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Positive Energy Representations of Affine Vertex Algebras

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Abstract: We construct new families of positive energy representations of affine vertex algebras together with their free field realizations by using localization technique. We introduce the *twisting functor* T_α on the category of modules over the *universal affine vertex algebra* $\mathcal{V}_\kappa(\mathfrak{g})$ of level κ for any positive root α of $\mathfrak{g}$, and the *Wakimoto functor* from a certain category of $\mathfrak{g}$-modules to the category of $\mathcal{V}_\kappa(\mathfrak{g})$-modules. These two functors commute (taking a proper restriction of T_α on $\mathfrak{g}$-modules) and the image of the Wakimoto functor consists of *relaxed Wakimoto* $\widehat{\mathfrak{g}}_\kappa$-modules. In particular, applying the twisting functor T_α to the relaxed Wakimoto $\widehat{\mathfrak{g}}_\kappa$-module whose top degree component is isomorphic to the Verma $\mathfrak{g}$-module $M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$, we obtain the relaxed Wakimoto $\widehat{\mathfrak{g}}_\kappa$-module whose top degree component is isomorphic to the α-Gelfand–Tsetlin $\mathfrak{g}$-module $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda, \alpha)$. We show that the relaxed Verma module and relaxed Wakimoto module whose top degree components are such α-Gelfand–Tsetlin modules, are isomorphic generically. This is an analogue of the result of E. Frenkel for Wakimoto modules both for critical and non-critical level. For a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ we construct a new large family of positive energy representations of the simple affine vertex algebra $\mathcal{L}_\kappa(\mathfrak{g})$ of admissible level κ by means of the twisting functor applied on generalized Verma modules for the parabolic subalgebra $\mathfrak{p}$.

Contents

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Introduction

Wakimoto modules and Verma modules for affine Kac–Moody algebras are the key objects in the theory of rational affine vertex algebras. Among others, Wakimoto modules provide a free field realization of affine Kac–Moody algebras. They were constructed in [Wak86,FF88,dBF97,Szc02]. Other free field realizations of affine Kac–Moody algebras associated with nonstandard Borel subalgebras were studied in [Cox05] (*imaginary Verma modules*), [CF04] (*intermediate Wakimoto modules*), [FKS19] (*generalized imaginary Verma modules*). Parabolic versions of Wakimoto modules (*generalized Wakimoto modules*) were introduced in [Fre05].

However, there is a growing interest in the study of non-highest weight modules, especially *relaxed* highest weight modules, over affine Kac–Moody algebras that play a significant role for non-rational affine vertex algebras. Relaxed highest weight modules were originally named in [FST98], where such modules over the affine Kac–Moody algebra $\widehat{\mathfrak{sl}_2}$ were used to study the $\mathcal{N} = 2$ superconformal vertex algebra. In [Ada16] relaxed highest weight modules over $\widehat{\mathfrak{sl}_3}$ were used to analyse the $\mathcal{N} = 4$ superconformal vertex algebra. Simple relaxed highest weight modules over the simple affine vertex algebra $\mathcal{L}_\kappa(\mathfrak{sl}_2)$ of admissible level κ were classified in [AM95]. For recent developments in classifications of relaxed highest weight modules we refer to [AFR17,ACR18,KR19a, KR19b].

The goal of the current paper is to give an explicit free field construction of new families of positive energy representations of the universal affine vertex algebras and simple affine vertex algebras using two main technical tools: the *twisting functor* on the category of modules over the (untwisted) affine Kac–Moody algebra $\widehat{\mathfrak{g}}_\kappa$ of level κ assigned to an arbitrary positive root α of a semisimple Lie algebra $\mathfrak{g}$, and the *Wakimoto functor* from a certain subcategory of $\mathfrak{g}$-modules to the category of positive energy $\widehat{\mathfrak{g}}_\kappa$-modules. In particular, the Wakimoto functor applied to Verma $\mathfrak{g}$-modules gives Verma $\widehat{\mathfrak{g}}_\kappa$-modules generically.

