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Photon emission in the graphene under the action of a quasiconstant external electric field

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Abstract Following a nonperturbative formulation of strong-field QED developed in our earlier works, and using the Dirac model of the graphene, we construct a reduced $QED_{3,2}$ to describe one species of the Dirac fermions in the graphene interacting with an external electric field and photons. On this base, we consider the photon emission in this model and construct closed formulas for the total probabilities. Using the derived formulas, we study probabilities for the photon emission by an electron and for the photon emission accompanying the vacuum instability in the quasiconstant electric field that acts in the graphene plane during the time interval T. We study angular and polarization distribution of the emission as well as emission characteristics in a high-frequency and low-frequency approximations. We analyze the applicability of the presented calculations to the graphene physics in laboratory conditions. In fact, we are talking about a possible observation of the Schwinger effect in these conditions.

1 Introduction

Graphene and similar nanostructures (topological insulators, etc.) belong to the class of so-called Dirac semimetals, the theoretical and experimental study of which has recently received much attention. In particular, this is due to the hopes for possible applications of these structures in the next generation of semiconducting devices. It should be noted that already in the early theoretical works [1–3] it was indicated that the physics of the graphene is not only described under certain conditions by QFT (quantum field theory), but can be a fertile area for the latter where its validity within extreme limits can be verified in laboratory conditions. This possibility is related to the fact that low-energy single-electron motion in graphene monolayers (at the charge neutrality point) and similar nanostructures is described by the Dirac model, being a 2 + 1 dimensional version of massless QED (quantum electrodynamics) with Fermi velocity $v_F \simeq 10^6$ m/s playing the role of the light speed in the relativistic dynamics of the corresponding Dirac fermions [1] (see also the review [4] for more details). Such a model is usually called reduced $QED_{3,2}$. It should be noted that in the $QED_{3,2}$ model the electromagnetic field itself is not confined to the graphene plane $z = 0$, but rather propagates (with the speed of light c) according to corresponding classical or quantum equations in the ambient 3 + 1-dimensional space-time. The electromagnetic field couples minimally to electrons situated on the graphene plane. Thus, we note once again that in the $QED_{3,2}$ there are two distinct velocities, one of charged particles (Dirac fermions) and another one of the electromagnetic fields. Since the Dirac fermions in the model are considered as almost massless, any low-frequency electric field is for them supercritical (the so-called Schwinger critical field $E_c = m^2 c^3 / e\hbar$ is almost zero). The latter fact allows one a laboratory verifying QED predictions for superstrong fields, in particular, real studying the Schwinger effect. From the theoretical point of view, what has been said means that the vacuum state in the model is sometimes unstable with respect to the Dirac fermion creation, such that the interaction with electric-like external field must be taken into account nonperturbatively. That is why the standard theory of the photon emission represented in QED text books cannot help. From our point of view, adequate nonperturbative calculations with respect to the external field can be done using a general approach to QED with strong external fields [5–8] (based on the existence of special exact solutions of the Dirac equation with these fields) and its further development [9, 10]. Thus, the $QED_{3,2}$ model with a part of electric-like electromagnetic field considering as an external classical one must always be treated by the abovementioned nonperturbative methods. Note that the effects due to high-frequency electromagnetic fields, which are often considered in connection with the optical response of Dirac

fermions in graphene, do not require the use of the mentioned nonperturbative methods. In the $QED_{3,2}$ model, there are actually two species of fermions corresponding to excitations about two distinct Dirac points in the Brillouin zone of the graphene. Taking into account the presence of two spin polarization of excitations of each kind, we have, in fact, four species of fermions in the model. Calculations of mean values can be done for one of the specie with some further extension to four species using the degeneracy factor $N_f = 4$.

Until now, in 3 + 1 QED, there were presented various nonperturbative calculations of zero-order processes in the framework of the general approach [5–10]; for example, see Refs. [11, 12] and references therein. These are processes of charged particle scattering, and processes of charge particle creation and annihilation related to the vacuum instability. In $QED_{3,2}$, the graphene conductivity modification due to the particle creation by external constant electric field (the Schwinger effect) was calculated as a zero-order process in Ref. [13]. Processes involving photon emission and annihilation in the presence of the vacuum instability are processes of higher order in radiative corrections. Their study is technically more complicated then the study of zero-order processes. Nevertheless, recently, there appeared publications devoted to the photon emission in the graphene due to external electric field in the framework of the Dirac model. In particular, a free electron--hole recombination was studied in Refs. [14] and [15] for the thermal equilibrium. A discussion of the photon emission by charged carries in the graphene due to constant uniform electric field was published in Ref. [16]. Due to the recent detection of an optical radiation in the graphene accompanying the creation of electron--hole pairs by a terahertz pulse [17, 18], it becomes possible to make a comparison of the corresponding theoretical calculations with experiments. It must be said that the emission of a photon by an electron moving in a constant electric field in 3 + 1 dimensions was studied first nonperturbatively by Nikishov [19, 20].

Peculiarities of physics of the graphene allows one studying the Schwinger effect in laboratory conditions. Theoretical calculations presented in the work [13] and their comparison with experimentally observed results of the *dc* conductivity in the graphene near the Dirac point testify in favor of the fact that it is the Schwinger effect that determines the nature of the conductivity. In addition, it has been found that the radiation of a time-dependent mean current, forming the backreaction to the electric field on the graphene plane, is emitted to the three-dimensional space in the form of linearly polarized and of very low-frequency plane electromagnetic waves. However, an observation of such waves and the *dc* conductivity is not a simple task in the presence of the background noise in the vicinity of the graphene sample. We believe that the emission and absorption of high-frequency photons accompanied the electronic quantum transport in the graphene are more realistic for possible experimental observations.

We stress that general equations allowing nonperturbative calculations of the higher-order processes in strong-field QED are clearly formulated in the Refs. [5–10]. In the present study, we specify these equations for the above described $QED_{3,2}$ model and with their help we consider processes of photon emission by the Dirac excitations in the graphene subjected by external constant electric field. In these calculations, effects of the vacuum instability are taken into account exactly, such that we study the process of the photon emission which is accompanied by creation from the vacuum additional Dirac excitations.

In contrast to the works known to us, in this article we consider effects in the $QED_{3,2}$ model with an intense external electric field, which is uniform and slowly varying, and which we interpret as a macroscopic external field. These effect differ principally from effects arising in magnetic-like fields or in fields of high-frequency electromagnetic waves. In condensed matter, the Dirac model is used primarily in the context of the relativistic quantum mechanics, or in the framework of the Matsubara's imaginary time formalism of QFT, where electrons are assumed to be in thermal equilibrium; see, e.g. Refs. [21–31] . However, a macroscopic electric field acting on charged particles may destroy their thermal equilibrium, such that conclusions based on the latter assumption may be not correct.

The article is organized as follows: In Sect. 2 we, following the general theory [5–8], construct the reduced $QED_{3,2}$ to describe one species of the Dirac fermions in the graphene interacting with an external electric field and photons. The required basic elements that we need to describe zero-order processes with respect to the electron--photon interaction are derived from Ref. [13]. Then, we consider the photon emission in this model and construct closed formulas for the total probabilities. In order to find the corresponding mean values in real graphene, results obtained for one species of the Dirac fermions must be multiplied by the factor $N_f = 4$ (the number of all charged species). In Sect. 3, we apply the developed formulation to calculating probabilities of the one-photon emission by an electron and of the one-photon emission accompanying the vacuum instability in a quasiconstant electric field that acts in the graphene plane during the time interval T. In Sect. 4, we analyze the obtained emissions characteristics in a high-frequency approximation. We study angular and polarization distribution of the emissions. The low-frequency approximation is considered in Appendix A. We analyze conditions of the applicability of the presented calculations in possible experimental conditions. In the last Sect. 5, we summarize the main results of the present work. Some useful mathematical details are placed in Appendix B.

2 The photon emission in the graphene in the framework of the $QED_{3,2}$ model

2.1 General

In this section, we consider general equations that will be used by us further to study the photon emission in a flat graphene monolayer in the framework of the nonperturbative approach [5–8] applied to the $QED_{3,2}$ model described schematically above.

We consider an infinite flat graphene sample to which an uniform electric field is applied, directed constantly along the axis x on the plane of the sample. We assume that the applied field is a strong external macroscopic low-frequency electric-like field that can treated as a quasiconstant one. We consider the case of zero temperature and chemical potential (i.e., at the charge neutrality point), so that the Dirac model can be used near the Dirac point.

As was already said the graphene sample in subjected to the action of a strong external macroscopic low-frequency electric-like field, some suppositions about which were already mentioned above. This field is parallel to the graphene plane, $z = 0$. By $\mathbf{r} = (x, y)$, we denote the two-dimensional position vector on the graphene plane. In which follows, we use boldface symbols for any two-dimensional vectors in $z = 0$ plane. The electromagnetic field couples minimally to the current of the Dirac fermions in the graphene plane. The external field can be given by two-dimensional vector potential $\mathbf{A}^{\text{ext}}(t, \mathbf{r}, z) = \left(A_x^{\text{ext}}, A_y^{\text{ext}} \right)$ (the scalar potential is chosen to be zero, $A_0 = 0$). In the model under consideration, charged particles of each kind in the graphene are described by the Dirac field which is two component spinor $\psi_\alpha(t, \mathbf{r})$, $\alpha = 1, 2$ on $2 + 1$ dimension. In this dimension, the algebra of the corresponding γ-matrices has two inequivalent representations,

$$\gamma^0 = \sigma^3, \ \gamma^1 = i\sigma^2, \ \gamma^2 = -i\varsigma\sigma^1, \tag{1}$$

where the σ^j are Pauli matrices, and by $\varsigma = \pm 1$ inequivalent representations are labeled. Distinct (pseudo spin) representations are associated with each Dirac point. For all integral quantities, since intervalley scattering can be neglected, the presence of two valleys related to each $\varsigma = \pm 1$ inequivalent representation is taken into account simply by multiplying by introducing the degeneracy factor 2. Taking into account the spin degeneracy factor 2, the total number of different species of Dirac fermions is $N_f = 4$. In order to find mean values of a physical quantity in the graphene, a mean value obtained for one species are multiplied by N_f. Remembering the origin of the Dirac model for the graphene description (see [1]), we believe that each component $\psi_\alpha(t, \mathbf{r})$ of the Dirac spinor is a projection of a Schrödinger wave function $\phi_\alpha(t, \mathbf{r}, z)$ in $3 + 1$ dimensions with a support in a specific sublattice of the honeycomb lattice of the graphene. These wave functions can be represented as:

$$\phi_\alpha(t, \mathbf{r}, z) = \psi_\alpha(t, \mathbf{r})\varphi(z) \, e^{ip_z z/\hbar}, \tag{2}$$

where the function $\varphi(z)$ describes the width of the graphene. A detailed description of $\varphi(z)$ is not necessary for our purposes, except for the fact that it decays rapidly outside the xy plane and is normalized according to $\int dz \|\varphi(z)\|^2$. In which follows, we assume the usual dipole approximation so that the exponential in Eq. (2) is approximated by the zeroth-order constant term. In this approximation, we replace $A^{\text{ext}}(t, \mathbf{r}, z)$ by its value $\mathbf{A}^{\text{ext}}(t, \mathbf{r}, 0)$ at $z = 0$. Then we can simplify the notation as follows: $\mathbf{A}^{\text{ext}}(t, \mathbf{r}, 0) = \mathbf{A}^{\text{ext}}(t, \mathbf{r})$. We allow the graphene sheet to have a global momentum p_z along the z axis, in order to account for the possibility of a momentum transfer in this direction with respect to some external system. The Dirac equation with an external field that couples minimally to electrons on graphene plane reads:

$$i\hbar \, \partial_t \psi(t, \mathbf{r}) = H^{\text{ext}}\psi(t, \mathbf{r}),$$
$$H^{\text{ext}} = v_F \gamma^0 \left\{ \boldsymbol{\gamma} \left[\mathbf{p} + \frac{e}{c}\mathbf{A}^{\text{ext}}(t, \mathbf{r}) \right] + m v_F \right\}, \tag{3}$$

where $\mathbf{p} = (p_x, p_y)$ is the in-plane component of the momentum operator, $\boldsymbol{\gamma} = (\gamma^1, \gamma^2)$, γ-matrices satisfy the standard anticommutation relations $\left[\gamma^\mu, \gamma^\nu \right]_+ = 2\eta^{\mu\nu}$, $\eta_{\mu\nu} = \text{diag}(+1, -1, -1)$, $\mu, \nu = 0, 1, 2$, and $e > 0$ is the absolute value of the electron charge.

In Eq. (3), a mass term in the Hamiltonian H^{ext} is introduced for one to be able to generalize the consideration to the presence of the possible mass gap $\Delta\varepsilon = m v_F^2$. Such a mass gap in the graphene band structure can appear in different ways. One of the examples is given by graphene nanoribbons (see [26] for a review). However, in our consideration below, we set $m = 0$.

Dirac Heisenberg operators $\hat{\Psi}(t, \mathbf{r})$ and $\hat{\Psi}^\dagger(t, \mathbf{r})$ are assigned to the Dirac fields $\psi(t, \mathbf{r})$ and $\psi^\dagger(t, \mathbf{r})$. These fields obey both the Dirac Eq. (3) with the potential $\mathbf{A}^{\text{ext}}(t, \mathbf{r})$] and the following nonvanishing equal time anticommutation relations:

$$\left[\hat{\Psi}(t, \mathbf{r}), \hat{\Psi}(t, \mathbf{r}') \right]_+, \ \left[\hat{\Psi}(t, \mathbf{r}), \hat{\Psi}^\dagger(t, \mathbf{r}') \right]_+ = \hbar\delta^{(2)}(\mathbf{r} - \mathbf{r}'). \tag{4}$$

The quantized free electromagnetic field is described by two-dimensional operators of vector potential $\hat{\mathbf{A}}(t, \mathbf{r}, z)$. As for the classical potentials, the dipole approximation allows us to replace $\hat{\mathbf{A}}(t, \mathbf{r}, z)$ by its value $\hat{\mathbf{A}}(t, \mathbf{r}) = \hat{\mathbf{A}}(t, \mathbf{r}, 0)$.

The total quantum Hamiltonian of the model reads:

$$\widehat{\mathcal{H}}(t) = \widehat{\mathcal{H}}_{\text{e},\mathbf{A}^{\text{ext}}} + \widehat{\mathcal{H}}_{\text{e},} + \widehat{\mathcal{H}},$$
$$\widehat{\mathcal{H}}_{\text{e},\mathbf{A}^{\text{ext}}} = \int \hat{\Psi}^\dagger(t, \mathbf{r}) H^{\text{ext}} \hat{\Psi}(t, \mathbf{r}) d\mathbf{r},$$
$$\widehat{\mathcal{H}}_{\text{e},} = -\int \hat{\mathbf{j}}(t, \mathbf{r})\hat{\mathbf{A}}(t, \mathbf{r}) d\mathbf{r}, \ \hat{\mathbf{j}}(t, \mathbf{r}) = -\frac{e v_F}{2c} \left[\hat{\Psi}^\dagger(t, \mathbf{r}), \gamma^0 \hat{\Psi}(t, \mathbf{r}) \right]_-, \tag{5}$$

where $\widehat{\mathcal{H}}_{\text{e},\mathbf{A}^{\text{ext}}}$ is the Hamiltonian of charged particles interacting with an external electric-like field given by the time-dependent potential $\mathbf{A}^{\text{ext}}(t, \mathbf{r})$, $\widehat{\mathcal{H}}_{\text{e},}$ is the Hamiltonian of the electron–photon interaction, and $\widehat{\mathcal{H}}$ is the free photon Hamiltonian. The integral

on the graphene plane is taken over an area S. We assume that the area S is sufficiently large to be macroscopic then boundary effects can be neglected.

