alpha_i} = a_{\alpha_i} - \sum_{\gamma\in\Delta^+\backslash\Pi} n_\gamma m_{\gamma,\alpha_i} \leq a_{\alpha_i} - n_\gamma m_{\gamma,\alpha_i}.$$

Moreover, for each $\gamma \in \Delta^+\backslash\Pi$ there exists $i \in \{1, 2, \ldots, r\}$ such that $\alpha_i \neq \alpha$ and $m_{\gamma,\alpha_i} \neq 0$, which implies that $\dim\,(U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}}))_{\mu_{a,\alpha}} < \infty$.

On the other hand, if $\alpha \in \Delta_{\mathfrak{u}}^+\backslash\Pi$ and $m_{\alpha,\gamma} = 0$ for $\gamma \in \Pi\backslash\Delta_{\mathfrak{u}}^+$, then $a + t - nt_\alpha^\alpha \in \mathbb{N}_0^{\Delta_{\mathfrak{u}}^+}$ for all $n \in \mathbb{N}_0$ provided $a + t \in \mathbb{N}_0^{\Delta_{\mathfrak{u}}^+}$, which implies that $\dim\,(U(\bar{\mathfrak{u}})_{(f_\alpha)}/U(\bar{\mathfrak{u}}))_{\mu_{a,\alpha}} = \infty$. This finishes the proof. □

The next theorem generalizes [16, Theorem 2.9, Theorem 2.11] for any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and for a positive root $\alpha \in \Delta_{\mathfrak{u}}^+$.

Theorem 4.3. *Let $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_{\mathfrak{u}}^+$. Then $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ is a locally $\mathfrak{t}_\alpha^-$-finite cyclic weight $\mathfrak{g}$-module with finite Γ_α-multiplicities, that is $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha) \in \mathcal{H}_{\rm fin}(\mathfrak{g}, \Gamma_\alpha) \cap \mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^-)$.*

Proof. It is an immediate consequence of Theorem 3.4(ii), since we have $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha) = T_\alpha(M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_{\mathfrak{u}}^+$. □

Let us recall that neither $\mathcal{H}_{\rm fin}(\mathfrak{g}, \Gamma_\alpha)$ nor $\mathcal{I}_f(\mathfrak{g}, \mathfrak{t}_\alpha^-)$ are tensor categories. On the other hand, we have that $\mathcal{I}(\mathfrak{g}, \mathfrak{t}_\alpha^-)$ is a tensor category by Proposition 2.5.

Remark. It was shown in [16, Theorem 2.16] that the $\mathfrak{g}$-module $W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda, \theta)$ is simple generically for $\mathfrak{g} = \mathfrak{sl}(3)$, where θ is the maximal root of $\mathfrak{g}$. Moreover, in this case Γ_θ is diagonalizable on $W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda, \theta)$. Therefore, it is natural to expect a similar behavior from $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_{\mathfrak{u}}^+$ in general.

The $\mathfrak{g}$-module $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_{\mathfrak{u}}^+$ is an α-Gelfand–Tsetlin module in the category $\mathcal{I}_{\rm fin}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$. On the other hand, by means of the natural duality we can analogously construct certain α-Gelfand–Tsetlin modules in the category $\mathcal{I}_{\rm fin}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$.

Furthermore, since the action of the center $\mathfrak{z}_{\mathfrak{g}}$ of $U(\mathfrak{g})$ on the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ is then given by

$$zv = \chi_{\lambda+\rho}(z)v \tag{4.5}$$

for all $z \in \mathfrak{z}_\mathfrak{g}$ and $v \in M^\mathfrak{g}_\mathfrak{p}(\lambda)$, in other words $M^\mathfrak{g}_\mathfrak{p}(\lambda)$ is a $\mathfrak{g}$-module with central character $\chi_{\lambda+\rho}$, it follows immediately from definition that $W^\mathfrak{g}_\mathfrak{p}(\lambda, \alpha)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^+_\mathfrak{u}$ is a $\mathfrak{g}$-module with the same central character $\chi_{\lambda+\rho}$.

A comparison of basic characteristics of Verma modules $M^\mathfrak{g}_\mathfrak{b}(\lambda)$ and α-Gelfand–Tsetlin modules $W^\mathfrak{g}_\mathfrak{b}(\lambda, \alpha)$ for $\lambda \in \Lambda^+(\mathfrak{b})$ and $\alpha \in \Delta^+$ is given in Table 1.

4.2. *Geometric realization of* $\boldsymbol{W^\mathfrak{g}_\mathfrak{b}(\lambda, \alpha)}$

We describe the geometric realization of α-Gelfand–Tsetlin modules $W^\mathfrak{g}_\mathfrak{b}(\lambda, \alpha)$ for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta^+$ using the theory of algebraic $\mathcal{D}$-modules on flag varieties without going into details. We are going to address the geometric realization of more general families of Gelfand–Tsetlin modules and the corresponding geometric induction in the subsequent paper.

Let G be a connected semisimple algebraic group over $\mathbb{C}$, H be a maximal torus of G and B be a Borel subgroup of G containing H with the unipotent radical N and the opposite unipotent radical $\bar{N}$. We denote by $\mathfrak{g}$, $\mathfrak{n}$, $\bar{\mathfrak{n}}$ and $\mathfrak{h}$ the Lie algebras of G, N, $\bar{N}$ and H, respectively. Then

$$X = G/B$$

is a smooth algebraic variety, the *flag variety* of G. Besides, we have the canonical G-equivariant projection

$$p : G \to G/B.$$

Following [25], for any $\lambda \in \mathfrak{h}^*$ there exists a G-equivariant sheaf of rings of twisted differential operators $\mathcal{D}^\lambda_X$ on X. Let us note that $\mathcal{D}^{-\rho}_X$ is the usual sheaf of rings of differential operators on X. Since $\mathcal{D}^\lambda_X$ is G-equivariant, we have a Lie algebra homomorphism $\mathfrak{g} \to \Gamma(X, \mathcal{D}^\lambda_X)$, which extends to a homomorphism

$$\Phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}^\lambda_X) \tag{4.6}$$

of $\mathbb{C}$-algebras. Hence, for any $\mathcal{D}^\lambda_X$-module $\mathcal{M}$ the vector space $\Gamma(X, \mathcal{M})$ of global sections of $\mathcal{M}$ has a natural $\mathfrak{g}$-module structure.

Let us denote by $\mathrm{Mod}_c(\mathcal{D}^\lambda_X)$ the category of coherent $\mathcal{D}^\lambda_X$-modules and by $\mathcal{M}_f(\mathfrak{g}, \chi_\lambda)$ the full subcategory of $\mathcal{M}(\mathfrak{g})$ consisting of finitely generated $\mathfrak{g}$-modules

Table 1. Comparison of Verma modules and Gelfand–Tsetlin modules.

Verma module $M^\mathfrak{g}_\mathfrak{b}(\lambda)$, $\alpha \in \Delta^+$	Gelfand–Tsetlin module $W^\mathfrak{g}_\mathfrak{b}(\lambda, \alpha)$, $\alpha \in \Delta^+ \cap \Pi$	Gelfand–Tsetlin module $W^\mathfrak{g}_\mathfrak{b}(\lambda, \alpha)$, $\alpha \in \Delta^+ \setminus \Pi$
central character $\chi_{\lambda+\rho}$	central character $\chi_{\lambda+\rho}$	central character $\chi_{\lambda+\rho}$
cyclic module	cyclic module	cyclic module
weight $\mathfrak{g}$-module with finite-dimensional weight spaces	weight $\mathfrak{g}$-module with finite-dimensional weight spaces	weight $\mathfrak{g}$-module with infinite-dimensional weight spaces
Γ_α-weight $\mathfrak{g}$-module with finite Γ_α-multiplicities	Γ_α-weight $\mathfrak{g}$-module with finite Γ_α-multiplicities	Γ_α-weight $\mathfrak{g}$-module with finite Γ_α-multiplicities

on which the center $\mathfrak{z}_{\mathfrak{g}}$ of $U(\mathfrak{g})$ acts via the central character χ_λ. The following remarkable theorem is due to Beilinson and Bernstein, see [5], [25].