In the finite-dimensional case the twisting functor was defined in [FK19b] following the work of Deodhar [Deo80]. If α is a simple root of a semisimple Lie algebra $\mathfrak{g}$, then the twisting functor T_α is related to the Arkhipov's twisting functor [Ark04] on the category $\mathcal{O}(\mathfrak{g})$. By applying the twisting functor T_α for a positive root α to the generalized Verma $\mathfrak{g}$-module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ induced from the simple finite-dimensional $\mathfrak{p}$-module with highest weight

λ one obtains the α-Gelfand–Tsetlin $\mathfrak{g}$-module $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda,\alpha)$ with finite Γ_α-multiplicities, where Γ_α is the commutative subalgebra of $U(\mathfrak{g})$ generated by the Cartan subalgebra $\mathfrak{h}$ and by the center of $U(\mathfrak{s}_\alpha)$, where $\mathfrak{s}_\alpha$ is the Lie subalgebra of $\mathfrak{g}$ given by the $\mathfrak{sl}_2$-triple for the root α. The $\mathfrak{g}$-modules $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\alpha)$ are cyclic weight modules with respect to the Cartan subalgebra $\mathfrak{h}$ with infinite-dimensional weight subspaces if α is not a simple root. On the other hand, if α is a simple root, then these modules are twisted Verma $\mathfrak{g}$-modules up to conjugation of the action of $\mathfrak{g}$, see [AL03,KS15,Mus19].

Modifying the construction given in [FK19a,FK19b] in the finite-dimensional setting, we define the twisting functor T_α assigned to a real root $\alpha \in \widehat{\Delta}^{\mathrm{re}}$ of $\widehat{\mathfrak{g}}_\kappa$ and obtain an endofunctor on the category of $\widehat{\mathfrak{g}}_\kappa$-modules. The important properties of the twisting functor T_α are summarized in the theorem below (cf. Theorems 4.9, 4.13, 4.14, 4.16).

Theorem A. (i) *For $\alpha \in \widehat{\Delta}^{\mathrm{re}}$, the twisting functor T_α preserves the category $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$. Moreover, the twisting functor T_α preserves also the category $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$ provided $\alpha \in \Delta$.*

(ii) *Let $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$ and let M be a smooth weight $\widetilde{\mathfrak{g}}_\kappa$-module on which the central element c acts as the identity. Then the $\widetilde{\mathfrak{g}}_\kappa$-module $T_\alpha(M)$ is a Gelfand–Tsetlin module with finite Γ_α-multiplicities if and only if the first cohomology group $H^1(\mathfrak{s}^-_\alpha; M)$ is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces.*

(iii) *For $\alpha \in \Delta \subset \widehat{\Delta}^{\mathrm{re}}$ there exists a natural isomorphism between $T_\alpha \circ \mathbb{M}_{\kappa,\mathfrak{g}}$ and $\mathbb{M}_{\kappa,\mathfrak{g}} \circ T^{\mathfrak{g}}_\alpha$, where $T^{\mathfrak{g}}_\alpha : \mathcal{M}(\mathfrak{g}) \to \mathcal{M}(\mathfrak{g})$ is the twisting functor for $\mathfrak{g}$ assigned to α. In particular, we have*

$$T_\alpha(\mathbb{M}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda))) \simeq \mathbb{M}_{\kappa,\mathfrak{g}}(W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda,\alpha))$$

for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^{\mathfrak{u}}_+ = \{\alpha \in \Delta_+;\ \mathfrak{g}_\alpha \subset \mathfrak{u}\}$, where $\mathfrak{p}$ is a standard parabolic subalgebra of $\mathfrak{g}$ with the nilradical $\mathfrak{u}$, $\Lambda^+(\mathfrak{p})$ is the set of $\mathfrak{p}$-dominant integral weights. Moreover, the twisting functor T_α commutes with tensoring by Weyl modules.