The decomposition of the operator $\hat{\mathbf{A}}(t, \mathbf{r})$ in terms of annihilation and creation operators of free photons, $C_{\mathbf{K}\vartheta}$ and $C_{\mathbf{K}\vartheta}^{\dagger}$ reads:

$$\hat{\mathbf{A}}(t, \mathbf{r}) = c \sum_{\mathbf{K},\vartheta} \sqrt{\frac{2\pi\hbar}{\varepsilon V \omega}} \boldsymbol{\epsilon}_{\mathbf{K}\vartheta} \left[C_{\mathbf{K}\vartheta}\, e^{i(\mathbf{kr}-\omega t)} + C_{\mathbf{K}\vartheta}^{\dagger}\, e^{-i(\mathbf{kr}-\omega t)} \right], \qquad (6)$$

where $\vartheta = 1, 2$ denotes a polarization index, $\boldsymbol{\epsilon}_{\mathbf{K}\vartheta}$ are mutual orthogonal unit polarization vectors transversal to three-dimensional wave vector $\mathbf{K} = (\mathbf{k}, k_z)$. The two-dimensional vector $\mathbf{k} = (k_x, k_y)$ is a projection of $\mathbf{K}$ on the graphene plane, $\omega = cK$, $K = |\mathbf{K}|$, V is the volume of the box regularization, and ε is the relative permittivity (for the graphene suspended in the vacuum $\varepsilon = 1$).

2.2 In- and out-states of charged particles

Following the general nonperturbative approach [5–8], we have to construct the corresponding in- and out-states. of charged particles of all the kinds with the help of exact solutions of Eq. (3) the electric-like external field. As was already mentioned above, the external field in the model is a slowly varying uniform electric-like field directed along the axis x. It is assumed that for $t < t_1$ and for $t > t_2$, the electric field is absent, therefore initial $|0, \text{in}\rangle_e$ and final $|0, \text{out}\rangle_e$ are vacuum state of free in- and out-charged particles, respectively. These vacua are different due to a difference of initial and final values of external electromagnetic field potentials. During the time interval $t_2 - t_1 = T$, the Dirac field interacts with the external field. There exists a set of creation and annihilation operators $a_n^{\dagger}(\text{in})$ and $a_n(\text{in})$ of in-particles (electrons), and operators $b_n^{\dagger}(\text{in})$ and $b_n(\text{in})$ of in-antiparticles (holes), at the same time there exists a set of creation and annihilation operators $a_n^{\dagger}(\text{out})$ and $a_n(\text{out})$ of out-electrons and operators $b_n^{\dagger}(\text{out})$ and $b_n(\text{out})$ of out-holes,

$$\begin{aligned} a_n(\text{in})|0, \text{in}\rangle_e &= b_n(\text{in})|0, \text{in}\rangle_e = 0, \ \forall n, \\ a_n(\text{out})|0, \text{out}\rangle_e &= b_n(\text{out})|0, \text{out}\rangle_e = 0, \ \forall n, \end{aligned} \qquad (7)$$

In both cases, by n we denote complete sets of quantum numbers describing in- and out- charged particles. As will be seen further in the case under consideration $n = \mathbf{p}$. The in- and out-operators obey the nonzero anticommutation relations:

$$\begin{aligned} [a_n(\text{in}), a_{n'}^{\dagger}(\text{in})]_+ &= [a_n(\text{out}), a_{n'}^{\dagger}(\text{out})]_+ = \hbar \delta_{n,n'}, \\ [b_n(\text{in}), b_{n'}^{\dagger}(\text{in})]_+ &= [b_n(\text{out}), b_{n'}^{\dagger}(\text{out})]_+ = \hbar \delta_{n,n'}. \end{aligned}$$

The in-operators are associated with a complete orthonormal set of solutions $\{ _{\zeta}\psi_n(t, \mathbf{r}) \}$ ($\zeta = +$ for electrons and $\zeta = -$ for holes) of the Dirac equation with an external electric field. Their asymptotics as $t < t_1$ can be classified as free particles and antiparticles. The out -operators are associated with a complete orthonormal out-set of solutions $\{ ^{\zeta}\psi_n(t, \mathbf{r}) \}$ of the Dirac equation with an external electric field. Their asymptotics as $t > t_2$ can be classified as free particles and antiparticles. The conserved inner product reads

$$\left(\psi, \psi' \right) = \int \psi^{\dagger}(t, \mathbf{r}) \psi'(t, \mathbf{r}) d\mathbf{r},$$

where the integration is over the finite area S of the standard box regularization. The orthonormality conditions are:

$$\left(_{\zeta}\psi_n, \ _{\zeta'}\psi_{n'} \right) = \delta_{\zeta,\zeta'}\delta_{n,n'}, \quad \left(^{\zeta}\psi_n, \ ^{\zeta'}\psi_{n'} \right) = \delta_{\zeta,\zeta'}\delta_{n,n'}. \qquad (8)$$

The in- and out-operators are defined by the two representations of the quantum Dirac field $\hat{\Psi}(t, \mathbf{r})$ in the Heisenberg representation (it means here: in the zero-order approximations with respect of interaction with photons)

$$\begin{aligned} \hat{\Psi}(t, \mathbf{r}) &= \sum_n \left[a_n(\text{in}) \, _+\psi_n(t, \mathbf{r}) + b_n^{\dagger}(\text{in}) \, _-\psi_n(t, \mathbf{r}) \right] \\ &= \sum_n \left[a_n(\text{out}) \, ^+\psi_n(t, \mathbf{r}) + b_n^{\dagger}(\text{out})) \, ^-\psi_n(t, \mathbf{r}) \right]. \end{aligned} \qquad (9)$$

The in- and out-solutions with given quantum numbers n are related by a linear transformation of the form:

$$\begin{aligned} ^{\zeta}\psi_n(t, \mathbf{r}) &= g_n\left(_+|^{\zeta} \right) \, _+\psi_n(t, \mathbf{r}) + g_n\left(_-|^{\zeta} \right) \, _-\psi_n(t, \mathbf{r}), \\ _{\zeta}\psi_n(t, \mathbf{r}) &= g_n\left(^+|_{\zeta} \right) \, ^+\psi_n(t, \mathbf{r}) + g_n\left(^-|_{\zeta} \right) \, ^-\psi_n(t, \mathbf{r}), \end{aligned} \qquad (10)$$

where the g's are some complex coefficients, $g\left(^{\zeta'}|_{\zeta} \right) = \left(_{\zeta}|^{\zeta'} \right)^{*}$. These coefficients obey the unitarity relations:

$$\begin{aligned} g_n\left(^{\zeta}|_+ \right) g_n\left(_+|^{\zeta} \right) + g_n\left(^{\zeta}|_- \right) g_n\left(_-|^{\zeta} \right) &= 1, \\ g_n\left(_{\zeta}|^+ \right) g_n\left(^+|_{\zeta} \right) + g_n\left(_{\zeta}|^- \right) g_n\left(^-|_{\zeta} \right) &= 1, \\ g_n\left(_+|^+ \right) g_n\left(^+|_- \right) + g_n\left(_+|^- \right) g_n\left(^-|_- \right) &= 0, \end{aligned}$$

$$g_n\left(^+|_+\right)g_n\left(_+|^-\right) + g_n\left(^+|_-\right)g_n\left(_-|^-\right) = 0, \tag{11}$$

which follow from the orthonormalization and completeness relations for the corresponding solutions. It is known that all the coefficients can be expressed in terms of two of them, e.g. of $g\left(_+|^+\right)$ and $g\left(_-|^+\right)$. However, even the latter coefficients are not completely independent,

$$\left|g_n\left(_-|^+\right)\right|^2 + \left|g_n\left(_+|^+\right)\right|^2 = 1. \tag{12}$$

Then a linear canonical transformation (Bogolubov transformation) between in- and out-operators which follows from Eq. (9) is defined by these coefficients

$$a_n(\text{out}) = g_n\left(^+|_+\right)a_n(\text{in}) + g_n\left(^+|_-\right)b_n^\dagger(\text{in}),$$
$$b_n^\dagger(\text{out}) = g_n\left(^-|_+\right)a_n(\text{in}) + g_n\left(^-|_-\right)b_n^\dagger(\text{in}). \tag{13}$$

Using relations (13), one finds that the differential mean numbers $N_n^{(\zeta)}$ of electrons (holes) created from the vacuum in the zero-order approximations with respect of the electron--photon interaction are:

$$N_n^{(+)} = {}_e\langle 0, \text{in}|a_n^\dagger(\text{out})a_n(\text{out})|0, \text{in}\rangle_e = \left|g_n\left(_-|^+\right)\right|^2,$$
$$N_n^{(-)} = {}_e\langle 0, \text{in}|b_n^\dagger(\text{out})b_n(\text{out})|0, \text{in}\rangle_e = \left|g_n\left(_+|^-\right)\right|^2. \tag{14}$$

We see that that the mean numbers of electrons (holes) created are equal and are also equal to the mean number of the pairs created, $N_n^{(+)} = N_n^{(-)} = N_n^{\text{cr}}$. All the information about electrons and holes creation, annihilation, and scattering in an electric field in the zero-order approximations with respect of the electron-photon interaction can be extracted from the coefficients can be extracted from the coefficients $g\left(_\zeta|^{\zeta'}\right)$ (see Ref. [5–8] for details).

2.3 In- and out-states with definite numbers of charged particles and photons

We note that the Fock space of the complete system under consideration is a tensor product of the Fock space of the electron subsystem and the Fock space of the free photon subsystem. As was pointed out above due to the vacuum instability, the in- and out-states of the electron subsystem are different in the general case. At the same time, the photon vacuum $|0\rangle$ remains unchanged. Denoting by $|0, \text{in}\rangle$ and $|0, \text{out}\rangle$ the initial and final vacuum states of the complete system, we can write:

$$|0, \text{in}\rangle = |0, \text{in}\rangle_e \otimes |0\rangle, \ |0, \text{out}\rangle = |0, \text{out}\rangle_e \otimes |0\rangle.$$

The initial and final states of the complete system with definite numbers of charged particles of all the kinds and photons have the form:

$$|\text{in}\rangle = C^\dagger \ldots b^\dagger(\text{in}) \ldots a^\dagger(\text{in}) \ldots |0, \text{in}\rangle,$$
$$|\text{out}\rangle = C^\dagger \ldots b^\dagger(\text{out}) \ldots a^\dagger(\text{out}) \ldots |0, \text{out}\rangle. \tag{15}$$

Probability amplitude of a transition from an initial to a final state (15) has the following form:

$$W = \langle\text{out}|\mathcal{S}|\text{in}\rangle, \tag{16}$$

where $\mathcal{S}$ is the scattering matrix in the external field,

$$\mathcal{S} = \mathcal{T} \exp\left\{-\frac{i}{\hbar}\int_{t_{in}}^{t_{out}} \mathcal{H}_{\text{int}}dt\right\}, \ \mathcal{H}_{\text{int}} \approx -\int \mathbf{j}(t, \mathbf{r})\mathbf{A}(t, \mathbf{r})d\mathbf{r},$$
$$\mathbf{j}(t, \mathbf{r}) = -\frac{ev_F}{2c}\left[\Psi^\dagger(t, \mathbf{r}), \gamma^0\Psi(t, \mathbf{r})\right]_-, \tag{17}$$

where $\Psi(t, \mathbf{r})$, $\Psi^\dagger(t, \mathbf{r})$, and $\mathbf{A}(t, \mathbf{r})$ are quantum field operators in the interaction representation, the symbol $\mathcal{T}$- is the chronological ordering operator and $t_{out} - t_{in} \to \infty$ is macroscopic time of the radiative interaction.

Below we are going to consider the emission of a photon from the vacuum and from a single-electron state. These processes will be studied in the first-order approximation for amplitudes which corresponds to the second-order approximation for the probabilities. In this case:

$$\mathcal{S} \approx 1 + i\Upsilon^{(1)}, \ \Upsilon^{(1)} = \frac{1}{\hbar}\int \mathbf{j}(t, \mathbf{r})\mathbf{A}(t, \mathbf{r})d\mathbf{r}dt$$
$$\implies \langle\text{out}|\mathcal{S}|\text{in}\rangle \approx i\langle\text{out}|\Upsilon^{(1)}|\text{in}\rangle \tag{18}$$

It is known that the QED$_{3,2}$ model is renormalizable. In the first-order approximation, we only have to believe that fields, the electric charge, and electron mass (if $m \neq 0$) are already given in the renormalized form, namely the charge e represents its physical value and the fine-structure constant is $\alpha = e^2/\hbar c \simeq 1/137$.

The probabilities of the one-photon emissions with quantum numbers $\mathbf{K}$, ϑ and at the same time with the production of $M \geq 1$ pairs of charged particles of one kind from the vacuum read:

$$\mathcal{P}_M(\mathbf{K},\vartheta) = \sum_{\{m\}\{n\}} (M!)^{-2} \big| \langle 0, \text{out}|b_{n_M}(\text{out}) \dots b_{n_1}(\text{out})$$
$$\times \, a_{m_M}(\text{out}) \dots a_{m_1}(\text{out}) C_{\mathbf{K}\vartheta} i \Upsilon^{(1)} |0, \text{in}\rangle \big|^2. \tag{19}$$

Summing these probabilities over M, we obtain the probability of the one-photon emission with full allowance for the possible instability of the vacuum with respect to the production of one kind of charged particles,

$$\mathcal{P}(\mathbf{K},\vartheta) = \sum_{M=1}^{\infty} \mathcal{P}_M(\mathbf{K},\vartheta). \tag{20}$$

The probabilities of the one-photon emission with quantum numbers $\mathbf{K}$, ϑ and at the same time with the production of $M \geq 0$ pairs of charged particles of one kind from a single-electron state which is characterized by a quantum numbers l reads:

$$\mathcal{P}_M\left(\mathbf{K}, \vartheta \,|\, \overset{+}{l} \right) = \sum_{\{m\}\{n\}} [M!\,(M+1)!]^{-1} \big| \langle 0, \text{out}|b_{n_M}(\text{out}) \dots b_{n_1}(\text{out})$$
$$\times \, a_{m_{M+1}}(\text{out}) \dots a_{m_1}(\text{out}) C_{\mathbf{K}\vartheta} i \Upsilon^{(1)} a_l^\dagger(\text{in})|0, \text{in}\rangle \big|^2. \tag{21}$$

The same probability from a single-hole state has the form:

$$\mathcal{P}_M\left(\mathbf{K}, \vartheta \,|\, \overset{-}{l} \right) = \sum_{\{m\}\{n\}} [M!\,(M+1)!]^{-1} \big| \langle 0, \text{out}|b_{n_M}(\text{out}) \dots b_{n_1}(\text{out})$$
$$\times \, a_{m_{M+1}}(\text{out}) \dots a_{m_1}(\text{out}) C_{\mathbf{K}\vartheta} i \Upsilon^{(1)} b_l^\dagger(\text{in})|0, \text{in}\rangle \big|^2. \tag{22}$$

Summing these probabilities over M, we obtain the probability of the one-photon emission from a single-electron (hole) state with full allowance for the possible instability of the vacuum with respect to the production of charged particles of one kind,

$$\mathcal{P}\left(\mathbf{K}, \vartheta \,|\, \overset{\pm}{l} \right) = \sum_{M=0}^{\infty} \mathcal{P}_M\left(\mathbf{K}, \vartheta \,|\, \overset{\pm}{l} \right). \tag{23}$$

If the probability of the creation of pairs from the vacuum is small, then main contributions to probabilities (23) and (20) are due minimal possible numbers of created pairs,

$$\mathcal{P}(\mathbf{K},\vartheta) \approx \mathcal{P}_1(\mathbf{K},\vartheta), \quad \mathcal{P}\left(\mathbf{K}, \vartheta \,|\, \overset{\pm}{l} \right) \approx \mathcal{P}_0\left(\mathbf{K}, \vartheta \,|\, \overset{\pm}{l} \right). \tag{24}$$

To construct an perturbation theory for the probability amplitudes, one needs to reduce the S-matrix to a generalized normal form with respect to the vacua $\langle 0, \text{out}|$ and $|0, \text{in}\rangle$ (see Ref. [5–8]).