Theorem 4.4. *Let $\lambda \in \mathfrak{h}^*$ be anti-dominant and regular, i.e. $\lambda(h_\alpha) \notin \mathbb{N}_0$ for $\alpha \in \Delta^+$. Then the functor of global sections*

$$\Gamma(X, \bullet) : \mathrm{Mod}_c(\mathcal{D}_X^\lambda) \to \mathcal{M}_f(\mathfrak{g}, \chi_\lambda)$$

induces an equivalence of abelian categories. The inverse is given through the localization functor $\Delta(M) = \mathcal{D}_X^\lambda \otimes_{U(\mathfrak{g})} M$ for $M \in \mathcal{M}_f(\mathfrak{g}, \chi_\lambda)$.

Since the α-Gelfand–Tsetlin module $W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda - \rho, \alpha)$ for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta^+$ is an object of the category $\mathcal{M}_f(\mathfrak{g}, \chi_\lambda)$, it corresponds to the coherent $\mathcal{D}_X^\lambda$-module on the flag variety X by Theorem 4.4. Our goal is to describe this coherent $\mathcal{D}_X^\lambda$-module explicitly.

First of all we recall the geometric realization of Verma modules $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda - \rho)$ for $\lambda \in \mathfrak{h}^*$. Let X_w for $w \in W \simeq N_G(H)/H$ be the N-orbit in X defined by

$$X_w = N\dot{w}B/B,$$

where $\dot{w} \in N_G(H)$ is a representative of w. Let us note that X_w is called a *Schubert cell.* Further, let us denote by

$$i_w : X_w \to X$$

the embedding of X_w into X for $w \in W$. Since i_w is an N-equivariant mapping, the pull-back $i_w^\sharp \mathcal{D}_X^\lambda$ for $\lambda \in \mathfrak{h}^*$ is an N-equivariant sheaf of rings of twisted differential operators on X_w, which is isomorphic to the sheaf of rings of differential operators $\mathcal{D}_{X_w}$ on X_w. As the structure sheaf $\mathcal{O}_{X_w}$ of X_w is a $\mathcal{D}_{X_w}$-module, we may consider its direct image

$$\mathcal{L}_\lambda(X_w, \mathcal{O}_{X_w}) = i_{w*}(\mathcal{D}_{X \leftarrow X_w}^\lambda \otimes_{i_w^\sharp \mathcal{D}_X^\lambda} \mathcal{O}_{X_w}), \tag{4.7}$$

where the $(i_w^{-1}\mathcal{D}_X^\lambda, i_w^\sharp \mathcal{D}_X^\lambda)$-bimodule $\mathcal{D}_{X \leftarrow X_w}^\lambda$ is the so-called transfer bimodule for i_w. Then for the unit element $e \in W$ we have

$$\Gamma(X, \mathcal{L}_\lambda(X_e, \mathcal{O}_{X_e})) \simeq M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda - \rho) \tag{4.8}$$

as $\mathfrak{g}$-modules for $\lambda \in \mathfrak{h}^*$, see e.g. [27].

To obtain the geometric realization of α-Gelfand–Tsetlin modules $W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda - \rho, \alpha)$ for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta^+$, we take into account different embeddings into X. Since $\mathfrak{s}_\alpha$ is a semisimple Lie subalgebra of $\mathfrak{g}$ for any $\alpha \in \Delta^+$, there is a unique closed connected semisimple algebraic subgroup G_α of G with the Lie algebra $\mathfrak{s}_\alpha$. Moreover, we have that $H_\alpha = H \cap G_\alpha$ is a maximal torus of G_α and $B_\alpha = B \cap G_\alpha$ is a Borel subgroup of G_α with the unipotent radical $N_\alpha = N \cap G_\alpha$ and the opposite unipotent radical $\bar{N}_\alpha = \bar{N} \cap G_\alpha$. Let us note that $\mathfrak{s}_\alpha^+$ and $\mathfrak{s}_\alpha^-$ are Lie algebras of N_α and $\bar{N}_\alpha$, respectively. Let us consider the closed embedding

$$i_\alpha : X_\alpha \to X$$

of the flag variety $X_\alpha = G_\alpha/B_\alpha$ into X. Moreover, since i_α is a G_α-equivariant mapping, the pull-back $i_\alpha^\sharp \mathcal{D}_X^\lambda$ is a G_α-equivariant sheaf of rings of twisted differential operators on X_α, which is isomorphic to $\mathcal{D}_{X_\alpha}^{\lambda_\alpha}$ with $\lambda_\alpha = \big(\lambda + \rho - \frac{1}{2}\alpha\big)|_{\mathfrak{h}\cap\mathfrak{s}_\alpha}$. Hence, for $\alpha \in \Delta^+$ we may introduce the geometric induction functor

$$\mathrm{GInd}_\alpha^\lambda : \mathcal{M}_f(\mathfrak{s}_\alpha, \chi_{\lambda_\alpha}) \to \mathcal{M}_f(\mathfrak{g}, \chi_\lambda)$$

provided $\lambda \in \mathfrak{h}^*$ and $\lambda_\alpha \in (\mathfrak{h}\cap\mathfrak{s}_\alpha)^*$ are anti-dominant and regular by

$$\mathrm{GInd}_\alpha^\lambda(M) = \Gamma(X, i_{\alpha*}(\mathcal{D}_{X\leftarrow X_\alpha}^\lambda \otimes_{i_\alpha^\sharp \mathcal{D}_X^\lambda} \Delta(M))), \tag{4.9}$$

where the $(i_\alpha^{-1}\mathcal{D}_X^\lambda, i_\alpha^\sharp \mathcal{D}_X^\lambda)$-bimodule $\mathcal{D}_{X\leftarrow X_\alpha}^\lambda$ is the transfer bimodule for i_α.

Let us consider an open subset $U_e = p(\bar{N})$ of X called the big cell. Then $\mathcal{D}_X^\lambda|_{U_e}$ and $\mathcal{D}_X|_{U_e}$ are isomorphic as sheaves of rings of twisted differential operators on U_e. Besides, as $\bar{\mathfrak{n}}$ is a nilpotent Lie algebra, the exponential mapping $\exp : \bar{\mathfrak{n}} \to \bar{N}$ is an isomorphism of algebraic varieties and induces a canonical isomorphism of algebraic varieties U_e and $\bar{\mathfrak{n}}$. Therefore, the homomorphism $\Phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\lambda)$ of $\mathbb{C}$-algebras gives rise to the homomorphism

$$\pi_\lambda : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\lambda) \to \Gamma(U_e, \mathcal{D}_X^\lambda) \xrightarrow{\sim} \Gamma(U_e, \mathcal{D}_X) \xrightarrow{\sim} \mathcal{A}_{\bar{\mathfrak{n}}} \tag{4.10}$$

of $\mathbb{C}$-algebras for $\lambda \in \mathfrak{h}^*$, where the Weyl algebra $\mathcal{A}_{\bar{\mathfrak{n}}}$ of the vector space $\bar{\mathfrak{n}}$ is defined through $\mathcal{A}_{\bar{\mathfrak{n}}} = \Gamma(\bar{\mathfrak{n}}, \mathcal{D}_{\bar{\mathfrak{n}}})$.