To this end, one has explicitly divide the Dirac field operators into parts, creative with respect to the vacuum $\langle 0, \text{out}|$ and annihilative with respect to the vacuum $|0, \text{in}\rangle$. In the first-order approximation, it is sufficient to reduce the operator $\mathbf{j}(t, \mathbf{r})$ to the generalized normal form,

$$\mathcal{N}_{out-in}\{\mathbf{j}(t, \mathbf{r})\} = \mathbf{j}(t, \mathbf{r}) - \langle \mathbf{j}(t, \mathbf{r})\rangle^c,$$
$$\langle \mathbf{j}(t, \mathbf{r})\rangle^c = \langle 0, \text{out}|\mathbf{j}(t, \mathbf{r})|0, \text{in}\rangle c_v^{-1}, \tag{25}$$

where $c_v = \langle 0, \text{out}|0, \text{in}\rangle$ is the vacuum to vacuum transition amplitude. Note that, in the general case, the vacuum polarization current $\langle \mathbf{j}(t, \mathbf{r})\rangle^c$ is not zero. It may contribute to a tadpolediagram. However, in the case under consideration (uniform and slowly varying external electric field), such a diagram has a notable value only in a very infrared range, which is not considered here. In particular, in the limiting case of a constant uniform field, this contribution can be safely neglected.[1]

Thus, the quantities under consideration can be represented as:

$$\mathcal{P}_0\left(\mathbf{K}, \vartheta \,|\, \overset{\pm}{l} \right) = P_v^{(0)} \sum_n \left| w^{(1)}\left(\overset{\pm}{n}; \mathbf{K},\vartheta \,|\, \overset{\pm}{l} \right) \right|^2, \quad P_v^{(0)} = |c_v|^2,$$
$$w^{(1)}\left(\overset{\pm}{n}; \mathbf{K},\vartheta \,|\, \overset{\pm}{l} \right) = \frac{ic}{\hbar}\sqrt{\frac{2\pi\hbar}{\varepsilon V \omega}} \int \boldsymbol{\epsilon}_{\mathbf{K}\vartheta} \mathbf{j}\left(\overset{\pm}{n} \,\Big|\, \overset{\pm}{l} \right) e^{i(\omega t - \mathbf{kr})} dt\,d\mathbf{r},$$

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[1] We note that this diagram can cause non-vanishing contributions when appearing as a part of a higer order diagram; see Ref. [32].

$$\mathbf{j}\left(\overset{+}{n}\Big|\overset{+}{l}\right) = c_v^{-1}\langle 0, \text{out}|a_n(\text{out})\mathcal{N}_{out-in}\{\mathbf{j}(t,\mathbf{r})\}a_l^\dagger(\text{in})|0, \text{in}\rangle,$$

$$\mathbf{j}\left(\overset{-}{n}\Big|\overset{-}{l}\right) = c_v^{-1}\langle 0, \text{out}|b_n(\text{out})\mathcal{N}_{out-in}\{\mathbf{j}(t,\mathbf{r})\}b_l^\dagger(\text{in})|0, \text{in}\rangle \ , \tag{26}$$

and

$$\mathcal{P}_1(\mathbf{K},\vartheta) = P_v^{(0)}\sum_{n,l}\left|w^{(1)}\left(\overset{-\,+}{n\,l}\ ; \mathbf{K},\vartheta\,|0\right)\right|^2,$$

$$w^{(1)}\left(\overset{-\,+}{n,l}; \mathbf{K},\vartheta\,|0\right) = \frac{ic}{\hbar}\sqrt{\frac{2\pi\hbar}{\varepsilon V\omega}}\int \boldsymbol{\epsilon}_{\mathbf{K}\vartheta}\mathbf{j}\left(\overset{-\,+}{n,l}\Big|0\right)e^{i(\omega t - \mathbf{kr})}dt d\mathbf{r} \ ,$$

$$\mathbf{j}\left(\overset{-\,+}{n,l}\Big|0\right) = c_v^{-1}\langle 0, \text{out}|b_n(\text{out})a_l(\text{out})\mathcal{N}_{out-in}\{\mathbf{j}(t,\mathbf{r})\}|0, \text{in}\rangle \ . \tag{27}$$

One can express matrix elements in Eqs. (27) and (26) via the solutions $_\zeta\psi_n(t,\mathbf{r})$ and $^\zeta\psi_n(t,\mathbf{r})$, and coefficients g as follows:

$$\mathbf{j}\left(\overset{+\,-}{n,l}\Big|0\right) = -\frac{ev_F}{c}g_l\left(_+|^+\right)^{-1}{}_+\bar\psi_l(t,\mathbf{r})\ _-\psi_n(t,\mathbf{r})g_n\left(^-|_-\right)^{-1},$$

$$\mathbf{j}\left(\overset{+}{n}\Big|l^+\right) = -\frac{ev_F}{c}g_n\left(_+|^+\right)^{-1}{}_+\bar\psi_n(t,\mathbf{r})\ ^+\psi_l(t,\mathbf{r})g_l\left(_+|^+\right)^{-1},$$

$$\mathbf{j}\left(\overset{-}{n}\Big|l^-\right) = \frac{ev_F}{c}g_l\left(^-|_-\right)^{-1}{}^-\bar\psi_l(t,\mathbf{r})\ _-\psi_n(t,\mathbf{r})g_n\left(^-|_-\right)^{-1} \ , \tag{28}$$

where $\bar\psi_n = \psi_n^\dagger\gamma^0$.

It seems that the matrix elements in Eqs. (19)-(23) can be written in a similar manner. However, this is only useful in the case of a not very strong electric field, when the approximation (24) is applicable. In the case of an intense external field, there exist many transition channels corresponding to the violation of the vacuum stability. Considering the photon emission by massless charged particles in the graphene any quasiconstant electric field has to be treated as a strong one. By this reason, it is effective to calculate mean characteristics of the emission using the unitarity condition for the S matrix as in the way described below.

Probabilities (20) and (23) can be represented as a trace of the operators $C_{\mathbf{K}\vartheta}\mathcal{S}|\text{in}\rangle\langle\text{in}|\mathcal{S}^{-1}C_{\mathbf{K}\vartheta}^\dagger$ with respect to the final basis,

$$\mathcal{P}(\mathbf{K},\vartheta\,|\text{in}) = \text{tr}\left[C_{\mathbf{K}\vartheta}\mathcal{S}|\text{in}\rangle\langle\text{in}|\mathcal{S}^{-1}C_{\mathbf{K}\vartheta}^\dagger\right], \tag{29}$$

where $|\text{in}\rangle$ is one of the following states: $|0, \text{in}\rangle$, $a_l^\dagger(\text{in})|0, \text{in}\rangle$, or $b_l^\dagger(\text{in})|0, \text{in}\rangle$. One can see that trace (29) can be written as a mean value of the photon number operator,

$$\mathcal{P}(\mathbf{K},\vartheta\,|\text{in}) = \langle\text{in}|\mathcal{S}^{-1}C_{\mathbf{k}\vartheta}^\dagger C_{\mathbf{k}\vartheta}\mathcal{S}|\text{in}\rangle \ . \tag{30}$$

In course of constructing a perturbation theory with respect to the radiative interaction, one needs to reorganize the S-matrix in the normal form $:\ldots:$ with respect to the in-vacuum (see Ref. [5–8]). In the first-order approximation, it is sufficient to represent only the operator $\mathbf{j}(t,\mathbf{r})$ in such a form,

$$\mathbf{j}(t,\mathbf{r}) = :\mathbf{j}(t,\mathbf{r}): + \langle\mathbf{j}(t,\mathbf{r})\rangle_{\text{in}} \ , \quad \langle\mathbf{j}(t,\mathbf{r})\rangle_{\text{in}} = \langle 0, \text{in}|\mathbf{j}(t,\mathbf{r})|0, \text{in}\rangle \ . \tag{31}$$

The vacuum mean current $\langle\mathbf{j}(t,\mathbf{r})\rangle_{\text{in}}$ is a sum of a vacuum polarization current and of a current of created charged particles. It is not zero in a slowly varying electric field and depends on the definition of the initial vacuum, $|0, \text{in}\rangle$ and on the evolution of the electric field from the initial time t_1 of switching on to the time instant t. After the time t_2 of switching the electric field off, the term $\langle\mathbf{j}(t,\mathbf{r})\rangle_{\text{in}}$ represents the current density of the created pairs of charged particles. This current is a source in the Maxwell equations for a mean electromagnetic field. Such a mean field is a slowly varying crossed field emitted perpendicular to the graphene plane (see Ref. [13] for details). One can see that in the frequency range of the photon emission $\omega \gg T^{-1}$, which is interesting to us, the contribution due to the current $\langle\mathbf{j}(t,\mathbf{r})\rangle_{\text{in}}$ can be neglected.

In particular, it follows from Eq. (30) that probabilities (20) and (23) read:

$$\mathcal{P}(\mathbf{K},\vartheta) = \sum_l\mathcal{P}(l; \mathbf{K},\vartheta\,|0), \ \ \mathcal{P}(l; \mathbf{K},\vartheta\,|0) = \sum_n\left|w_{\text{in}}^{(1)}\left(\overset{-\,+}{n\,l}; \mathbf{K},\vartheta\,|0\right)\right|^2,$$

$$w_{\text{in}}^{(1)}\left(\overset{-\,+}{n,l}; \mathbf{K},\vartheta\,|0\right) = \frac{ic}{\hbar}\sqrt{\frac{2\pi\hbar}{\varepsilon V\omega}}\int \boldsymbol{\epsilon}_{\mathbf{K}\vartheta}\mathbf{j}_{\text{in}}\left(\overset{-\,+}{n,l}\Big|0\right)e^{i(\omega t - \mathbf{kr})}dt d\mathbf{r} \ ,$$

$$\mathbf{j}_{\text{in}}\left(\overset{-\,+}{n,l}\Big|0\right) = \langle 0, \text{in}|b_n(\text{in})a_l(\text{in}):\mathbf{j}(t,\mathbf{r}):|0, \text{in}\rangle \ , \tag{32}$$

where $\mathcal{P}(l;\mathbf{K},\vartheta\,|0)$ is the probability of one photon emission with quantum numbers $\mathbf{K}$, ϑ which is accompanied by the production of pairs of one kind with a quantum number l , and

$$\mathcal{P}\left(\mathbf{K},\vartheta\,\Big|{\overset{\pm}{l}}\right) = \sum_n \left| w_{\text{in}}^{(1)}\left({\overset{\pm}{n}};\ \mathbf{K},\vartheta\,\Big|{\overset{\pm}{l}}\right)\right|^2,$$

$$w_{\text{in}}^{(1)}\left({\overset{\pm}{n}};\ \mathbf{K},\vartheta\,\Big|{\overset{\pm}{l}}\right) = \frac{ic}{\hbar}\sqrt{\frac{2\pi\hbar}{\varepsilon V \omega}}\int \boldsymbol{\epsilon}_{\mathbf{K}\vartheta}\mathbf{j}_{\text{in}}\left({\overset{\pm}{n}}\Big|{\overset{\pm}{l}}\right)e^{i(\omega t - \mathbf{k}\mathbf{r})}dt d\mathbf{r},$$

$$\mathbf{j}_{\text{in}}\left({\overset{+}{n}}\Big|{\overset{+}{l}}\right) = \langle 0,\text{in}|a_n(\text{in}) : \mathbf{j}(t,\mathbf{r}) : a_l^{\dagger}(\text{in})|0,\text{in}\rangle,$$

$$\mathbf{j}_{\text{in}}\left({\overset{-}{n}}\Big|{\overset{-}{l}}\right) = \langle 0,\text{in}|b_n(\text{in}) : \mathbf{j}(t,\mathbf{r}) : b_l^{\dagger}(\text{in})|0,\text{in}\rangle. \tag{33}$$

In order to find the probability of one photon emission with quantum numbers $\mathbf{K}$, ϑ which is accompanied by the production of pairs of all the kinds in the graphene, the probability (32) is multiplied by the number of species $N_f = 4$,

$$\mathcal{P}_{N_f}(\mathbf{K},\vartheta) = N_f \mathcal{P}(\mathbf{K},\vartheta). \tag{34}$$

One can express matrix elements in Eqs. (32) and (33) via the solutions $_\zeta\psi_n(t,\mathbf{r})$ as follows:

$$\mathbf{j}_{\text{in}}\left({\overset{-}{n},\overset{+}{l}}\,\Big|0\right) = -\frac{ev_F}{c}\,_+\bar{\psi}_l(t,\mathbf{r})\,_-\psi_n(t,\mathbf{r}),$$

$$\mathbf{j}_{\text{in}}\left({\overset{\pm}{n}}\Big|{\overset{\pm}{l}}\right) = \mp\frac{ev_F}{c}\,_\pm\bar{\psi}_n(t,\mathbf{r})\,_\pm\psi_l(t,\mathbf{r}). \tag{35}$$

Note that, in the general case, the matrix elements $w_{\text{in}}^{(1)}\left({\overset{-}{n},\overset{+}{l}};\mathbf{K},\vartheta\,|0\right)$ and $w_{\text{in}}^{(1)}\left({\overset{\pm}{n}};\mathbf{K},\vartheta\,\Big|{\overset{\pm}{l}}\right)$ considered separately are quite different from the amplitudes of the relative probabilities $w^{(1)}\left({\overset{-}{n},\overset{+}{l}};\mathbf{K},\vartheta\,|0\right)$ and $w^{(1)}\left({\overset{\pm}{n}};\mathbf{K},\vartheta\,\Big|{\overset{\pm}{l}}\right)$ given by Eqs. (27) and (26), respectively. Only if the mean number of the pairs created are sufficiently small, $N_n, N_l \ll 1$, the difference between the solutions $_\zeta\psi_n(t,\mathbf{r})$ and $^\zeta\psi_n(t,\mathbf{r})$ at a given ζ can be neglected.

3 Photon emission in a constant electric field

3.1 Exact solutions

Next, we proceed to direct calculations of the photon emission in graphene induced by applied external electric field. The electric field acts in the graphene plane during the time interval $t_2 - t_1 = T$ as a constant field E and vanishes out the interval. Such a field is often called T-constant electric field. In the $QED_{3,2}$ model which describes massless charged particles, even a seemingly weak electric field E, if it remains unchanged for a sufficiently long time, creates electron--hole pairs from the vacuum. The vacuum instability problem in the graphene in T-constant electric field was studied in detail in Ref. [13]. Switching on and off effects of in the latter field can be neglected if we suppose that the time interval T is sufficiently large, namely

$$T/\Delta t_{st} \gg \max\left\{1, \frac{m^2 v_F^3}{|eE|\hbar}\right\}, \tag{36}$$

where Δt_{st} is a big characteristic time scale Δt_{st} in the graphene physics,

$$\Delta t_{st} = (|eE|v_F/\hbar)^{-1/2} \gg t_\gamma . \tag{37}$$

and $t_\gamma = \hbar/\gamma \simeq 0.24\text{fs}$ is the microscopic time scale with $\gamma = 2.7\text{eV}$ being the hopping energy. Then the perturbation theory with respect to electric field breaks down and the dc response changes from the linear in E time-independent regime to a nonlinear in E and time-dependent regime (see Ref. [33]). This regime was recently observed in measurements of $I - V$ curves of graphene devices near the Dirac point (see Ref. [36]).

We recall that the T-constant electric field can be described by the vector potential with only one nonzero component $A_x^{\text{ext}}(t)$,

$$A_x^{\text{ext}}(t) = -cE\begin{cases} t_1 & t \in \text{I} = (-\infty, t_{in}),\ t_1 = -T/2 \\ t, & t \in \text{Int} = [t_1, t_2] \\ t_2, & t \in \text{II} = (t_2, \infty),\ t_2 = T/2 \end{cases},$$

such that the corresponding electric field has also only one nonzero component, $E_x(t)$,

$$E_x(t) = \begin{cases} E > 0, \ t \in \text{Int} \\ 0, \ t \in \text{I} \cup \text{II} \end{cases}.$$

The time scale (37) is specific to the graphene physics. It plays the role of a stabilization time after which differential mean numbers of created pairs take the form:

$$N_{\mathbf{p}}^{\text{cr}} \simeq e^{-\pi\lambda}, \quad \lambda = \frac{v_F p_y^2 + m^2 v_F^3}{eE\hbar}. \tag{38}$$

which is the same for the case of the constant electric field in the finite momentum range

$$D: \sqrt{\frac{v_F}{eE\hbar}}|p_y| < (T/\Delta t_{st} - \tau)^{1/2}, \ \sqrt{\frac{v_F}{eE\hbar}}|p_x| < \frac{1}{2}T/\Delta t_{st} - \tau, \tag{39}$$

where τ is an arbitrary number satisfying the condition

$$T/\Delta t_{st} \gg \tau \gg \max\left\{1, \frac{m^2 v_F^3}{eE\hbar}\right\}; \tag{40}$$

(see Ref. [13] for details). The total number density of electron–hole pairs created by the electric field (multiplied by a degeneracy factor $N_f = 4$) is:

$$n_g^{cr} = r_g^{cr} T, \quad r_g^{cr} = N_f (2\pi)^{-2} \left(v_F \hbar^3\right)^{-1/2} |eE|^{3/2}. \tag{41}$$

The QED with an external constant electric field is a consistent model as long as the low-frequency radiation field produced by an induced current is negligible compared to the external field, which is supported by external sources to remain fixed. In the graphene with zero mass carries, this imposes the consistency restriction [13]:

$$T/\Delta t_{st} \ll \pi/4\alpha, \tag{42}$$

where α is the fine-structure constant.

In this case under consideration, the external field can be considered as a good approximation of the effective mean field. In the presence of the mass gap $\Delta\varepsilon = m v_F^2$, restriction (42) is attenuated by a factor $\exp[\pi(\Delta\varepsilon)^2/e|E|v_F\hbar]$; see Ref. [37]. We call the typical time scale related to Eq. (42), $\Delta t_{br} = \Delta t_{st}\pi/4\alpha$, the time of backreaction. On the other hand, the dimensionless parameter in the lhs of Eq. (42) satisfies the condition given by Eq. (36). Thus, there is a window in the parameter range where the model is consistent, $t_\gamma \ll \Delta t_{st} \ll \Delta t_{br}$. Moreover, this restriction corresponds to a specific regime, which might be relevant to some known experiments in the graphene physics; see [13, 17] for details. The time T could be treated as a typical time-scale, which we call the effective time duration T_{eff}, $T = T_{eff}$ in what follows. Some kind of dissipation process may truncate the motion of a particle at T_{dis}, in which case $T_{eff} = T_{dis}$. In the absence of the dissipation, the transport is ballistic; then, considering a strip with a lateral infinite width and a finite length L_x, we assume the ballistic flight time $T_{bal} = L_x/v_F$ to be the effective time duration, $T_{eff} = T_{bal}$. In typical experiments, $L_x \sim 1^-$m, which gives $T_{bal} \sim 10^{-12}$s. We note that the experimentally terahertz pulses [17] are characterized by a similar to T_{bal} period. Taking $T = T_{bal}$ in Eqs. (40) and (42), we obtain the following restrictions on the constant electric field under consideration:

$$E = aE_0, \quad E_0 = 1 \times 10^6 \text{V/m}, \quad 7 \times 10^{-4} \ll a \ll 8. \tag{43}$$

Since the voltage is $V = EL_x$, one finds the inequalities

$$7 \times 10^{-4} \, \text{V} \ll V \ll 8 \, \text{V}. \tag{44}$$

These voltages are in the range used in experiments in graphene physics.

Solutions of the Dirac Eq. (3) with the T-constant field were studied in detail in Ref. [11]. It was demonstrated that the corresponding initial set $\{_\zeta \psi_n(t, \mathbf{r})\}$ and final set $\{^\zeta \psi_n(t, \mathbf{r})\}$ can be chosen in the form:

$$_{\pm}\psi_{\mathbf{p}}(t, \mathbf{r}) = (i\hbar\,\partial_t + H^{\text{ext}})\,_{\pm}\phi_{\mathbf{p},\pm 1}(t, \mathbf{r}), \quad _{\pm}\phi_{\mathbf{p},\pm 1}(t, \mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\,_{\pm}\varphi_{\mathbf{p},\pm 1}(t)U_{\pm 1},$$
$$^{\pm}\psi_{\mathbf{p}}(t, \mathbf{r}) = (i\hbar\,\partial_t + H^{\text{ext}})\,^{\pm}\phi_{\mathbf{p},\mp 1}(t, \mathbf{r}), \quad ^{\pm}\phi_{\mathbf{p},\mp}(t, \mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\,^{\pm}\varphi_{\mathbf{p},\mp 1}(t)U_{\mp 1}, \tag{45}$$

where U_s are constant orthonormalized spinors

$$U_{+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ U_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

At early ($t < t_1$ -region I) and late ($t > t_2$ -region II) times, Eq. (3) has plane wave solutions $_{\pm}\varphi_{\mathbf{p},s}(t)$ and $^{\pm}\varphi_{\mathbf{p},s}(t)$, respectively, which satisfy simple dispersion relations:

$$\text{I}: \ _\zeta\varphi_{\mathbf{p},s}(t) \sim e^{-i\zeta\varepsilon_{in}t/\hbar}, \quad \text{II}: \ ^\zeta\varphi_{\mathbf{p},s}(t) \sim e^{-i\zeta\varepsilon_{out}t/\hbar},$$

$$\varepsilon_{in/out} = v_F \sqrt{(p_x - eEt_{1/2})^2 + p_y^2 + m^2 v_F^2} \,. \tag{46}$$

For $t \in \text{Int}$, if the electric field satisfies condition (36), it is enough to use solutions (45) with momenta in range (39). In this range of momenta, the function $_\pm\varphi_{\mathbf{p},s}(t)$ and $^\pm\varphi_{\mathbf{p},s}(t)$ have the form of the Weber parabolic cylinder functions (WPCF's):

$$_+^-\varphi_{\mathbf{p},s}(t) = C D_{\nu - \frac{1+s}{2}}[\pm(1-i)\xi], \,_-^+\varphi_{\mathbf{p},s}(t) = C D_{-\nu - \frac{1-s}{2}}[\pm(1+i)\xi]\,,$$

$$\xi = \sqrt{\frac{v_F}{eE\hbar}}(eEt - p_x), \ \nu = \frac{i\lambda}{2}, \ C = (2eE\hbar v_F S)^{-1/2} \exp\left(-\frac{\pi\lambda}{8}\right), \tag{47}$$

where S is the graphene area. An initial state $_\pm\psi_{\mathbf{p}}(t, \mathbf{r})$ describes a particle/hole with a well-defined energy ε_{in} at the distant past. Similarly, a final state $^\pm\psi_{\mathbf{p}}(t, \mathbf{r})$ describes a particle/hole with a well-defined energy ε_{out} at the distant future. Then, the probability of the emission of a photon in the T-constant electric field during the time interval T is indistinguishable from the one in the constant field ($T \to \infty$). Therefore, we assume in what follows that $T \to \infty$.

We note that calculating amplitudes (35), it is convenient to represent solutions $_\pm\psi_{\mathbf{p}}(t, \mathbf{r})$ in a different form. By inserting Eq. (47) in Eq. (45), and taking explicitly derivatives, we find this form:

$$_\pm\psi_{\mathbf{p}}(t, \mathbf{r}) = e^{i\mathbf{p}\mathbf{r}/\hbar} \,_\pm\psi_{\mathbf{p}}(t),$$

$$_\pm\psi_{\mathbf{p}}(t) = v_F\left[(mv_F \mp i\zeta p_y) \,_\pm\varphi_{\mathbf{p},\pm1}(t)U_{\mp1} + (\pm1+i)\sqrt{\frac{eE\hbar}{v_F}} \,_\pm\varphi_{\mathbf{p},\mp1}(t)U_{\pm1}\right]. \tag{48}$$

3.2 Probabilities

The differential probability of one photon emission with a given polarization ϑ and the wave vector situated in the range from $\mathbf{K}$ to $\mathbf{K}+d\mathbf{K}$, which is accompanied by the pair production of one kind from the vacuum, reads:

$$d\mathcal{P}(\mathbf{K},\vartheta) = \mathcal{P}_{N_f}(\mathbf{K},\vartheta)\frac{Vd\mathbf{K}}{(2\pi)^3}, \tag{49a}$$

where the quantity $\mathcal{P}_{N_f}(\mathbf{K},\vartheta)$ is given by Eqs. (32) and (34).

The differential probability of one photon emission with a given polarization ϑ and a wave vector situated in the range from $\mathbf{K}$ to $\mathbf{K}+d\mathbf{K}$ from a single-electron (hole) state is

$$d\mathcal{P}\left(\mathbf{K},\vartheta\,|_{\mathbf{p}}^{\pm}\right) = \mathcal{P}\left(\mathbf{K},\vartheta\,|_{\mathbf{p}}^{\pm}\right)\frac{Vd\mathbf{K}}{(2\pi)^3}, \tag{50}$$

where the probability $\mathcal{P}\left(\mathbf{K},\vartheta\,|_{\mathbf{p}}^{\pm}\right)$ is given by Eq. (33).

Using the parametrization by frequency ω and solid angle $d\Omega$, $d\mathbf{K} = c^{-3}\omega^2 d\omega d\Omega$, one can write the probabilities per unit frequency and solid angle as

$$\frac{d\mathcal{P}(\mathbf{K}, \vartheta)}{d\omega d\Omega} = N_f \sum_{\mathbf{p}} \frac{d\mathcal{P}(\mathbf{p}; \mathbf{K}, \vartheta|0)}{d\omega d\Omega},$$

$$\frac{d\mathcal{P}(\mathbf{p}; \mathbf{K}, \vartheta|0)}{d\omega d\Omega} = \frac{V\omega^2}{(2\pi c)^3} \sum_{\mathbf{p}'} \left|w_{in}^{(1)}\left(_{\mathbf{p}}^-{}_{\mathbf{p}'}^+; \mathbf{K}, \vartheta|0\right)\right|^2; \tag{51}$$

$$\frac{d\mathcal{P}\left(\mathbf{K},\vartheta\,|_{\mathbf{p}}^{\pm}\right)}{d\omega d\Omega} = \frac{V\omega^2}{(2\pi c)^3} \sum_{\mathbf{p}'} \left|w_{in}^{(1)}\left(_{\mathbf{p}'}^{\pm}; \mathbf{K},\vartheta\,|_{\mathbf{p}}^{\pm}\right)\right|^2, \tag{52}$$

where the amplitudes are given by Eqs. (32) and (33), respectively. Integrating over the area S, we obtain that

$$w_{in}^{(1)}\left(_{\mathbf{p}}^-{}_{\mathbf{p}'}^+; \mathbf{K},\vartheta|0\right) = iev_F\Delta t_{st}\sqrt{\frac{2\pi}{\hbar\varepsilon V\omega}}\delta_{\mathbf{p}', \mathbf{p}-\hbar\mathbf{k}}M_{\mathbf{p}'\mathbf{p}}^0\,,$$

$$M_{\mathbf{p}'\mathbf{p}}^0 = -\frac{S}{\Delta t_{st}}\int_{t_1}^{t_2} \,_+\bar{\psi}_{\mathbf{p}'}(t)\boldsymbol{\epsilon}_{\mathbf{K}\vartheta} \,_-\psi_{\mathbf{p}}(t)e^{i\omega t}dt\,;$$

$$w_{in}^{(1)}\left(_{\mathbf{p}'}^{\pm}; \mathbf{K},\vartheta\,|_{\mathbf{p}}^{\pm}\right) = iev_F\Delta t_{st}\sqrt{\frac{2\pi}{\hbar\varepsilon V\omega}}\delta_{\mathbf{p}', \mathbf{p}-\hbar\mathbf{k}}M_{\mathbf{p}'\mathbf{p}}^{\pm}\,,$$

$$M_{\mathbf{p}'\mathbf{p}}^{\pm} = \mp\frac{S}{\Delta t_{st}}\int_{t_1}^{t_2} \,_{\pm}\bar{\psi}_{\mathbf{p}'}(t)\boldsymbol{\epsilon}_{\mathbf{K}\vartheta} \,_{\pm}\psi_{\mathbf{p}}(t)e^{i\omega t}dt\,, \tag{53}$$

where $\delta_{\mathbf{p}',\pm\mathbf{p}-\hbar\mathbf{k}}$ is the Kronecker symbol, the spinor $_{\pm}\psi_{\mathbf{p}}(t)$ is given by Eq. (48), and it is taken into account that the contribution to the integral over times $t \in \mathrm{I} \cup \mathrm{II}$ is zero due to the absence of the electric field. Squaring the amplitudes (53) and summing over the momenta $\mathbf{p}'$, we represent the probability densities (51) and (52) as

$$\frac{d\mathcal{P}(\mathbf{p}; \mathbf{K},\vartheta|0)}{d\omega d\Omega} = \frac{\alpha}{\varepsilon}\left(\frac{v_F}{c}\right)^2 \frac{\omega \Delta t_{st}^2}{(2\pi)^2} \left|M_{\mathbf{p}'\mathbf{p}}^0\right|^2\Bigg|_{\mathbf{p}'=\mathbf{p}-\hbar\mathbf{k}}, \tag{54}$$

$$\frac{d\mathcal{P}\left(\mathbf{K}, \vartheta|\overset{\pm}{\mathbf{p}}\right)}{d\omega d\Omega} = \frac{\alpha}{\varepsilon}\left(\frac{v_F}{c}\right)^2 \frac{\omega \Delta t_{st}^2}{(2\pi)^2} \left|M_{\mathbf{p}'\mathbf{p}}^\pm\right|^2\Bigg|_{\mathbf{p}'=\mathbf{p}-\hbar\mathbf{k}}. \tag{55}$$