In addition, there is a nice explicit description of the $\mathbb{C}$-algebra homomorphism (4.10). For that reason, let $\{f_\gamma;\ \gamma \in \Delta^+\}$ be a root basis of the opposite nilradical $\bar{\mathfrak{n}}$. We denote by $\{x_\gamma;\ \gamma \in \Delta^+\}$ the linear coordinate functions on $\bar{\mathfrak{n}}$ with respect to the basis $\{f_\gamma;\ \gamma \in \Delta^+\}$ of $\bar{\mathfrak{n}}$. Then the Weyl algebra $\mathcal{A}_{\bar{\mathfrak{n}}}$ is generated by $\{x_\gamma, \partial_{x_\gamma};\ \gamma \in \Delta^+\}$ together with the canonical commutation relations. The homomorphism $\pi_\lambda : U(\mathfrak{g}) \to \mathcal{A}_{\bar{\mathfrak{n}}}$ of $\mathbb{C}$-algebras for $\lambda \in \mathfrak{h}^*$ is then given by the formula (4.16), see [27].

Let $\mathcal{M}$ be a $i_\alpha^\sharp \mathcal{D}_X^\lambda$-module for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta^+$. Since $i_\alpha : X_\alpha \to X$ is a closed embedding, $\mathcal{D}_X^\lambda|_{U_e} \simeq \mathcal{D}_X|_{U_e}$ as sheaves of rings of twisted differential operators on U_e and $U_e \cap X_\alpha \simeq \mathfrak{s}_\alpha^-$, we immediately obtain

$$i_{\alpha*}(\mathcal{D}_{X\leftarrow X_\alpha}^\lambda \otimes_{i_\alpha^\sharp \mathcal{D}_X^\lambda} \mathcal{M})|_{U_e} \simeq \mathbb{C}[\partial_{x_\gamma}, \gamma \in \Delta_\alpha^+] \otimes_{\mathbb{C}} i_{\alpha*}(\mathcal{M})|_{U_e \simeq \bar{\mathfrak{n}}}, \tag{4.11}$$

where $\Delta_\alpha^+ = \Delta^+ \backslash \{\alpha\}$ and the action of the Lie algebra $\mathfrak{g}$ on the right-hand side is given through the homomorphism $\pi_\lambda : U(\mathfrak{g}) \to \mathcal{A}_{\bar{\mathfrak{n}}}$ of $\mathbb{C}$-algebras.

If we denote by $M_{\mathfrak{b}\cap\mathfrak{s}_\alpha}^{\mathfrak{s}_\alpha}((\lambda+\rho-\alpha)_{|\mathfrak{h}\cap\mathfrak{s}_\alpha})$ the Verma module with highest weight $(\lambda+\rho-\alpha)_{|\mathfrak{h}\cap\mathfrak{s}_\alpha}$ for the Borel subalgebra $\mathfrak{b}\cap\mathfrak{s}_\alpha$ and by $N_{\mathfrak{b}\cap\mathfrak{s}_\alpha}^{\mathfrak{s}_\alpha}((\lambda+\rho)_{|\mathfrak{h}\cap\mathfrak{s}_\alpha})$ the contragredient Verma module with highest weight $(\lambda+\rho)_{|\mathfrak{h}\cap\mathfrak{s}_\alpha}$ for the opposite Borel subalgebra $\bar{\mathfrak{b}}\cap\mathfrak{s}_\alpha$, then we have

$$M_{\mathfrak{b}\cap\mathfrak{s}_\alpha}^{\mathfrak{s}_\alpha}((\lambda+\rho-\alpha)_{|\mathfrak{h}\cap\mathfrak{s}_\alpha}) \simeq \Gamma(X_\alpha, \mathcal{L}_{\lambda_\alpha}(X_{\alpha,N,e}, \mathcal{O}_{X_{\alpha,N,e}})) \simeq \mathbb{C}[\partial_{x_\alpha}]$$

and

$$N_{\mathfrak{b}\cap\mathfrak{s}_\alpha}^{\mathfrak{s}_\alpha}((\lambda+\rho)_{|\mathfrak{h}\cap\mathfrak{s}_\alpha}) \simeq \Gamma(X_\alpha, \mathcal{L}_{\lambda_\alpha}(X_{\alpha,\bar{N},e}, \mathcal{O}_{X_{\alpha,\bar{N},e}})) \simeq \mathbb{C}[x_\alpha]$$

as $\mathfrak{s}_\alpha$-modules, where $X_{\alpha,N,e} = N\dot{e}B_\alpha/B_\alpha$ and $X_{\alpha,\bar{N},e} = \bar{N}\dot{e}B_\alpha/B_\alpha$ are the N-orbit and $\bar{N}$-orbit in X_α, respectively. Let us denote by

$$\mathcal{K}_\lambda(X_{\alpha,N,e}, \mathcal{O}_{\alpha,N,e}) = i_{\alpha*}(\mathcal{D}^\lambda_{X\leftarrow X_\alpha} \otimes_{i^\sharp_\alpha \mathcal{D}^\lambda_X} \mathcal{L}_{\lambda_\alpha}(X_{\alpha,N,e}, \mathcal{O}_{X_{\alpha,N,e}})) \tag{4.12}$$

and

$$\mathcal{K}_\lambda(X_{\alpha,\bar{N},e}, \mathcal{O}_{\alpha,\bar{N},e}) = i_{\alpha*}(\mathcal{D}^\lambda_{X\leftarrow X_\alpha} \otimes_{i^\sharp_\alpha \mathcal{D}^\lambda_X} \mathcal{L}_{\lambda_\alpha}(X_{\alpha,\bar{N},e}, \mathcal{O}_{X_{\alpha,\bar{N},e}})) \tag{4.13}$$

for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta^+$ coherent $\mathcal{D}^\lambda_X$-modules.

The following theorem gives us the geometric realization of Verma modules $M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ and of α-Gelfand–Tsetlin modules $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda, \alpha)$ for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta^+$.

Theorem 4.5. *Let $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta^+$. Then we have*

$$\Gamma(X, \mathcal{K}_{\lambda+\rho}(X_{\alpha,N,e}, \mathcal{O}_{\alpha,N,e})) \simeq M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda) \tag{4.14}$$

and

$$\Gamma(X, \mathcal{K}_{\lambda+\rho}(X_{\alpha,\bar{N},e}, \mathcal{O}_{\alpha,\bar{N},e})) \simeq W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda, \alpha) \tag{4.15}$$

as $\mathfrak{g}$-modules.