Using the explicit representations (1) and (48) as well as the substitutions

$$u = \sqrt{\frac{v_F}{eE\hbar}}\left[eEt - \frac{1}{2}(p_x + p_x')\right], \quad u(t_{1,2}) = u|_{t=t_{1,2}},$$

$$u_x = \sqrt{\frac{v_F}{eE\hbar}}(p_x' - p_x), \quad u_\pm = u \pm u_x/2, \ u_0 = \Delta t_{st}\omega, \tag{56}$$

we obtain

$$M_{\mathbf{p}'\mathbf{p}}^0 = -\exp\left(i\omega\frac{p_x + p_x'}{2eE}\right)\exp\left[-\frac{\pi(\lambda + \lambda')}{8}\right]$$

$$\times \left\{i\chi_\vartheta^{0,1}\tilde{Y}_{00} + (2eE\hbar)^{-1}v_F(mv_F + i\zeta p_y')(mv_F + i\zeta p_y)\chi_\vartheta^{1,0}\tilde{Y}_{11}\right.$$

$$+ e^{-i\pi/4}\sqrt{\frac{v_F}{2eE\hbar}}\left[-(mv_F + i\zeta p_y')\chi_\vartheta^{1,1}\tilde{Y}_{10} + (mv_F + i\zeta p_y)\chi_\vartheta^{0,0}\tilde{Y}_{01}\right]\right\},$$

$$M_{\mathbf{p}'\mathbf{p}}^+ = -\exp\left(i\omega\frac{p_x + p_x'}{2eE}\right)\exp\left[-\frac{\pi(\lambda + \lambda')}{8}\right]$$

$$\times \left\{\chi_\vartheta^{0,0}Y_{00} + (2eE\hbar)^{-1}v_F(mv_F + i\zeta p_y')(mv_F - i\zeta p_y)\chi_\vartheta^{1,1}Y_{11}\right.$$

$$+ \sqrt{\frac{v_F}{2eE\hbar}}e^{i\pi/4}\left[(mv_F + i\zeta p_y')\chi_\vartheta^{1,0}Y_{10} - i(mv_F - i\zeta p_y)\chi_\vartheta^{0,1}Y_{01}\right]\right\}, \tag{57}$$

where

$$Y_{j'j}(t_2, t_1) = \int_{u(t_1)}^{u(t_2)} D_{-\nu'-j'}[-(1+i)u_-]D_{\nu-j}[-(1-i)u_+]e^{iu_0u}du, \tag{58}$$

$$\tilde{Y}_{j'j}(t_2, t_1) = \int_{u(t_1)}^{u(t_2)} D_{-\nu'-j'}[-(1+i)u_-]D_{-\nu-j}[-(1+i)u_+]e^{iu_0u}du, \tag{59}$$

and

$$\chi_\vartheta^{(1-s')/2,(1-s)/2} = U_{s'}^\dagger\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_{\mathbf{K}\vartheta}U_s, \ \nu' = \frac{i\lambda'}{2}, \quad \lambda' = \lambda|_{p_y \to p_y'}.$$

One can check that the probability density for an electron and hole in Eq. (55) are easily related by the replacement of $\mathbf{p}' \rightleftarrows \mathbf{p}$,

$$M_{\mathbf{p}'\mathbf{p}}^- = M_{\mathbf{p}\mathbf{p}'}^+. \tag{60}$$

To evaluate the angular matrix element $\chi_\vartheta^{j',j}$, we adopt the convention used in Ref. [14] and define an orthonormal triple

$$\mathbf{K}/K = (\sin\theta\cos\phi, \ \sin\theta\sin\phi, \ \cos\theta),$$

$$\boldsymbol{\epsilon}_{\mathbf{K}1} = \mathbf{e}_z \times \mathbf{K}/|\mathbf{e}_z \times \mathbf{K}|, \quad \boldsymbol{\epsilon}_{\mathbf{K}2} = \mathbf{K}\times\boldsymbol{\epsilon}_{\mathbf{K}1}/|\mathbf{K}\times\boldsymbol{\epsilon}_{\mathbf{K}1}|. \tag{61}$$

Then

$$\boldsymbol{\epsilon}_{\mathbf{K}1} = (-\sin\phi, \ \cos\phi, \ 0),$$

$$\boldsymbol{\epsilon}_{\mathbf{K}2} = (-\cos\theta\cos\phi, \ -\cos\theta\sin\phi, \ \sin\theta) \tag{62}$$

for $\mathbf{K}$ in the upper spatial region, $k_z \geq 0$. Thereby, we obtain:

$$\chi_1^{1,1} = -\chi_1^{0,0} = \sin\phi, \ \chi_1^{0,1} = -\chi_1^{1,0} = i\zeta\cos\phi;$$

$$\chi_2^{1,1} = -\chi_2^{0,0} = \cos\theta\cos\phi, \ \chi_2^{0,1} = -\chi_2^{1,0} = -i\zeta\cos\theta\sin\phi. \tag{63}$$

For the momenta p_x and p'_x satisfying condition (39) and for finite $u_0 \ll \min(u(t_1), |u(t_2)|)$, it is possible to consider limits $T \to \infty$ in integrals (58) and (59). We denote the corresponding limits as:

$$Y_{j'j} = Y_{j'j}(t_2, t_1)\big|_{T \to \infty}, \quad \tilde{Y}_{j'j} = \tilde{Y}_{j'j}(t_2, t_1)\big|_{T \to \infty}. \tag{64}$$

These limits can be simplified using the hyperbolic coordinates ρ and φ,

$$\rho = \sqrt{u_0^2 - u_x^2}, \ \tanh \varphi = \frac{u_x}{u_0} \text{ if } u_0^2 - u_x^2 > 0. \tag{65}$$

We note that in both cases of the emission, we have $p'_x = p_x - \hbar k_x$. Therefore, in any frequency range the ratio $|u_x|/u_0$ is very small,

$$\frac{|u_x|}{u_0} = \frac{|k_x|}{K} \frac{v_F}{c} \le \frac{v_F}{c}, \tag{66}$$

and the condition $u_0^2 - u_x^2 > 0$ is fulfilled. This feature of photon emission is due to the fact that the Fermi velocity v_F in graphene is much smaller than the speed of light c.

The φ dependence of integrals (64) can be factorized with the help of Eq. (B13) (see Appendix B) and takes the form:

$$Y_{j'j} = \exp\left[\left(i\frac{\lambda' - \lambda}{2} + j' + j - 1\right)\varphi\right] \mathcal{J}_{j',j}(\rho),$$
$$\mathcal{J}_{j',j}(\rho) = \int_{-\infty}^{\infty} D_{-\nu'-j'}[-(1+i)u]D_{\nu-j}[-(1-i)u]e^{i\rho u}du; \tag{67}$$

$$\tilde{Y}_{j'j} = \exp\left[\left(i\frac{\lambda' - \lambda}{2} + j' - j\right)\varphi\right] \tilde{\mathcal{J}}_{j',j}(\rho),$$
$$\tilde{\mathcal{J}}_{j',j}(\rho) = \int_{-\infty}^{\infty} D_{-\nu'-j'}[-(1+i)u]D_{-\nu-j}[-(1+i)u]e^{i\rho u}du. \tag{68}$$

These integrals can be expressed via the confluent hypergeometric function Ψ as

$$\mathcal{J}_{j',j}(\rho) = (-1)^j \sqrt{\frac{2}{\pi}} \Gamma(\nu - j + 1)e^{i\pi(\nu'+j'-1)/2} \sinh \frac{\pi\lambda}{2} I_{j',1-j}(\rho), \tag{69}$$

$$\tilde{\mathcal{J}}_{j',j}(\rho) = e^{i\pi(\nu+\nu'+j+j')/2} I_{j',j}(\rho), \tag{70}$$

$$I_{j',j}(\rho) = \sqrt{\pi} \exp\left[\left(\ln\frac{\rho}{\sqrt{2}} - \frac{i\pi}{4}\right)(\nu - \nu' + j - j') + i\frac{\rho^2}{2} - \frac{i\pi}{4}\right]$$
$$\times \Psi\left(\nu + j, 1 + \nu - \nu' + j - j'; -i\frac{\rho^2}{2}\right), \tag{71}$$

where Γ is the gamma function (see Appendix B for details).

3.3 Natural limits of parameters

In the case under consideration, one should take into account the existence of natural limits of physical parameters characterizing both the charged particles themselves and their radiation.

Let us consider the domain of the applicability of the perturbation theory with respect of the photon emission in the case under consideration. In the T-constant electric field there is the natural range of the very low frequency of emission, $\omega \lesssim \omega^{IR} = 2\pi T^{-1}$. In this range the perturbation theory works if the total number of photons is small enough. Otherwise, the radiation must be treated in the mean field approximation. For our purposes, it is enough to restrict the applicability of the perturbation theory with respect of the photon emission by the condition $\omega > \omega^{IR}$, which is convenient to represent as:

$$u_0 > u_0^{IR}, \ u_0^{IR} = 2\pi \Delta t_{st} T^{-1}. \tag{72}$$

It should be recalled that, in a number of cases, the need to cut off from below the region of the radiation frequencies is often encountered in QED problems. This makes it possible to deal with divergences (the well-known infrared catastrophe) whose nature is associated with the impossibility of separating a charged particle from its radiation field; see Appendix A for details. It is known that such soft photons carry away only a negligibly small part of the energy of emission, so that the corresponding back reaction is also negligible. In the case under consideration an estimation of the corresponding cut off parameter shows that its value is much less than the quantity u_0^{IR}, which means, in turn, that the domain of the applicability of the perturbation theory is bigger than the one that follows from the inequality (72). Therefore, condition (72) provides the possibility of applicability of the perturbation theory in the problem we are considering. Thus, we believe that results obtained in section (3.2) may be considered credible in all the frequency range (72).

We note now that condition (72) is not a significant limitation when applying our approach to a wide class of similar physical problems. Indeed, as has been shown in Ref. [13] nonlinear and linear $I - V$ graphs experimentally observed in low and high-mobility graphene samples [36], can be explained in the framework of strong-field QED$_{3,2}$ in the mean field approximation, taking into account the backreaction of the mean current of created carriers to the applied electric field which is set by a constant voltage. In addition, it has been found that the radiation of a time-dependent mean current, forming the backreaction to the electric field on the graphene plane, is emitted to the three-dimensional space in the form of very low frequency, $\omega \lesssim \omega^{\text{IR}}$, linearly polarized plane electromagnetic waves. It can be seen that the backreaction of the mean current can be neglected on the big time intervals of the time scale Δt_{st} order, which is equivalent to the assumption that the electric field is constant. Thus, the backreaction does not effect the emission of high-frequency photons, because the corresponding formation interval Δt is of the order Δt_{st} (the latter will be demonstrated below).

Maximum possible values of particle momenta in the T-constant field were determined by the Eq. (39). The finite dimensions of graphene samples do not allow us to consider the spectrum of small momenta to be continuous. However, the dependence on the longitudinal impulses p_x in the expressions (57), (58), and (59) is such that the discreteness of these momenta can simply be ignored. Nevertheless, for small lateral width L_y, only those momenta p_y that are not very small, namely, satisfy the condition

$$|p_y| \gg \Delta p_y, \ \ \Delta p_y = \frac{2\pi \hbar}{L_y}.$$

In turn, this this limits from below the admissible values of the dimensionless parameter λ,

$$\sqrt{\lambda} \gg \sqrt{\lambda_{\min}}, \ \ \sqrt{\lambda_{\min}} = \sqrt{\frac{v_F \hbar}{eE}} \frac{2\pi}{L_y}. \tag{73}$$

In the absence of nanoribbons, we may assume that $L_y \sim 1^-$m , then

$$\lambda_{\min} \sim \frac{2.7}{a} \times 10^{-2}, \tag{74}$$

where the range of allowable values of dimensionless parameter a is given by Eq. (43). In the case of nanoribbons one has to take into account that typical width of a nanoribbon l_y is $l_y \sim 1$nm and the parameter λ is quantized,

$$\sqrt{\lambda_n} = \sqrt{\frac{v_F \hbar}{eE}} \frac{2\pi}{l_y} n, \ n = 0, 1, 2, \ldots . \tag{75}$$

Thus, for the smallest nonzero value of λ_1 we have: $\sqrt{\lambda_1} \sim 27/a$.

The large time scale is:

$$\Delta t_{st} \approx (a)^{-1/2} 2.6 \times 10^{-14} \text{s} . \tag{76}$$

Characteristic frequency

$$\omega_{sc} = \Delta t_{st}^{-1} \approx \sqrt{a} \times 0.39 \times 10^{14} \text{s}^{-1}, \tag{77}$$

provides a value of one for dimensionless parameter u_0 and, therefore, specifies a frequency scale against which high-frequency and low-frequency emissions regions can be defined. Note that it depends on electric field E value. Characteristic wavelength scale is:

$$l_{sc} = \frac{2\pi c}{\omega_{sc}} \approx \frac{48}{\sqrt{a}} \times 10^{-6} \text{m} \tag{78}$$

For example, in the case of the typical voltage ~ 1 V ($a \sim 1$) the corresponding high-frequency range is a mid-wavelength infrared. We stress that terahertz-field induced spontaneous optical emission in the range of 340–600 nm was observed from a monolayer graphene on a glass substrate [17, 18].

The differential probabilities (49a) and (50) can be integrated over $\mathbf{K}$ only between such limits that leave the integral probability much smaller than unity. Let us demonstrate that for the integration over ω there is a natural cutoff from above. Let us consider the high-frequency case,

$$u_0 \gtrsim \tau_\gamma, \ \ \tau_\gamma \gg 1, \tag{79}$$

where τ_γ is an arbitrary given number. For probability densities of the photon emission with $\mathbf{K}$, given by Eqs. (54) and (55), an important role is played by definite time intervals (58) and (59). These are intervals where main contributions to the integrals are formed. On the same intervals the main contributions are formed to probability densities of the photon absorption. We can find this intervals using the saddle-point method.

Let us consider integral (59). Under condition (79), the mentioned saddle-point is situated in the range where absolute values of arguments of both WPCF's involved in integral (59) are big,

$$|u + u_x/2| \gg \max\{1, \lambda\} \text{ and } |u - u_x/2| \gg \max\{1, \lambda\}. \tag{80}$$

In this case, if $u \pm u_x/2 < 0$, one uses the following asymptotic expansion:

$$D_p(z) = e^{-z^2/4} z^p \left[1 + O\left(|z|^{-2}\right) \right] \text{ if } |\arg z| < \frac{3\pi}{4}. \tag{81}$$

If $u \pm u_x/2 > 0$, applying Eq. (81), one uses a relation between WPCF's (see (2.8.2.(7)) in Ref. [44]),

$$D_p(z) = e^{-i\pi p} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-i\pi(p+1)/2} D_{-p-1}(iz). \tag{82}$$

Thus one finds that the saddle-point is $u = u_0/2$. Since u_0 is positive, the saddle-point can be situated only in the range $u \pm u_x/2 > 0$. Following the same way one finds that the saddle-point of the kernel in integral (58) is $u = u_0/2$ and is also situated in the range $u \pm u_x/2 > 0$.

Using substitutions (56) one can see that the saddle-point equation represents a conservation law of the kinetic energy,

$$v_F \left[2eEt - \left(p_x + p'_x \right) \right] = \hbar\omega. \tag{83}$$

In the neighborhood of the saddle-point the corresponding kernels have Gaussian forms with maxima at the time instant

$$t_c = \frac{1}{2}\left(\Delta t_{st}^2 \omega + \frac{p_x + p'_x}{eE} \right) \tag{84}$$

and with the standard deviation

$$\Delta t_{sd} = \Delta t_{st}/\sqrt{2}. \tag{85}$$

The time t_c corresponds to the position of the center of the formation interval Δt for given ω, p_x, and p'_x. The width of the formation interval Δt must be large enough to accommodate the points $u + u_x/2$ and $u - u_x/2$. In addition, the formation interval must overlap the interval Δt_{sd}, $\Delta t_{sd} < \Delta t$. It is natural to assume that $\Delta t \sim \Delta t_{st}$. It implies the following condition:

$$|u_x| < 1. \tag{86}$$

With account taken of the relation $p'_x = p_x - \hbar k_x$ one can see that inequality (86) implies:

$$v_F \Delta t_{st} |k_x| = \frac{|k_x|}{K} \frac{v_F}{c} u_0 < 1. \tag{87}$$

Thus, in the case of high frequencies, the width of the formation interval does not depend on the frequency ω and on the momentum of the particle and is determined entirely by the electric field E. Thus, the variation of the external electric field acting on the particle within the formation length can be neglected, which allows us to use the locally constant field approximation. By the same reason, the obtained results can be easily extended to the study of the emission in any slowly varying field configuration. Assuming that the electric field E decreases quickly enough beyond the formation interval (for example, as a result of the backreaction of created pairs, see details in Ref. [13]), the upper limitation (43) to the intensity of the constant electric field can be significantly weakened. This means, for example, that the above considerations may be extended to terahertz pulses of intensity from 100 to 250 kV/cm, which are used in the existing experiments [17, 18].

The saddle-point of the kernel involved in integral (58) is located in the range $u \pm u_x/2 > 0$, that is, longitudinal kinetic momenta of an emitting electron are negative, $P_x(t) = p_x - eEt < 0$ and $P'_x(t) = p'_x - eEt < 0$ (longitudinal kinetic momenta of a hole are positive, $-P_x(t)$ and $-P'_x(t)$). According to Eq. (66), P_x and P'_x differ little in magnitude, so we can neglect the contribution from the small longitudinal component k_x in Eq. (83), which gives: $2v_F|P_x(t)| \approx \hbar\omega$. Note that in the T-constant field, the range of p_x, given by Eq. (39), implies that the initial kinetic momenta of an electron under consideration are always positive, $P_x(t_1) > 0$ (initial kinetic momenta of a hole are negative). In the case of the photon emission which accompanies the pair production from the initial vacuum, the saddle-point of the kernel involved in integral (59) is located in the same range $u \pm u_x/2 > 0$, that is, the longitudinal kinetic momentum of the electron of a pair is negative, $P'_x(t) < 0$, while the longitudinal kinetic momentum of the hole of a pair is positive, $P_x^{(h)}(t) = -P_x(t) > 0$. The conservation law for the kinetic energy at the saddle-point, given by Eq. (83), can be written in terms of these momenta as $v_F \left[P_x^{(h)}(t) + \left| P'_x(t) \right| \right] = \hbar\omega$. It follows from Eq. (84) that

$$t_c \approx \frac{1}{2} \Delta t_{st}^2 \omega + \frac{p_x}{eE}. \tag{88}$$

It means that for the photon emission of a given frequency ω dependence of the effect on p_x comes down to just shifting of the center of the formation interval. On the other hand, for a given momentum p_x photons with high frequencies are formed later.

It follows from Eq. (88) that for any given p_x the high-frequency emission, $\omega/\omega_{sc} \gtrsim \tau_\gamma$, starts when the longitudinal kinetic momentum $P_x(t)$ reaching its threshold value at $t_c \sim t_0$ according to condition (79),

$$\frac{2|P_x(t_0)|}{eE\Delta t_{st}} \approx \tau_\gamma. \tag{89}$$

The minimal frequency where the region of high frequencies starts is:

$$\omega_{\min} = \frac{2|P_x(t_0)|}{eE\Delta t_{st}}\omega_{sc} \approx \tau_\gamma \omega_{sc} \,. \tag{90}$$

The smallest possible value of the moment t_0, at which Eq. (90) is satisfied, is achieved at the smallest possible momentum value p_x from the finite range (39). Taking it into account, we find

$$t_0 - t_1 \sim (\tau_\gamma/2 + \tau)\Delta t_{st}. \tag{91}$$

The frequency ω grows from the minimum value $\omega_{\min}$ as long as the electric field is acting and reaches the maximum possible for a given p_x frequency ω_2 at the time instant $t_c \sim t_2$, when the electric field switches off. Photon with such a frequency is emitted during the formation interval preceding the moment t_2 of switching off the electric field. It follows from Eq. (88) that

$$\omega_2 \approx \frac{2|P_x(t_2)|}{eE\Delta t_{st}}\omega_{sc}. \tag{92}$$

Absolute maximum among all possible frequencies ω_2 with different momenta p_x satisfying Eq. (39) is:

$$\omega_{\max} \approx 2\Delta t_{st}^{-2}[t_2 + \max(-p_x/eE)] \approx 2(T/\Delta t_{st})\omega_{sc} \,. \tag{93}$$

A frequency range between $\omega_{\max}$ and $\omega_{\min}$ does exists if

$$\frac{\omega_{\max}}{\omega_{\min}} \approx \frac{2T}{\Delta t_{st}\tau_\gamma} > 1, \tag{94}$$

which means that the field duration time T satisfying Eqs. (36) and (42) is sufficiently large.

In particular, it follows from the estimation (93) that dimensionless parameter u_0 is restricted from above,

$$u_0 < \omega_2\Delta t_{st} < u_0^{\max}, \quad u_0^{\max} = \omega_{\max}/\omega_{sc} \approx 2T/\Delta t_{st} \,. \tag{95}$$

Note, that if

$$\frac{v_F}{c}u_0^{\max} < 1, \tag{96}$$

inequality (87) always holds true. Besides, the lower bound of the range of the frequency, given by Eq. (72), is also defined by the quantity $u_0^{\max}$,

$$u_0^{IR} = 4\pi/u_0^{\max}. \tag{97}$$

Choosing $T = T_{bal} \sim 10^{-12}$s and taking into account the estimation for the quantity Δt_{st} given by Eq.(76), we find

$$u_0^{\max} \approx 78\sqrt{a}, \ \omega_{\max} \approx 3.0 \times 10^{15}a \ s^{-1} \,. \tag{98}$$

We believe that at $\tau_\gamma \sim 3$ one can confident enough to identify the range of high frequencies. It follows from estimation (98) and from restrictions on the parameter a given by Eq. (43) that the high-frequency range $u_0 \gtrsim \tau_\gamma$ definitely exists for the fields under consideration.

4 High-frequency approximation

In section 3.2, we have obtained characteristics of the one-photon emission probabilities that are valid in range (72). In the general case, angular and polarization distributions of the emitted photons have quite complicated form. Nevertheless, their analysis is greatly simplified in the range of high frequencies (we recall that this range is defined by relation (79)). One can see that namely the emission in this range makes the main contribution to the one-photon emission considered by us. This is explained by the fact that in this case, the width of the formation interval is small $\Delta t \sim \Delta t_{st}$ and does not depend on the frequency ω and on the particles momenta and is determined entirely by the electric field E. Thus, the obtained results can be extended to the study of the emission in any slowly varying field configuration. For this reason the emission of high-frequency photons which accompanies the electronic quantum transport in the graphene is more realistic for possible experimental observations. In which follows, we assume that the mass gap in the graphene is absent, $m = 0$, and we neglect the small terms depending on φ. At high frequencies,

$$\rho \approx u_0 > \tau_\gamma, \tag{99}$$

and using an asymptotic behavior of the function Ψ given by Eq. (6.13.1.(1)) in Ref. [38], we find:

$$\Psi\left(\nu + j, 1 + \nu - \nu' + j - j'; -i\frac{\rho^2}{2}\right) = \left(-i\frac{\rho^2}{2}\right)^{-\nu-j}\left[1 + O\left(\rho^{-2(j+1)}\right)\right].$$

Whence it follows that

$$I_{j'j}(\rho) \approx \sqrt{\pi} \left(\frac{\rho}{\sqrt{2}} \right)^{-\nu'-\nu-j'-j} \exp\left\{ \frac{i}{4} \left[-\pi + \rho^2 + \pi \left(\nu' + \nu + j' + j \right) \right] \right\} \tag{100}$$

We see that the leading contribution to the amplitude $M_{\mathbf{p'p}}^+$ given by Eq. (57) is due to the term with Y_{01}. Using representations (67), (70), and (100), we find:

$$\left| M_{\mathbf{p'p}}^+ \right|^2 \bigg|_{\mathbf{p'}=\mathbf{p}-\hbar\mathbf{k}} \approx f(\lambda, \lambda') \left| \chi_{\vartheta}^{0,1} \right|^2,$$

$$f(\lambda, \lambda') = 2\pi \sinh \frac{\pi\lambda}{2} \exp\left[-\frac{\pi}{4} \left(5\lambda + 7\lambda' \right) \right], \tag{101}$$

where $\chi_{\vartheta}^{0,1}$ is given by Eq. (63) and

$$\lambda' = (v_F \Delta t_{st})^2 \left| k_y - p_y/\hbar \right|^2. \tag{102}$$

Thus, we find that the asymptotic behavior of the probability of the one photon emission with a given polarization ϑ from a single-electron (hole) state per unit frequency and solid angle is:

$$\frac{d\mathcal{P}\left(\mathbf{K}, \vartheta | \overset{\pm}{\mathbf{p}} \right)}{d\omega d\Omega} = \frac{\alpha}{\varepsilon} \left(\frac{v_F}{c} \right)^2 \frac{\omega \Delta t_{st}^2}{(2\pi)^2} \left| M_{\mathbf{p'p}}^{\pm} \right|^2 \bigg|_{\mathbf{p'}=\mathbf{p}-\hbar\mathbf{k}}, \tag{103}$$

where $\left| M_{\mathbf{p'p}}^+ \right|^2$ is given by Eq. (101) and $\left| M_{\mathbf{p'p}}^- \right|^2 = \left| M_{\mathbf{pp'}}^+ \right|^2$. Summing probabilities (103) over the polarizations, we obtain the probability of unpolarized emission from a single-electron (hole) state per unit frequency and solid angle as:

$$\frac{d\mathcal{P}\left(\mathbf{K} | \overset{\pm}{\mathbf{p}} \right)}{d\omega d\Omega} = \frac{\alpha}{\varepsilon} \left(\frac{v_F}{c} \right)^2 \frac{\omega \Delta t_{st}^2}{(2\pi)^2} \left| \tilde{M}_{\mathbf{p'p}}^{\pm} \right|^2 \bigg|_{\mathbf{p'}=\mathbf{p}-\hbar\mathbf{k}}, \quad \left| \tilde{M}_{\mathbf{p'p}}^- \right|^2 = \left| \tilde{M}_{\mathbf{pp'}}^+ \right|^2,$$

$$\left| \tilde{M}_{\mathbf{p'p}}^+ \right|^2 \bigg|_{\mathbf{p'}=\mathbf{p}-\hbar\mathbf{k}} \approx f(\lambda, \lambda') \left[1 - \sin^2 \phi \left(1 - \cos^2 \theta \right) \right]. \tag{104}$$

Probabilities (103) and (104) increase monotonically with increasing the frequency ω and reach their maxima, given by Eqs. (92) and (93), respectively, as $\omega \to \omega_2 < \omega_{\max}$. One can find the probability of one-photon emission from given distributions of electrons and holes of one kind per unit frequency and solid angle as follows:

$$\frac{d\mathcal{P}\left(\mathbf{K}, \vartheta | \overset{\pm}{\text{in}} \right)}{d\omega d\Omega} = \frac{S}{(2\pi\hbar)^2} \int \frac{d\mathcal{P}\left(\mathbf{K}, \vartheta | \overset{\pm}{\mathbf{p}} \right)}{d\omega d\Omega} N_{\mathbf{p}}^{(\pm)}(\text{in}) dp_x dp_y, \tag{105}$$

where $N_{\mathbf{p}}^{(\pm)}(\text{in})$ are some initial differential mean numbers of electrons (+) and holes (−). If the numbers $N_{\mathbf{p}}^{(\pm)}(\text{in})$ are the same for all the charge species, the final probability is given by Eq. (105) multiplying it by the number N_f of the species.

One can see that for given angles θ and ϕ the function defined by Eq. (101) has the Gaussian form as a function of the wave number k_y, and besides $p_y/\hbar$ is the position of its maximum and $\left(\sqrt{7\pi} v_F \Delta t_{st}/\sqrt{2} \right)^{-1}$ is the corresponding standard deviation. Note that this deviation increases with the intensity of the electric field. We note that the emission from a one-electron state depends essentially on the electron transversal momentum p_y. This emission takes place only if the latter momentum differs from zero. For big $\lambda \gg 1$ the probability of the emission decreases exponentially as λ increases. The emission is maximum at $\tanh \frac{\pi\lambda}{2} = \frac{2}{5}$, that is, the main contribution to it comes from electrons with moderate magnitude $\lambda \sim 1/\pi$. In this case function (101) as the function of the wave number k_y reaches its maximum at $\lambda' = 0$. It is convenient to introduce the quantity $\omega_y = c|k_y|$, which represents a corresponding contribution to the frequency ω. It is obviously that $\omega_y < \omega$. Since $\omega < \omega_{\max}$, where $\omega_{\max}$ is given by Eq. (93), we obtain the restriction $\omega_y < \omega_{\max}$. The condition $\lambda' = 0$ implies:

$$\frac{v_F}{c} \frac{\omega_y}{\omega_{sc}} = \sqrt{\lambda}, \tag{106}$$

where ω_{sc} is given by Eq. (77). Assuming $\sqrt{\lambda} \sim 1$ one can satisfy condition (106) only in the case when $\omega_{\max}$ is big enough, such that

$$\frac{2T v_F}{\Delta t_{st} c} \sim 1. \tag{107}$$

It is possible if the dimensionless parameter $T/\Delta t_{st}$ is big enough as well. Otherwise the condition $\lambda' = 0$ is unreachable if $\sqrt{\lambda} \sim 1$.