Proof. Let $\mathcal{M} = \mathcal{K}_\lambda(X_{\alpha,N,e}, \mathcal{O}_{\alpha,N,e})$. Then we may write

$$\begin{aligned}\Gamma(X, \mathcal{M}) &\simeq \Gamma(U_e, \mathcal{M}) \\ &\simeq \mathbb{C}[\partial_{x_\gamma},\ \gamma \in \Delta^+_\alpha] \otimes_{\mathbb{C}} \Gamma(U_e \cap X_\alpha, \mathcal{L}_{\lambda_\alpha}(X_{\alpha,N,e}, \mathcal{O}_{X_{\alpha,N,e}})) \\ &\simeq \mathbb{C}[\partial_{x_\gamma},\ \gamma \in \Delta^+_\alpha] \otimes_{\mathbb{C}} \mathbb{C}[\partial_{x_\alpha}] \simeq \mathbb{C}[\partial_{x_\gamma},\ \gamma \in \Delta^+],\end{aligned}$$

where used (4.11) and (4.12). Analogously, for $\mathcal{M} = \mathcal{K}_\lambda(X_{\alpha,\bar{N},e}, \mathcal{O}_{\alpha,\bar{N},e})$ we have

$$\begin{aligned}\Gamma(X, \mathcal{M}) &\simeq \Gamma(U_e, \mathcal{M}) \\ &\simeq \mathbb{C}[\partial_{x_\gamma}, \gamma \in \Delta^+_\alpha] \otimes_{\mathbb{C}} \Gamma(U_e \cap X_\alpha, \mathcal{L}_{\lambda_\alpha}(X_{\alpha,\bar{N},e}, \mathcal{O}_{X_{\alpha,\bar{N},e}})) \\ &\simeq \mathbb{C}[\partial_{x_\gamma}, \gamma \in \Delta^+_\alpha] \otimes_{\mathbb{C}} \mathbb{C}[x_\alpha] \simeq \mathbb{C}[x_\alpha, \partial_{x_\gamma}, \gamma \in \Delta^+_\alpha].\end{aligned}$$

By [27] we have that the $\mathfrak{g}$-module $\mathbb{C}[\partial_{x_\alpha},\ \alpha \in \Delta^+]$ is isomorphic to the Verma module $M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda - \rho)$ and by Theorem 4.10 we get that the $\mathfrak{g}$-module $\mathbb{C}[x_\alpha, \partial_{x_\gamma}, \gamma \in \Delta^+_\alpha]$ corresponds to the α-Gelfand–Tsetlin module $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda - \rho, \alpha)$. □

4.3. *Geometric realization of $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$ — Explicit formulas*

In this section, we describe an explicit form of the geometric realization of α-Gelfand–Tsetlin modules $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^+_{\mathfrak{u}}$. In the case when $\mathfrak{g}$ is a simple Lie algebra, $\mathfrak{p} = \mathfrak{b}$ and α is the maximal root of $\mathfrak{g}$ such realization was constructed in [16, Theorem 2.14].

Let $\{f_\alpha;\ \alpha \in \Delta^+_{\mathfrak{u}}\}$ be a basis of the opposite nilradical $\bar{\mathfrak{u}}$. We denote by $\{x_\alpha;\ \alpha \in \Delta^+_{\mathfrak{u}}\}$ the linear coordinate functions on $\bar{\mathfrak{u}}$ with respect to the basis $\{f_\alpha;\ \alpha \in \Delta^+_{\mathfrak{u}}\}$

of $\bar{\mathfrak{u}}$. Then the Weyl algebra $\mathcal{A}_{\bar{\mathfrak{u}}}$ of the vector space $\bar{\mathfrak{u}}$ is generated by $\{x_\alpha, \partial_{x_\alpha};\ \alpha \in \Delta_{\mathfrak{u}}^+\}$ together with the canonical commutation relations. For $\lambda \in \Lambda^+(\mathfrak{p})$ there is a homomorphism

$$\pi_\lambda : U(\mathfrak{g}) \to \mathcal{A}_{\bar{\mathfrak{u}}} \otimes_{\mathbb{C}} \operatorname{End} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}}$$

of $\mathbb{C}$-algebras given through

$$\pi_\lambda(a) = -\sum_{\alpha \in \Delta_{\mathfrak{u}}^+} \left[\frac{\operatorname{ad}(u(x)) e^{\operatorname{ad}(u(x))}}{e^{\operatorname{ad}(u(x))} - \operatorname{id}} \, (e^{-\operatorname{ad}(u(x))} a)_{\bar{\mathfrak{u}}} \right]_\alpha \partial_{x_\alpha} + \sigma_{\lambda+\rho_{\mathfrak{u}}}((e^{-\operatorname{ad}(u(x))} a)_{\mathfrak{p}}) \tag{4.16}$$

for all $a \in \mathfrak{g}$, where the Weyl vector $\rho_{\mathfrak{u}} \in \Lambda^+(\mathfrak{p})$ is defined by

$$\rho_{\mathfrak{u}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\mathfrak{u}}^+} \alpha,$$

$[a]_\alpha$ denotes the αth coordinate of $a \in \bar{\mathfrak{u}}$ with respect to the basis $\{f_\alpha;\ \alpha \in \Delta_{\mathfrak{u}}^+\}$ of $\bar{\mathfrak{u}}$, $a_{\bar{\mathfrak{u}}}$ and $a_{\mathfrak{p}}$ are $\bar{\mathfrak{u}}$-part and $\mathfrak{p}$-part of $a \in \mathfrak{g}$ with respect to the decomposition $\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{p}$, and finally the element $u(x) \in \mathbb{C}[\bar{\mathfrak{u}}] \otimes_{\mathbb{C}} \mathfrak{g}$ is given by

$$u(x) = \sum_{\alpha \in \Delta_{\mathfrak{u}}^+} x_\alpha f_\alpha.$$

Let us note that $\mathbb{C}[\bar{\mathfrak{u}}] \otimes_{\mathbb{C}} \mathfrak{g}$ has the natural structure of a Lie algebra. Hence, we have a well-defined $\mathbb{C}$-linear mapping $\operatorname{ad}(u(x)) : \mathbb{C}[\bar{\mathfrak{u}}] \otimes_{\mathbb{C}} \mathfrak{g} \to \mathbb{C}[\bar{\mathfrak{u}}] \otimes_{\mathbb{C}} \mathfrak{g}$. In particular, we have

$$\pi_\lambda(a) = -\sum_{\alpha \in \Delta_{\mathfrak{u}}^+} \left[\frac{\operatorname{ad}(u(x))}{e^{\operatorname{ad}(u(x))} - \operatorname{id}} \, a \right]_\alpha \partial_{x_\alpha} \tag{4.17}$$

for $a \in \bar{\mathfrak{u}}$ and

$$\pi_\lambda(a) = \sum_{\alpha \in \Delta_{\mathfrak{u}}^+} [\operatorname{ad}(u(x))a]_\alpha \partial_{x_\alpha} + \sigma_{\lambda+\rho_{\mathfrak{u}}}(a) \tag{4.18}$$

for $a \in \mathfrak{l}$.

Let us note that the same construction of the homomorphism $\pi_\lambda : U(\mathfrak{g}) \to \mathcal{A}_{\bar{\mathfrak{u}}} \otimes_{\mathbb{C}} \operatorname{End} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}}$ of $\mathbb{C}$-algebras, as was described in the previous section, works for any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and any 1-dimensional $\mathfrak{p}$-module $\mathbb{F}_{\lambda+\rho_{\mathfrak{u}}}$, see [27]. For a general simple finite-dimensional $\mathfrak{p}$-module $\mathbb{F}_{\lambda+\rho_{\mathfrak{u}}}$, the expression for π_λ may be extracted from the formula for the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ associated to $\mathfrak{g}$ since $\mathfrak{g}$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$, see [18].

The generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ is realized as $\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+2\rho_{\mathfrak{u}}}$, where $\mathcal{I}_{\mathrm{V}}$ is the left ideal of $\mathcal{A}_{\bar{\mathfrak{u}}}$ defined by $\mathcal{I}_{\mathrm{V}} = (x_\alpha;\ \alpha \in \Delta_{\mathfrak{u}}^+)$, see e.g. [27] and [18]. The $\mathfrak{g}$-module structure on $\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+2\rho_{\mathfrak{u}}}$ is then given through the homomorphism $\pi_{\lambda+\rho_{\mathfrak{u}}} : U(\mathfrak{g}) \to \mathcal{A}_{\bar{\mathfrak{u}}} \otimes_{\mathbb{C}} \operatorname{End} \mathbb{F}_{\lambda+2\rho_{\mathfrak{u}}}$ of $\mathbb{C}$-algebras.

The $\mathbb{C}$-algebra $\mathbb{C}[\bar{\mathfrak{u}}]$ of polynomial functions on $\bar{\mathfrak{u}}$ has the natural structure of a Q_+-graded $\mathbb{C}$-algebra uniquely determined by the requirement $\deg(x_\alpha) = \alpha$ for $\alpha \in \Delta_{\mathfrak{u}}^+$. Therefore, we have a direct sum decomposition

$$\mathbb{C}[\bar{\mathfrak{u}}] = \bigoplus_{\gamma \in Q_+} \mathbb{C}[\bar{\mathfrak{u}}]_\gamma.$$

We say that a differential operator $P \in \mathcal{A}_{\bar{\mathfrak{u}}}$ has degree $\delta \in Q$ if $P(\mathbb{C}[\bar{\mathfrak{u}}]_\gamma) \subset \mathbb{C}[\bar{\mathfrak{u}}]_{\gamma+\delta}$ for all $\gamma \in Q_+$. The degree of a differential operator defines on the Weyl algebra $\mathcal{A}_{\bar{\mathfrak{u}}}$ the canonical structure of a Q-graded $\mathbb{C}$-algebra, which gives us a direct sum decomposition

$$\mathcal{A}_{\bar{\mathfrak{u}}} = \bigoplus_{\gamma \in Q} \mathcal{A}_{\bar{\mathfrak{u}},\gamma}.$$

Further, for $\alpha \in \Delta_{\mathfrak{u}}^+$ we introduce a differential operator $p_\alpha \in \mathcal{A}_{\bar{\mathfrak{u}}}$ by

$$\begin{aligned} p_\alpha &= -\sum_{\gamma \in \Delta_{\mathfrak{u}}^+} \left[\frac{\mathrm{ad}(u(x))}{e^{\mathrm{ad}(u(x))} - \mathrm{id}}\ f_\alpha \right]_\gamma \partial_{x_\gamma} \\ &= -\sum_{\gamma \in \Delta_{\mathfrak{u}}^+} \partial_{x_\gamma} \left[\frac{\mathrm{ad}(u(x))}{e^{\mathrm{ad}(u(x))} - \mathrm{id}}\ f_\alpha \right]_\gamma, \end{aligned} \tag{4.19}$$

where $u(x) = \sum_{\gamma \in \Delta_{\mathfrak{u}}^+} x_\gamma f_\gamma$. Besides, we may write

$$p_\alpha = -\partial_{x_\alpha} + q_\alpha$$

with

$$\begin{aligned} q_\alpha &= -\sum_{\gamma \in \Delta_{\mathfrak{u}}^+} \left[\left(\frac{\mathrm{ad}(u(x))}{e^{\mathrm{ad}(u(x))} - \mathrm{id}} - \mathrm{id} \right) f_\alpha \right]_\gamma \partial_{x_\gamma} \\ &= -\sum_{\gamma \in \Delta_{\mathfrak{u}}^+} \partial_{x_\gamma} \left[\left(\frac{\mathrm{ad}(u(x))}{e^{\mathrm{ad}(u(x))} - \mathrm{id}} - \mathrm{id} \right) f_\alpha \right]_\gamma \end{aligned} \tag{4.20}$$

for $\alpha \in \Delta_{\mathfrak{u}}^+$. Let us note that the differential operators p_α, q_α and ∂_{x_α} have degree $-\alpha$ for $\alpha \in \Delta_{\mathfrak{u}}^+$. We denote by $\mathcal{N}_\alpha$ the Lie subalgebra of $\mathcal{A}_{\bar{\mathfrak{u}}}$ generated by the subset $\{\mathrm{ad}(\partial_{x_\alpha})^n(q_\alpha);\ n \in \mathbb{N}_0\}$ and by $\mathcal{U}_\alpha$ the $\mathbb{C}$-subalgebra of $\mathcal{A}_{\bar{\mathfrak{u}}}$ generated by $\mathcal{N}_\alpha$.

As ∂_{x_α} and p_α for $\alpha \in \Delta_{\mathfrak{u}}^+$ are locally ad-nilpotent regular elements of $\mathcal{A}_{\bar{\mathfrak{u}}}$, we may construct left rings of fractions for $\mathcal{A}_{\bar{\mathfrak{u}}}$ with respect to the multiplicative sets $\{\partial_{x_\alpha}^n;\ n \in \mathbb{N}_0\}$ and $\{p_\alpha^n; n \in \mathbb{N}_0\}$. Let us recall that for $\lambda \in \Lambda^+(\mathfrak{p})$ we have an isomorphism

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda - \rho_{\mathfrak{u}}) \simeq \mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}} \tag{4.21}$$

of $U(\mathfrak{g})$-modules, which gives rise to an isomorphism $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda - \rho_{\mathfrak{u}})_{(f_\alpha)} \simeq (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}})_{(f_\alpha)}$ of $U(\mathfrak{g})_{(f_\alpha)}$-modules. Since the $U(\mathfrak{g})$-module structure on $\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}}$ is given through the homomorphism

$$\pi_\lambda : U(\mathfrak{g}) \to \mathcal{A}_{\bar{\mathfrak{u}}} \otimes_{\mathbb{C}} \mathrm{End}\, \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}} \tag{4.22}$$

of $\mathbb{C}$-algebras and $\pi_\lambda(f_\alpha) = p_\alpha$ for $\lambda \in \Lambda^+(\mathfrak{p})$, we obtain that

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda - \rho_{\mathfrak{u}})_{(f_\alpha)} \simeq (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(p_\alpha)} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}} \tag{4.23}$$

as $U(\mathfrak{g})_{(f_\alpha)}$-modules, where the $U(\mathfrak{g})_{(f_\alpha)}$-module structure on $(\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(p_\alpha)} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}}$ is given by the uniquely determined homomorphism

$$\pi_{\lambda,\alpha} : U(\mathfrak{g})_{(f_\alpha)} \to (\mathcal{A}_{\bar{\mathfrak{u}}})_{(p_\alpha)} \otimes_{\mathbb{C}} \operatorname{End} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}}$$

of $\mathbb{C}$-algebras.

Let us note that the structure of an $\mathcal{A}_{\bar{\mathfrak{u}}}$-module on $(\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(p_\alpha)}$ is quite complicated in general. However, we show that

$$(\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(p_\alpha)} \simeq (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$$

as modules over the Weyl algebra $\mathcal{A}_{\bar{\mathfrak{u}}}$.

Lemma 4.6. *The element $q_\alpha \partial_{x_\alpha}^{-1} \in (\mathcal{A}_{\bar{\mathfrak{u}}})_{(\partial_{x_\alpha})}$ acts locally nilpotently on $(\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$ for $\alpha \in \Delta_{\mathfrak{u}}^+$.*

Proof. From (4.20) we have $\operatorname{ad}(\partial_{x_\alpha})^n(q_\alpha) \in \mathcal{I}_{\mathrm{V}}$ for $n \in \mathbb{N}_0$, which gives us $\mathcal{N}_\alpha \subset \mathcal{I}_{\mathrm{V}}$ and $\mathcal{U}_\alpha \subset \mathcal{I}_{\mathrm{V}}$ since $\mathcal{I}_{\mathrm{V}}$ is a left ideal of $\mathcal{A}_{\bar{\mathfrak{u}}}$. Further, let $\{\mathcal{F}_k\mathcal{A}_{\bar{\mathfrak{u}}}\}_{k\in\mathbb{N}_0}$ be a filtration of $\mathcal{A}_{\bar{\mathfrak{u}}}$ given by the order of a differential operator. Then we have

$$\mathcal{N}_\alpha(\mathcal{F}_k\mathcal{A}_{\bar{\mathfrak{u}}} + \mathcal{I}_{\mathrm{V}}) \subset \operatorname{ad}(\mathcal{N}_\alpha)(\mathcal{F}_k\mathcal{A}_{\bar{\mathfrak{u}}}) + \mathcal{I}_{\mathrm{V}} \subset \mathcal{F}_k\mathcal{A}_{\bar{\mathfrak{u}}} + \mathcal{I}_{\mathrm{V}}$$

for $k \in \mathbb{N}_0$. Besides, we have

$$\mathcal{N}_\alpha = \bigoplus_{n\in\mathbb{N}_0} \mathcal{N}_{\alpha,-n\alpha}$$

with $\mathcal{N}_{\alpha,0} = 0$ and $\mathcal{N}_{\alpha,-\alpha} = \mathbb{C}q_\alpha$, which together with the fact that for each $\gamma \in Q$ there exists an integer $n_{k,\gamma} \in \mathbb{N}_0$ such that $\mathcal{F}_k\mathcal{A}_{\bar{\mathfrak{u}},\gamma-n\alpha} = 0$ for all $n > n_{k,\gamma}$ implies that

$$\mathcal{U}_{\alpha,-n\alpha}(\mathcal{F}_k\mathcal{A}_{\bar{\mathfrak{u}},\gamma} + \mathcal{I}_{\mathrm{V}}) \subset \mathcal{I}_{\mathrm{V}}$$

for all $n > n_{k,\gamma}$. In other words, for each $q \in \mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}}$ there exists an integer $n_q \in \mathbb{N}_0$ such that $\mathcal{U}_{\alpha,-n\alpha}q = 0$ for all $n > n_q$.

Further, we show by induction on k that

$$(q_\alpha\partial_{x_\alpha}^{-1})^k = \sum_{j=k}^{\infty} \partial_{x_\alpha}^{-j} p_{\alpha,k,j},$$

where $p_{\alpha,k,j} \in \mathcal{U}_{\alpha,-j\alpha}$ for $j \geq k$. Moreover, there exists an integer $j_k \in \mathbb{N}_0$ such that $p_{\alpha,k,j} = 0$ for $j > j_k$. For $k = 0$ we have

$$p_{\alpha,k,j} = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

Let assume that it holds for some $k \in \mathbb{N}_0$. Then we may write

$$\begin{aligned}(q_\alpha \partial_{x_\alpha}^{-1})^{k+1} &= q_\alpha \partial_{x_\alpha}^{-1} \sum_{k=j}^{\infty} \partial_{x_\alpha}^{-j} p_{\alpha,k,j} = \sum_{k=j}^{\infty} \partial_{x_\alpha}^{-j-1} \partial_{x_\alpha}^{j+1} q_\alpha \partial_{x_\alpha}^{-j-1} p_{\alpha,k,j} \\ &= \sum_{j=k}^{\infty} \partial_{x_\alpha}^{-j-1} \sum_{\ell=0}^{\infty} \binom{j+\ell}{\ell} \partial_{x_\alpha}^{-\ell} \operatorname{ad}(\partial_{x_\alpha})^\ell (q_\alpha) p_{\alpha,k,j} \\ &= \sum_{j=k+1}^{\infty} \partial_{x_\alpha}^{-j} \sum_{\ell=0}^{j-k-1} \binom{j-1}{\ell} \operatorname{ad}(\partial_{x_\alpha})^\ell (q_\alpha) p_{\alpha,k,j-\ell-1},\end{aligned}$$

where we used

$$\partial_{x_\alpha}^n q \partial_{x_\alpha}^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \partial_{x_\alpha}^{-k} \operatorname{ad}(\partial_{x_\alpha})^k (q)$$

for $n \in \mathbb{N}_0$ and $q \in \mathcal{A}_{\bar{\mathfrak{u}}}$ in the third equality, which gives us

$$p_{\alpha,k+1,j} = \sum_{\ell=0}^{j-k-1} \binom{j-1}{\ell} \operatorname{ad}(\partial_{x_\alpha})^\ell (q_\alpha) p_{\alpha,k,j-\ell-1}$$

for $j \geq k+1$. Therefore, we have $p_{\alpha,k+1,j} \in \mathcal{U}_{\alpha,-j\alpha}$ for $j \geq k+1$ and $j_{k+1} = j_k + m + 1$, where $m \in \mathbb{N}_0$ satisfies $\operatorname{ad}(\partial_{x_\alpha})^m (q_\alpha) = 0$.

Furthermore, for $n \in \mathbb{N}_0$ and $q \in \mathcal{A}_{\bar{\mathfrak{u}}}$ we have

$$\begin{aligned}p_{\alpha,k,j} \partial_{x_\alpha}^{-n} q &= \partial_{x_\alpha}^{-n} \partial_{x_\alpha}^{n} p_{\alpha,k,j} \partial_{x_\alpha}^{-n} q \\ &= \partial_{x_\alpha}^{-n} \sum_{\ell=0}^{\infty} \binom{n+\ell-1}{\ell} \partial_{x_\alpha}^{-\ell} \operatorname{ad}(\partial_{x_\alpha})^\ell (p_{\alpha,k,j}) q\end{aligned}$$

for $j,k \in \mathbb{N}_0$ and $j \geq k$. As we have $p_{\alpha,k,j} \in \mathcal{U}_{\alpha,-j\alpha}$, we obtain $\operatorname{ad}(\partial_{x_\alpha})^\ell (p_{\alpha,k,j}) \in \mathcal{U}_{\alpha,-(\ell+j)\alpha}$ for $\ell \in \mathbb{N}_0$. Since for $q \in \mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}}$ there exists an integer $k_q \in \mathbb{N}_0$ such that $\mathcal{U}_{\alpha,-k\alpha} q = 0$ for all $k > k_q$, we get that $p_{\alpha,k,j} \partial_{x_\alpha}^{-n} q = 0$ for all $k > k_q$ and $j \geq k$. Therefore, we have $(q_\alpha \partial_{x_\alpha}^{-1})^k \partial_{x_\alpha}^{-n} q = 0$ for $k > k_q$. Hence, we have the required statement. ☐

Lemma 4.7. *For $\alpha \in \Delta_{\mathfrak{u}}^+$ the $\mathbb{C}$-linear mapping*

$$\varphi_\alpha \colon (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})} \to (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$$

defined by

$$\varphi_\alpha(a) = -\partial_{x_\alpha}^{-1} \sum_{k=0}^{\infty} (q_\alpha \partial_{x_\alpha}^{-1})^k a \tag{4.24}$$

for $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$ satisfies the relations

$$p_\alpha \varphi_\alpha(a) = a \quad \textit{and} \quad \varphi_\alpha(p_\alpha a) = a$$

for $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$.