We see that the leading contribution to the amplitude $M^0_{\mathbf{p}'\mathbf{p}}$ given by Eq. (57) arrises from the term with $\tilde{Y}_{00}$. Using representations (68), (70), and (100), we find:

$$\left|M^0_{\mathbf{p}'\mathbf{p}}\right|^2\bigg|_{\mathbf{p}'=-\mathbf{p}-\hbar\mathbf{k}} \approx \tilde{f}(\lambda,\lambda')\left|\chi^{0,1}_\vartheta\right|^2, \;\; \tilde{f}(\lambda,\lambda') = \pi e^{-\pi(\lambda+\lambda')}, \tag{108}$$

where $\chi^{0,1}_\vartheta$ is given by Eq. (63). We note that the quantity $\tilde{f}(\lambda,\lambda')$ is proportional to the product of differential mean numbers of electron and hole of a pair created, respectively. Thus, we find that the asymptotic behavior of the probability of the one-photon emission with a given polarization ϑ, which accompanies the production from the initial vacuum state of pairs of charged species with a given momentum $\mathbf{p}$ per unit frequency and solid angle reads:

$$\frac{d\mathcal{P}(\mathbf{p};\mathbf{K},\vartheta|0)}{d\omega d\Omega} \approx \frac{\alpha}{\varepsilon}\left(\frac{v_F}{c}\right)^2 \frac{\omega\Delta t_{st}^2}{(2\pi)^2}\left|M^0_{\mathbf{p}'\mathbf{p}}\right|^2\bigg|_{\mathbf{p}'=\mathbf{p}-\hbar\mathbf{k}}, \tag{109}$$

where $\left|M^0_{\mathbf{p}'\mathbf{p}}\right|^2$ is given by Eq. (108). The total probability of the one-photon emission with quantum numbers $\mathbf{K}$ and ϑ, which accompanies the pair production of all N_f species from the initial vacuum state per unit frequency and solid angle is presented by an integral over the finite momentum range D, given by Eq. (39),

$$\frac{d\mathcal{P}(\mathbf{K},\vartheta)}{d\omega d\Omega} = \frac{N_f S}{(2\pi\hbar)^2}\int_D \frac{d\mathcal{P}(\mathbf{p};\mathbf{K},\vartheta|0)}{d\omega d\Omega}dp_x dp_y. \tag{110}$$

Its asymptotic behavior has the form:

$$\frac{d\mathcal{P}(\mathbf{K},\vartheta)}{d\omega d\Omega} \approx \mathcal{R}(\mathbf{K},\vartheta)ST, \;\; \mathcal{R}(\mathbf{K},\vartheta) = d(\omega,\omega_y)\left|\chi^{0,1}_\vartheta\right|^2,$$

$$d(\omega,\omega_y) = \frac{\alpha N_f}{\varepsilon 2^{5/2}\pi}\frac{\omega}{\omega_{sc}l_{sc}^2}\exp\left[-\frac{\pi}{2}\left(\frac{v_F\omega_y}{c\omega_{sc}}\right)^2\right], \tag{111}$$

where l_{sc} is the characteristic wavelength scale given by Eq. (78). Note that probability (111) is proportional to the total number density of electron–hole pairs created, given by Eq. (41). Summing the total probabilities (111) over the polarizations, we obtain the probability of unpolarized emission which accompanies the pair production from the vacuum per unit frequency and solid angle:

$$\frac{d\mathcal{P}(\mathbf{K})}{d\omega d\Omega} \approx \mathcal{R}(\mathbf{K})ST, \;\; \mathcal{R}(\mathbf{K}) = d(\omega,\omega_y)\left[1 - \sin^2\phi\left(1 - \cos^2\theta\right)\right]. \tag{112}$$

The formula (112) was previously obtained in Ref. [16] for $\varepsilon = 1$ in the framework of many-body quantum mechanics, where the interaction with external electric field is treated nonperturbatively.

Note that the frequency ω in Eqs. (109), (111), and (112) is restricted from the above, $\omega < \omega_{max}$, where ω_{max} is given by Eq. (93). This implies the restriction $\omega_y < \omega < \omega_{max}$ for ω_y. Therefore,

$$\frac{v_F\omega_y}{c\omega_{sc}} < \frac{2Tv_F}{\Delta t_{st}c}.$$

One can see that the argument of the exponential function in Eq. (111) can significantly affect the value of the probability only under the condition

$$\frac{2Tv_F}{\Delta t_{st}c} \gtrsim 1. \tag{113}$$

In this case one can see that this probability decreases exponentially if $\omega_y \to \omega$ and $\omega \to \omega_{max}$, that is, for frequencies close to the maximum ω_{max}, the emission in the y-axis direction is suppressed. The probabilities (111) and (112) increase monotonically with increasing frequency ω and reach their maximum as $\omega \to \omega_{max}$.

We see that the asymptotic behavior of angular and polarization distributions from one-electron (hole) state and from the vacuum state are very similar. Nevertheless, one can distinguish between these two types of the radiation. Indeed, probabilities (111) and (112) are proportional to macroscopic duration time T, which is a consequence of the integration over the large range D of the momentum p_x variation, while probability (105) does not depend on T. Therefore, by studying the dependence of the radiation on T, it is possible, in principle, to identify its origin. In addition, one can stress that the main contribution to the emission is due to probabilities (111) and (112), if the density of initial electrons (holes) is much less than density (41) of created electron–hole pairs.

The angular distribution is determined by the factor $\left|\chi^{0,1}_\vartheta\right|^2$ given by Eq. (63). It is quite different for the polarization $\vartheta = 1$ (polarization in the XY plane) and for the polarization $\vartheta = 2$ (polarization in the perpendicular to vector $\boldsymbol{\epsilon}_{\mathbf{K}1}$ direction). These factors are:

$$\left|\chi^{0,1}_1\right|^2 = \cos^2\phi = \frac{k_x^2}{k_x^2 + k_y^2}, \;\; \left|\chi^{0,1}_2\right|^2 = |\cos\theta\sin\phi|^2 = \frac{k_z^2 k_y^2}{K^2\left(k_x^2 + k_y^2\right)}. \tag{114}$$

Thus, the emission with the polarization $\vartheta = 1$ has the same probability along all the directions belonging to the plane which is perpendicular to the one XY and is tilted at the angle ϕ with respect to the axis x. The maximum of the probability takes place for a small angle ϕ, $\cos^2 \phi \rightarrow 1$. The emission with the polarization $\vartheta = 1$ is absent in the YZ plane, $\cos \phi = 0$. The emission with the polarization $\vartheta = 2$ is also absent in the XZ and XY planes. The maximum of the emission is observed in the YZ plane in the directions close to z-axis, $|\cos \theta \sin \phi|^2 \rightarrow 1$. Therefore, the emission in the YZ, XZ, and XY planes is highly polarized. We see that the unpolarized emission is maximal in the directions defined by the relations $\cos^2 \phi \rightarrow 1$ and $\cos^2 \theta \rightarrow 1$. An emission in the y -axis, $\cos \phi = 0$ and $\cos \theta = 0$, is absent.

The above calculations of the emission are made in the first order of the perturbation theory. This approximation is reasonable in case if total emission probabilities from a given initial state are small. In this relation, let us consider probability (104). One can see that

$$\max \left| \tilde{M}^{\pm}_{\mathbf{p'p}} \right|^2 \bigg|_{\mathbf{p'} = \mathbf{p} - \hbar \mathbf{k}} \sim 1,$$

Integrating probability (104) all the frequencies in the domain where high-frequency approximation holds true, that is, from $\omega_{\min}$ to $\omega_{\max}$ ($\omega_{\max}$ is given by Eq. (93)), we can estimate the maximum total emission probability from a one-particle state. The smallness of this probability implies applicability condition of the perturbation theory:

$$\frac{\alpha}{\varepsilon} \left(\frac{v_F T}{c \Delta t_{st}} \right)^2 \ll 1. \tag{115}$$

Since $v_F / c \sim 1/300$, condition (115) represents weak enough restriction on the field parameter $T / \Delta t_{st}$ satisfying conditions (36) and (42).

Let us consider probability (112). The function $\mathcal{R}(\mathbf{K})$ is restricted from above by the quantity $d(\omega, 0)$. Integrating probability (112) over the angles θ and ϕ and over the all frequencies in the domain where high-frequency approximation holds true, that is, from $\omega_{\min}$ to $\omega_{\max}$ ($\omega_{\min}$ and $\omega_{\max}$ are given by Eqs. (90) and (93), respectively), we can estimate the maximum total emission probability $\mathcal{P}_{\max}$ which accompanies the pair production from the vacuum state as:

$$\mathcal{P}_{\max} \approx \frac{\alpha}{\varepsilon} \frac{2^{7/2} N_f S}{3 l_{sc}^2} \left(\frac{T}{\Delta t_{st}} \right)^3. \tag{116}$$

Probabilities (112) and (116) grow with the increase of the intensity of the electric field. At the same time, the total probability $P_{\max}$ grows especially fast due to the linear growth of the frequency range. So, if the electric field increases by q times, then the probability $P_{\max}$ increases by $q^{5/2}$ times.

The smallness of probability (116) implies also the applicability condition of the perturbation theory:

$$\mathcal{P}_{\max} \ll 1. \tag{117}$$

The typical quantity is $S \sim \left(10^{-6} \mathrm{m} \right)^2$. Using estimation of l_{sc} given by Eq. (78), we obtain $S l_{sc}^{-2} \sim a(48)^{-2}$. This parameter can be considered as a small one. In existing experiment conditions, where $T / \Delta t_{st}$ and a satisfy conditions (36) and (43), respectively, restriction (117) may impose an essential limit on the applicability of the perturbation theory. However, let us note that relation (43) follows from the assumption that during the time T the electric field remains constant. Assuming that the electric field E decreases quickly enough beyond the formation interval, the upper limitation (43) to the intensity of the electric field can be significantly weakened.

5 Summary

In the present work, we have constructed an appropriate calculation techniques in the framework of the reduced $QED_{3,2}$ to describe one species of Dirac fermions interacted with an external electric field and photons in the graphene. In these techniques, effects of the vacuum instability due to the particle creation by the external electric field are taken into account nonperturbatively. In such a way, we consider the photon emission in the graphene in the first-order approximation, taking into account a vacuum instability by using the unitarity relation, and construct closed formulas for total probabilities. Namely, we have calculated the probabilities for the photon emission by an electron and for the photon emission accompanying the vacuum instability in a quasiconstant electric field that acts in the graphene plane during macroscopic time interval T. In order to find corresponding mean values in the real graphene, results obtained for one species of the Dirac fermions are multiplied by the number of species $N_f = 4$. It has been shown that the frequency of emission grows in proportion to the duration of the electric field and reaches a final maximum value before the electric field is turned off. The lower limit of applicability of the perturbation theory with respect of the photon emission is established and showed that the contribution of soft photons beyond this boundary can be neglected. The obtained emission characteristics are analyzed in a high-frequency approximation which is more suitable for possible experimental observations. The angular and polarization distributions of the emission are also studied. The asymptotic behavior of the unpolarized photon emission

accompanying the vacuum instability matches with the previous calculation in Ref. [16], based on many-body quantum mechanics. We see that the asymptotic behavior of angular and polarization distributions from one-electron (hole) state and from the vacuum state are very similar. Nevertheless, we point out that one can distinguish between these two types of the radiation by considering the emission under electric fields with different duration times T. The applicability of the presented calculations to the graphene physics in existing experimental conditions is shown. This implies also a general possibility of laboratory verifying strong-field QED predictions, and, in particular, real studying the Schwinger effect.

It was shown that in a high-frequency approximation the variation of the external electric field acting on the particle within the formation length can be neglected, which justify the applicability of the locally constant field approximation. Thus, the developed approach can be easily extended to study the emission in any slowly varying field configuration.

In the single graphene sheet there are actually two species of fermions in the Dirac model of graphene. In our consideration, it is assumed that the two cones of graphene are decupled and the system behaves like two copies of a single Dirac cone. Topological insulators are characterized by a single Dirac cone on each surface; see [26, 39, 40] for a review. Thus, the results obtained in the present study could be relevant for a single Dirac cone on a surface of a topological insulator.

Declarations

Appendix A: Low-frequency approximation

Let us consider the probability densities, given by Eqs. (54), (55), and (57) in the range of low frequencies,

$$u_0 \ll 1. \tag{A1}$$

The ratio $|u_x|/u_0$, given by Eq. (66), is very small such that

$$\rho \approx u_0 \ll 1. \tag{A2}$$

In this limit, the behavior of the function $I_{j',j}(\rho)$, given by Eq. (71), can be found using properties of the confluent hypergeometric function Ψ; see Eqs. (6.8(2)) - (6.8(4)) from Ref. [38]. The only functions $I_{0,1}(\rho)$ and $I_{1,0}(\rho)$ grow as $\rho \to 0$,

$$\left| I_{0,1}(\rho) \right| \sim \left| I_{1,0}(\rho) \right| \sim \rho^{-1}. \tag{A3}$$

Thus, the leading contribution to the amplitude $M_{\mathbf{p'p}}^+$ given by Eq. (57) is due to the terms $Y_{00} \approx \mathcal{J}_{0,0}(\rho) \sim \rho^{-1}$ and $Y_{11} \approx \mathcal{J}_{1,1}(\rho) \sim \rho^{-1}$. The leading contribution to the amplitude $M_{\mathbf{p'p}}^0$ given by Eq. (57) is due to the terms $\tilde{Y}_{01} \approx \tilde{\mathcal{J}}_{0,1}(\rho) \sim \rho^{-1}$ and $\tilde{Y}_{10} \approx \tilde{\mathcal{J}}_{1,0}(\rho) \sim \rho^{-1}$. Therefore, the both modules squares amplitudes square grow proportionally to u_0^{-2},

$$\left| M_{\mathbf{p'p}}^0 \right|^2 \Big|_{\mathbf{p'}=-\mathbf{p}-\hbar\mathbf{k}} \sim u_0^{-2}, \quad \left| M_{\mathbf{p'p}}^\pm \right|^2 \Big|_{\mathbf{p'}=\mathbf{p}-\hbar\mathbf{k}} \sim u_0^{-2}. \tag{A4}$$

At the same time, the both probability densities (54) and (55) are divergent functions of the order u_0^{-1} as $u_0 \to 0$,

$$\frac{d\mathcal{P}(\mathbf{p}; \mathbf{K}, \vartheta | 0)}{du_0 d\Omega} \sim \frac{\alpha}{\varepsilon} \left(\frac{v_F}{c} \right)^2 \frac{1}{u_0}, \quad \frac{d\mathcal{P}\left(\mathbf{K}, \vartheta | \overset{\pm}{\mathbf{p}} \right)}{du_0 d\Omega} \sim \frac{\alpha}{\varepsilon} \left(\frac{v_F}{c} \right)^2 \frac{1}{u_0}. \tag{A5}$$

Such a behavior is an indication that the perturbative description of such soft photons, does not work. Namely for photons with frequencies less than a threshold frequency u_0^{soft}, $u_0 \lesssim u_0^{\text{soft}}$, in case when functions (A5) becomes of the order of unity. This makes it possible to evaluate the quantity u_0^{soft},

$$u_0^{\text{soft}} \sim \frac{\alpha}{\varepsilon} \left(\frac{v_F}{c} \right)^2. \tag{A6}$$

The number of such soft photons may be big enough. However, the only physically measurable quantity is the emitted energy. This energy is a negligibly small in the domain $u_0 \lesssim u_0^{\text{soft}}$. This case is called the infrared catastrophe whose nature is associated with the impossibility of separating a charged particle from its radiation field; see, e.g., section 98 in Ref. [41] and sections 46 and 50.3 in Ref. [42]. The case of the strong-electric field QED is considered in Ref. [43]. The infrared divergences of QED are essentially classical, and depend on the nature of the external current and on the experimental resolution. The infrared catastrophe is absent from the complete nonperturbative solution. Thus, one sees that the domain of the applicability of the perturbation theory is $u_0 > u_0^{\text{soft}}$ and a contribution from the domain $u_0 \lesssim u_0^{\text{soft}}$ is negligible.