Proof. By Lemma 4.6 we have that the $\mathbb{C}$-linear mapping φ_α is well defined since the sum on the right-hand side is finite. Further, we may write

$$\begin{aligned} p_\alpha \varphi_\alpha(a) &= (\partial_{x_\alpha} - q_\alpha)\partial_{x_\alpha}^{-1} \sum_{k=0}^{\infty} (q_\alpha \partial_{x_\alpha}^{-1})^k a \\ &= \sum_{k=0}^{\infty} (q_\alpha \partial_{x_\alpha}^{-1})^k a - \sum_{k=0}^{\infty} (q_\alpha \partial_{x_\alpha}^{-1})^{k+1} a = a \end{aligned}$$

and

$$\begin{aligned} \varphi_\alpha(p_\alpha a) &= \partial_{x_\alpha}^{-1} \sum_{k=0}^{\infty} (q_\alpha \partial_{x_\alpha}^{-1})^k (\partial_{x_\alpha} - q_\alpha) a \\ &= \partial_{x_\alpha}^{-1} \sum_{k=0}^{\infty} (q_\alpha \partial_{x_\alpha}^{-1})^k (1 - q_\alpha \partial_{x_\alpha}^{-1}) \partial_{x_\alpha} a \\ &= \partial_{x_\alpha}^{-1} \bigg(\sum_{k=0}^{\infty} (q_\alpha \partial_{x_\alpha}^{-1})^k - \sum_{k=0}^{\infty} (q_\alpha \partial_{x_\alpha}^{-1})^{k+1} \bigg) \partial_{x_\alpha} a = a \end{aligned}$$

for all $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$. □

Lemma 4.8. *Let $\alpha \in \Delta_{\mathfrak{u}}^+$. Then for all $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$ we have*

(i) $\varphi_\alpha^n(p_\alpha^m a) = p_\alpha^m \varphi_\alpha^n(a) = \varphi_\alpha^{n-m}(a)$ *for* $n, m \in \mathbb{N}_0$ *and* $n \geq m$;
(ii) $p_\alpha^n q \varphi_\alpha^n(a) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \varphi_\alpha^k(\mathrm{ad}(p_\alpha)^k(q)a)$ *for* $n \in \mathbb{N}_0$ *and* $q \in \mathcal{A}_{\bar{\mathfrak{u}}}$.

Proof. The first statement follows immediately from Lemma 4.7. Further, for $n \in \mathbb{N}_0$ and $q \in \mathcal{A}_{\bar{\mathfrak{u}}}$ we have

$$p_\alpha q \varphi_\alpha(a) = \sum_{k=0}^{n} \varphi_\alpha^k(\mathrm{ad}(p_\alpha)^k(q)a) + \varphi_\alpha^n(\mathrm{ad}(p_\alpha)^{n+1}(q)\varphi_\alpha(a))$$

for $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$, which follows by induction on n. Indeed, for $n = 0$ we have

$$p_\alpha q \varphi_\alpha(a) = q p_\alpha \varphi_\alpha(a) + \mathrm{ad}(p_\alpha)(q)\varphi_\alpha(a) = qa + \mathrm{ad}(p_\alpha)(q)\varphi_\alpha(a)$$

for $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$. Let us assume that it holds for some $n \in \mathbb{N}_0$. Then we may write

$$\begin{aligned} p_\alpha q \varphi_\alpha(a) &= \sum_{k=0}^{n} \varphi_\alpha^k(\mathrm{ad}(p_\alpha)^k(q)a) + \varphi_\alpha^n(\mathrm{ad}(p_\alpha)^{n+1}(q)\varphi_\alpha(a)) \\ &= \sum_{k=0}^{n} \varphi_\alpha^k(\mathrm{ad}(p_\alpha)^k(q)a) + \varphi_\alpha^{n+1}(p_\alpha\, \mathrm{ad}(p_\alpha)^{n+1}(q)\varphi_\alpha(a)) \\ &= \sum_{k=0}^{n} \varphi_\alpha^k(\mathrm{ad}(p_\alpha)^k(q)a) + \varphi_\alpha^{n+1}(\mathrm{ad}(p_\alpha)^{n+1}(q)a) \\ &\quad + \varphi_\alpha^{n+1}(\mathrm{ad}(p_\alpha)^{n+2}(q)\varphi_\alpha(a)) \end{aligned}$$

for $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$. As there exists an integer $n_q \in \mathbb{N}_0$ such that $\mathrm{ad}(p_\alpha)^n(q) = 0$ for all $n > n_q$, we obtain

$$p_\alpha q \varphi_\alpha(a) = \sum_{k=0}^{\infty} \varphi_\alpha^k(\mathrm{ad}(p_\alpha)^k(q)a)$$

for $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$. Therefore, we proved the second statement for $n = 1$. The rest of the proof is by induction on n. Let us assume that it holds for some $n \in \mathbb{N}_0$. Then we may write

$$\begin{aligned} p_\alpha^{n+1} q \varphi_\alpha^{n+1}(a) &= p_\alpha \sum_{k=0}^{\infty} \binom{n+k-1}{k} \varphi_\alpha^k(\mathrm{ad}(p_\alpha)^k(q)\varphi_\alpha(a)) \\ &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \varphi_\alpha^k(p_\alpha \,\mathrm{ad}(p_\alpha)^k(q)\varphi_\alpha(a)) \\ &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \sum_{\ell=0}^{\infty} \varphi_\alpha^{k+\ell}(\mathrm{ad}(p_\alpha)^{k+\ell}(q)a) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{n+j-1}{j} \varphi_\alpha^k(\mathrm{ad}(p_\alpha)^k(q)a) \\ &= \sum_{k=0}^{\infty} \binom{n+k}{k} \varphi_\alpha^k(\mathrm{ad}(p_\alpha)^k(q)a) \end{aligned}$$

for $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$. □

The previous lemma enables us to define the structure of an $(\mathcal{A}_{\bar{\mathfrak{u}}})_{(p_\alpha)}$-module on $(\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$ for $\alpha \in \Delta_{\mathfrak{u}}^+$ by

$$p_\alpha^{-n} qa = \varphi_\alpha^n(qa),$$

where $n \in \mathbb{N}_0$, $q \in \mathcal{A}_{\bar{\mathfrak{u}}}$ and $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$.

Proposition 4.9. *Let $\alpha \in \Delta_{\mathfrak{u}}^+$. Then the $\mathbb{C}$-linear mapping*

$$\Phi_\alpha \colon (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(p_\alpha)} \to (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$$

defined through

$$\Phi_\alpha(p_\alpha^{-n} a) = \varphi_\alpha^n(a) \tag{4.25}$$

for $n \in \mathbb{N}_0$ and $a \in \mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}}$ is an isomorphism of $(\mathcal{A}_{\bar{\mathfrak{u}}})_{(p_\alpha)}$-modules.