In the case under consideration, the quantity u_0^{soft} is very small, $u_0^{\text{soft}} \sim 10^{-7}$. It follows from estimation (98) and from restrictions on the parameter u_0^{IR} given by Eqs. (72) and (97) that in the realistic values of the parameters $u_0^{\text{soft}} \ll u_0^{\text{IR}}$.

Appendix B: Fourier transformation of the product of two WPC functions

Integrals (64) can be reduced to a more simple form using the Nikishov's representations given by Eq. (65)) for the hyperbolic coordinates ρ and φ (see Ref. [19, 20]). To demonstrate how it works, we note that the integrals represent particular cases of the more general integrals

$$J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho, \varphi) = \int_{-\infty}^{+\infty} du \, f_{\Lambda'}^{\zeta'}(u - u_x/2) f_{\Lambda}^{\zeta}(u + u_x/2) e^{iu_0 u}, \tag{B7}$$

where $f_{\Lambda}^{\zeta}(z)$ are WPCF's satisfying the differential equation

$$\left(\frac{d^2}{dz^2} + z^2 + \Lambda \right) f_{\Lambda}^{\zeta}(z) = 0, \tag{B8}$$

and u_0 and u_x, defined by Eq. (56), are:

$$u_0 = \rho \cosh \varphi, \ u_x = \rho \sinh \varphi \ \text{if} \ u_0^2 > u_x^2. \tag{B9}$$

The functions $f_{\Lambda}^{\zeta}(z)$ with different values of $\zeta = \pm$ are solutions of Eq. (B8) with some complex parameters Λ. In particular,

$$\begin{aligned} J_{\Lambda'\Lambda}^{-+}(\rho, \varphi) &= Y_{j'j}, \ \Lambda = \lambda + i(2j - 1), \ \Lambda' = \lambda' + i(1 - 2j'), \\ J_{\Lambda'\Lambda}^{--}(\rho, \varphi) &= \tilde{Y}_{j'j}, \ \Lambda = \lambda + i(1 - 2j), \ \Lambda' = \lambda' + i(1 - 2j'). \end{aligned} \tag{B10}$$

Calculating the derivative of integral (B7) with respect to the hyperbolic angle φ, we find:

$$\frac{\partial J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho, \varphi)}{\partial \varphi} = W + \int_{-\infty}^{+\infty} iu_x f_{\Lambda'}^{\zeta'}(u - u_x/2) f_{\Lambda}^{\zeta}(u + u_x/2) e^{iu_0 u} du,$$

$$W = \frac{u_0}{2} \int_{-\infty}^{+\infty} \left[f_{\Lambda'}^{\zeta'}(u - u_x/2) \left. \frac{\partial f_{\Lambda}^{\zeta}(z)}{\partial z} \right|_{z=u+u_x/2} \right.$$

$$\left. - \left. \frac{\partial f_{\Lambda'}^{\zeta'}(z)}{\partial z} \right|_{z=u-u_x/2} f_{\Lambda}^{\zeta}(u + u_x/2) \right] e^{iu_0 u} du, \ u_x = \frac{\partial u_0}{\partial \varphi}, \ u_0 = \frac{\partial u_x}{\partial \varphi}.$$

Integrating by parts and neglecting boundary terms, we can transform W to the following form:

$$W = \frac{i}{2} \int_{-\infty}^{+\infty} \left[f_{\Lambda'}^{\zeta'}(u - u_x/2) \left. \frac{\partial^2 f_{\Lambda}^{\zeta}(z)}{\partial z^2} \right|_{z=u+u_x/2} \right.$$

$$\left. - \left. \frac{\partial^2 f_{\Lambda'}^{\zeta'}(z)}{\partial z^2} \right|_{z=u-u_x/2} f_{\Lambda}^{\zeta}(u + u_x/2) \right] e^{iu_0 u} du. \tag{B11}$$

Using Eq. (B8) in integral (B11), we find:

$$\frac{\partial J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho, \varphi)}{\partial \varphi} = \frac{i}{2} (\Lambda' - \Lambda) J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho, \varphi). \tag{B12}$$

Solutions of this equation are:

$$J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho, \varphi) = e^{\frac{i}{2}(\Lambda'-\Lambda)\varphi} J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho, 0). \tag{B13}$$

We use the notation $J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho) = J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho, 0)$ in what follows. This function satisfies the differential Eq. (19)

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \frac{(\Lambda - \Lambda')^2}{4\rho^2} + \frac{\rho^2}{4} - \frac{\Lambda + \Lambda'}{2}\right] J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho) = 0. \tag{B14}$$

This fact can be verified performing integrations by parts with account taken of Eq. (B8). The differential Eq. (B14) can be reduced to a confluent hypergeometric equation. Using two linearly independent solutions of such an equation, we find general solution of the differential Eq. (B14)

$$J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho) = e^{-\eta/2}\left[C_1\eta^{i\beta}\Phi\left(\frac{i\Lambda}{2} + \frac{1}{2}, 1 + 2i\beta; \eta\right) + C_2\eta^{-i\beta}\Phi\left(\frac{i\Lambda'}{2} + \frac{1}{2}, 1 - 2i\beta; \eta\right)\right],$$
$$\eta = -i\rho^2/2, \ \beta = (\Lambda - \Lambda')/4, \tag{B15}$$

where the C_1 and C_2 are some undetermined coefficients, which must be fixed by appropriate boundary conditions, so that solution (B15) corresponds to the original integral (B7).

The confluent hypergeometric function $\Phi(a, c; \eta)$ is entire in η and a, and is a meromorphic function of c. Note that $\Phi(a, c; 0) = 1$. WPCF's are entire functions of Λ and Λ'. One can see that the integrals $J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho)$ are entire functions of Λ and Λ' and meromorphic functions of $\Lambda - \Lambda'$. Then one can find a boundary condition $J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho)$ at $\rho \to 0$ for some convenient values of j and j'. The remaining integrals $J_{\Lambda'\Lambda}^{\zeta'\zeta}(\rho)$ can be obtained extending domains of Λ and Λ' by an analytic continuation.

Let us start with $\tilde{J}_{0,0}(\rho)$ given by Eq. (68). This integral can be represented as a solution of Eq. (B14) where $\Lambda' = \lambda' + i$ and $\Lambda = \lambda + i$. The coefficients C_1 and C_2 in Eq. (B15) can be fixed by a comparison with the $\rho \to 0$ limit of integral (68). Let us first represent this integral as follows:

$$\tilde{J}_{0,0}(\rho) = F^0 + F^+ + F^-, \ F^+ = \int_0^\infty f^+(u)e^{i\rho u}du, \ F^- = \int_{-\infty}^0 f^-(u)e^{i\rho u}du,$$
$$F^0 = \int_0^\infty f(u)\big[f(u) - f^+(u)\big]e^{i\rho u}du + \int_{-\infty}^0 f(u)\big[f(u) - f^-(u)\big]e^{i\rho u}du, \tag{B16}$$
$$f(u) = D_{-\nu'}[-(1+i)u]D_{-\nu}[-(1+i)u],$$

where $f^\pm(u) = f(u)|_{u\to\pm\infty}$. It can be seen that function (B15) is reduced to the oscillations $C_1\eta^{i\beta} + C_2\eta^{-i\beta}$ as $\rho \to 0$. Then ρ-independent terms do not contribute to the integrals F^0 and $F^\pm$. Taking into account that $\lim_{\rho\to 0} F^0$ and $\lim_{\rho\to 0} F^-$ do not depend on ρ, one sees that the oscillation terms of F^+ are only essential. Using relations (8.2.(7)) and (8.4.(1)) from Ref. [44], one finds:

$$\tilde{J}_{0,0}(\rho) = \sqrt{\pi}e^{i\pi(\nu+\nu'-1)/4}\left[e^{i\pi\nu/2}\frac{\Gamma(\nu-\nu')}{\Gamma(\nu)}\left(\frac{\rho}{\sqrt{2}}\right)^{\nu'-\nu}\right.$$
$$\left.+e^{i\pi\nu'/2}\frac{\Gamma(\nu'-\nu)}{\Gamma(\nu')}\left(\frac{\rho}{\sqrt{2}}\right)^{\nu-\nu'} \text{ as } \rho \to 0\right]. \tag{B17}$$

Comparing Eqs. (B15) and (B17), we obtain:

$$C_1 = \sqrt{\pi}e^{i\pi(\nu+\nu'-1/2)/2}\frac{\Gamma(\nu'-\nu)}{\Gamma(\nu')}, \ C_2 = \sqrt{\pi}e^{i\pi(\nu+\nu'-1/2)/2}\frac{\Gamma(\nu-\nu')}{\Gamma(\nu)}. \tag{B18}$$

Using relation (6.5.(7)) from Ref. [38], one can represent function given by Eqs. (B15) and (B18) as

$$\tilde{J}_{0,0}(\rho) = \sqrt{\pi}e^{i\pi(\nu+\nu'-1/2)/2}e^{-\eta/2}\eta^{(\nu-\nu')/2}\Psi(\nu, 1 + \nu - \nu'; \eta), \tag{B19}$$

where $\Psi(\nu, 1 + \nu - \nu'; \eta)$ is the confluent hypergeometric function,

$$\Psi(\nu, 1 + \nu - \nu'; \eta) = \frac{\Gamma(\nu'-\nu)}{\Gamma(\nu')}\Phi(\nu, 1 + \nu - \nu'; \eta)$$
$$+ \frac{\Gamma(\nu-\nu')}{\Gamma(\nu)}\eta^{\nu'-\nu}\Phi(\nu', 1 + \nu' - \nu; \eta). \tag{B20}$$

Using transformation $\nu \to \nu + j$ and $\nu' \to \nu' + j'$ in Eq. (B19), one obtains the final form (70) for integral (68).

The integral $\mathcal{J}_{j',j}(\rho)$ given by Eq. (67) can be represented as the solution of Eq. (B14) where $\Lambda' = \lambda' + i\left(1 - 2j'\right)$ and $\Lambda = \lambda + i$ $(2j-1)$. Using relation (8.2.(6)) from Ref. [44], we transform one of the WPCF's in Eq. (67) to obtain convenient representations:

$$\mathcal{J}_{j',j}(\rho) = \frac{\Gamma(\nu - j + 1)}{\sqrt{2\pi}}\left[e^{i\pi(\nu-j)/2}\tilde{\mathcal{J}}_{j',1-j}(\rho) + e^{-i\pi(\nu-j)/2}\mathcal{J}'_{j',1-j}(\rho)\right],\tag{B21}$$

$$\mathcal{J}'_{j',1-j}(\rho) = \int_{-\infty}^{\infty} D_{-\nu'-j'}[-(1+i)u]D_{-\nu+j-1}[(1+i)u]e^{i\rho u}du,\tag{B22}$$

where $\tilde{\mathcal{J}}_{j',1-j}(\rho)$ is given by Eq. (70). The integral $\mathcal{J}'_{j',1-j}(\rho)$ is represented by function (B15) where some coefficients C'_1 and C'_2 can be fixed by the comparison with $\rho \to 0$ limit of the integral $\mathcal{J}'_{j',1-j}(\rho)$.

Let us start with $\mathcal{J}'_{0,0}(\rho)$, where $\Lambda' = \lambda' + i$ and $\Lambda = \lambda + i$. In this case, it can be seen that function (B15) takes the form $C'_1\eta^{i\beta} + C'_2\eta^{-i\beta}$ as $\rho \to 0$. Hence all ρ-independent terms of $\mathcal{J}'_{0,0}(\rho)$, given by Eq. (B22), can be ignored at $\rho \to 0$ limit and only the oscillation terms of following integrals

$$G^+ = \int_0^{\infty} g^+(u)e^{i\rho u}du,\ \ F^- = \int_{-\infty}^0 g^-(u)e^{i\rho u}du,$$
$$g^{\pm}(u) = g(u)|_{u\to\pm\infty},\ \ g(u) = D_{-\nu'}[-(1+i)u]D_{-\nu}[(1+i)u]\tag{B23}$$

are essential. Using relations (8.2.(7)) and (8.4.(1)) from Ref. [44], one finds:

$$\mathcal{J}'_{0,0}(\rho) = \sqrt{\pi}e^{i\pi(\nu'-\nu-1/2)/2}\left[e^{i\pi(\nu-\nu')/4}\frac{\Gamma(\nu-\nu')}{\Gamma(\nu)}\left(\frac{\rho}{\sqrt{2}}\right)^{\nu'-\nu}\right.$$
$$\left. + e^{-i\pi(\nu-\nu')/4}\frac{\Gamma(\nu'-\nu)}{\Gamma(\nu')}\left(\frac{\rho}{\sqrt{2}}\right)^{\nu-\nu'}\ \text{as}\ \rho \to 0\right].\tag{B24}$$

Comparing Eqs. (B15) and (B24) we obtain:

$$C'_1 = \sqrt{\pi}e^{i\pi(\nu'-\nu-1/2)/2}\frac{\Gamma(\nu'-\nu)}{\Gamma(\nu')},\ \ C'_2 = \sqrt{\pi}e^{i\pi(\nu'-\nu-1/2)/2}\frac{\Gamma(\nu-\nu')}{\Gamma(\nu)}.\tag{B25}$$

Using relation (6.5.(7)) from Ref. [38], the function given by Eqs. (B15) and (B25) can be represented as:

$$\mathcal{J}'_{0,0}(\rho) = \sqrt{\pi}e^{i\pi(\nu'-\nu-1/2)/2}e^{-\eta/2}\eta^{(\nu-\nu')/2}\Psi(\nu, 1 + \nu - \nu'; \eta).\tag{B26}$$

Using the transformations $\nu \to \nu + 1 - j$ and $\nu' \to \nu' + j'$ in Eq. (B26), we obtain the following representation for integral (B22):

$$\mathcal{J}'_{j',1-j}(\rho) = e^{-i\pi(\nu-\nu'+1-j-j')/2}I_{j',1-j}(\rho),\tag{B27}$$

where $I_{j',j}(\rho)$ is given by Eq. (71). Substituting representations (70) and (B27) into Eq. (B21) we find the final form (69).

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