Proof. Since Φ_α is a homomorphism of $(\mathcal{A}_{\bar{\mathfrak{u}}})_{(p_\alpha)}$-modules based on the previous considerations, we only need to prove that the $\mathbb{C}$-linear mapping Φ_α is injective and surjective. We may write

$$\Phi_\alpha\Bigg(\prod_{\gamma \in \Delta_{\mathfrak{u}}^+} \partial_{x_\gamma}^{n_\gamma} p_\alpha^{-n} 1\Bigg) = \prod_{\gamma \in \Delta_{\mathfrak{u}}^+} \partial_{x_\gamma}^{n_\gamma} \varphi_\alpha^n(1) = (-1)^n \prod_{\gamma \in \Delta_{\mathfrak{u}}^+} \partial_{x_\gamma}^{n_\gamma} \partial_{x_\alpha}^{-n}$$

for $n \in \mathbb{N}_0$ and $n_\gamma \in \mathbb{N}_0$ for $\gamma \in \Delta_{\mathfrak{u}}^+$, which implies that Φ_α is surjective. To prove the injectivity of Φ_α let us assume that $\Phi_\alpha(a) = 0$ for some $a \in (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(p_\alpha)}$. Then there exists an integer $n \in \mathbb{N}_0$ such that $p_\alpha^n a \in \mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}} \subset (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(p_\alpha)}$. Hence, we have

$$0 = p_\alpha^n \Phi_\alpha(a) = \Phi_\alpha(p_\alpha^n a) = p_\alpha^n a \in \mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}} \subset (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})},$$

which gives us $a = 0$. Therefore, we proved that Φ_α is an isomorphism of $(\mathcal{A}_{\bar{\mathfrak{u}}})_{(p_\alpha)}$-modules. □

Let us recall that for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_{\mathfrak{u}}^+$ we have an isomorphism

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda - \rho_{\mathfrak{u}})_{(f_\alpha)} \simeq (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(p_\alpha)} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}}$$

of $U(\mathfrak{g})$-modules, where the $U(\mathfrak{g})$-module structure on $(\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(p_\alpha)} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}}$ is given through the homomorphism (4.22) of $\mathbb{C}$-algebras. By Proposition 4.9 we have an isomorphism $(\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(p_\alpha)} \simeq (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})}$ of $\mathcal{A}_{\bar{\mathfrak{u}}}$-modules, which gives rise to an isomorphism

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda - \rho_{\mathfrak{u}})_{(f_\alpha)} \simeq (\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}} \tag{4.26}$$

of $U(\mathfrak{g})$-modules, where the action of $U(\mathfrak{g})$ on $(\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}})_{(\partial_{x_\alpha})} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+\rho_{\mathfrak{u}}}$ is given by the homomorphism (4.22) of $\mathbb{C}$-algebras.

We denote by $\Delta_{\mathfrak{u},\alpha}^+$ the set $\Delta_{\mathfrak{u}}^+ \backslash \{\alpha\}$ for $\alpha \in \Delta_{\mathfrak{u}}^+$. Further, let us consider $\mathcal{A}_{\bar{\mathfrak{u}}}$-modules

$$\mathcal{F}_{\bar{\mathfrak{u}}} = \mathbb{C}[\partial_{x_\gamma}, \gamma \in \Delta_{\mathfrak{u}}^+] \quad \text{and} \quad \mathcal{F}_{\bar{\mathfrak{u}},\alpha} = \mathbb{C}[x_\alpha, \partial_{x_\gamma}, \gamma \in \Delta_{\mathfrak{u},\alpha}^+] \tag{4.27}$$

for $\alpha \in \Delta_{\mathfrak{u}}^+$, where $\mathcal{F}_{\bar{\mathfrak{u}}}$ and $\mathcal{F}_{\bar{\mathfrak{u}},\alpha}$ are endowed with the structure of $\mathcal{A}_{\bar{\mathfrak{u}}}$-modules by means of the canonical isomorphisms of vector spaces

$$\mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{\mathrm{V}} \simeq \mathcal{F}_{\bar{\mathfrak{u}}} \quad \text{and} \quad \mathcal{A}_{\bar{\mathfrak{u}}}/\mathcal{I}_{GT,\alpha} \simeq \mathcal{F}_{\bar{\mathfrak{u}},\alpha}. \tag{4.28}$$

The left ideals $\mathcal{I}_{\mathrm{V}}$ and $\mathcal{I}_{\mathrm{GT},\alpha}$ of the Weyl algebra $\mathcal{A}_{\bar{\mathfrak{u}}}$ are defined through $\mathcal{I}_{\mathrm{V}} = (x_\gamma, \gamma \in \Delta_{\mathfrak{u}}^+)$ and $\mathcal{I}_{\mathrm{GT},\alpha} = (\partial_{x_\alpha}, x_\gamma, \gamma \in \Delta_{\mathfrak{u},\alpha}^+)$. In addition, we have a short exact sequence

$$0 \to \mathcal{F}_{\bar{\mathfrak{u}}} \to (\mathcal{F}_{\bar{\mathfrak{u}}})_{(\partial_{x_\alpha})} \to \mathcal{F}_{\bar{\mathfrak{u}},\alpha} \to 0 \tag{4.29}$$

of $\mathcal{A}_{\bar{\mathfrak{u}}}$-modules, where the surjective homomorphism of $\mathcal{A}_{\bar{\mathfrak{u}}}$-modules from $(\mathcal{F}_{\bar{\mathfrak{u}}})_{(\partial_{x_\alpha})}$ to $\mathcal{F}_{\bar{\mathfrak{u}},\alpha}$ is given by

$$\prod_{\gamma \in \Delta_{\mathfrak{u}}^+} \partial_{x_\gamma}^{n_\gamma} \mapsto \begin{cases} \dfrac{x_\alpha^{-n_\alpha-1}}{(-n_\alpha-1)} \displaystyle\prod_{\gamma \in \Delta_{\mathfrak{u},\alpha}^+} \partial_{x_\gamma}^{n_\gamma} & \text{if } n_\alpha < 0, \\ 0 & \text{if } n_\alpha \geq 0 \end{cases}$$

for $n_\alpha \in \mathbb{Z}$ and $n_\gamma \in \mathbb{N}_0$, $\gamma \in \Delta_{\mathfrak{u},\alpha}^+$.

Theorem 4.10. *Let $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_{\mathfrak{u}}^+$. Then we have*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \simeq \frac{\mathcal{A}_{\bar{\mathfrak{u}}}}{\mathcal{I}_{\mathrm{V}}} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+2\rho_{\mathfrak{u}}} \simeq \mathcal{F}_{\bar{\mathfrak{u}}} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+2\rho_{\mathfrak{u}}} \tag{4.30}$$

and

$$W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda,\alpha) \simeq \frac{\mathcal{A}_{\bar{\mathfrak{u}}}}{\mathcal{I}_{GT,\alpha}} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+2\rho_{\mathfrak{u}}} \simeq \mathcal{F}_{\bar{\mathfrak{u}},\alpha} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+2\rho_{\mathfrak{u}}}, \tag{4.31}$$

where $\mathcal{I}_{\mathrm{V}}$ *and* $\mathcal{I}_{\mathrm{GT},\alpha}$ *are the left ideals of the Weyl algebra* $\mathcal{A}_{\bar{\mathfrak{u}}}$ *defined by* $\mathcal{I}_{\mathrm{V}} = (x_\gamma, \gamma \in \Delta_{\mathfrak{u}}^+)$ *and* $\mathcal{I}_{\mathrm{GT},\alpha} = (\partial_{x_\alpha}, x_\gamma, \gamma \in \Delta_{\mathfrak{u},\alpha}^+)$.

Proof. From the previous considerations and definition of $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda,\alpha)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_{\mathfrak{u}}^+$ we have

$$W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda,\alpha) \simeq \frac{M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)_{(f_\alpha)}}{M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)} \simeq \left(\frac{(\mathcal{F}_{\bar{\mathfrak{u}}})_{(\partial_{x_\alpha})}}{\mathcal{F}_{\bar{\mathfrak{u}}}}\right) \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+2\rho_{\mathfrak{u}}} \simeq \mathcal{F}_{\bar{\mathfrak{u}},\alpha} \otimes_{\mathbb{C}} \mathbb{F}_{\lambda+2\rho_{\mathfrak{u}}},$$

where we used (4.26), (4.28) and (4.29). □

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