In the previous theorem, the commutative subalgebra Γ_α of $U_c(\widetilde{\mathfrak{g}}_\kappa) = U(\widetilde{\mathfrak{g}}_\kappa)/(c-1)$ for $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$ is generated by the Cartan subalgebra $\mathfrak{h}$ of the extended affine Kac–Moody algebra $\widetilde{\mathfrak{g}}_\kappa$ and by the center $Z(\mathfrak{s}_\alpha)$ of $U(\mathfrak{s}_\alpha)$, where $\mathfrak{s}_\alpha$ is the Lie subalgebra of $\widetilde{\mathfrak{g}}_\kappa$ given by the $\mathfrak{sl}_2$-triple for the root $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$. We denote $\mathfrak{s}^-_\alpha = \mathfrak{s}_\alpha \cap \widehat{\mathfrak{n}}_{\mathrm{st}}$, where $\widetilde{\mathfrak{g}}_\kappa = \widehat{\mathfrak{n}}_{\mathrm{st}} \oplus \mathfrak{h} \oplus \widehat{\mathfrak{n}}_{\mathrm{st}}$ is the standard triangular decomposition of $\widetilde{\mathfrak{g}}_\kappa$. Further, $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$ and $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$ stand for the categories of smooth $\widehat{\mathfrak{g}}_\kappa$-modules (Definition 2.1) and the category of positive energy $\widehat{\mathfrak{g}}_\kappa$-modules (Definition 2.2), respectively, on which the central element c acts as the identity.

In particular, by applying the twisting functor T_α to the relaxed Verma module $\mathbb{M}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda))$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^{\mathfrak{u}}_+$ we obtain a new class of positive energy $\widehat{\mathfrak{g}}_\kappa$-modules, *relaxed Verma modules* induced from the α-Gelfand–Tsetlin $\mathfrak{g}$-module $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda,\alpha)$.

The Feigin–Frenkel homomorphism between the universal affine vertex algebra $\mathcal{V}_\kappa(\mathfrak{g})$ and the tensor product of the Weyl vertex algebra $\mathcal{M}_{\overline{\mathfrak{n}}}$ with the Heisenberg vertex algebra $\mathcal{V}_{\kappa-\kappa_c}(\mathfrak{h})$ gives an explicit free field construction of Wakimoto modules, see [Fre05]. We use the Feigin–Frenkel homomorphism to get a free field realization of relaxed Verma modules in Theorems 5.5 and 5.12. The obtained $\widehat{\mathfrak{g}}_\kappa$-modules are *relaxed Wakimoto modules*, they are images of Verma modules $M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ and α-Gelfand–Tsetlin modules $W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\alpha)$ under the Wakimoto functor $\mathbb{W}_{\kappa,\mathfrak{g}}$. The most important properties of the Wakimoto functor are collected in the theorem below (cf. Theorem 5.14, Corollaries 5.15, 5.16).

Theorem B. *Let $\lambda \in \mathfrak{h}^*$ and let $\alpha \in \Delta_+ \subset \widehat{\Delta}_+^{\mathrm{re}}$ be a positive root.*

(i) *The functors $T_\alpha \circ \mathbb{W}_{\kappa,\mathfrak{g}}$ and $\mathbb{W}_{\kappa,\mathfrak{g}} \circ T_\alpha^{\mathfrak{g}}$ are naturally isomorphic. In particular, we have*

$$T_\alpha(\mathbb{W}_{\kappa,\mathfrak{g}}(M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda))) \simeq \mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha)).$$

(ii) *If κ is a non-critical level, then*

$$\mathbb{M}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha)) \simeq \mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha)),$$

provided the Verma module $\mathbb{M}_{\kappa,\mathfrak{g}}(M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda))$ is a simple $\widehat{\mathfrak{g}}_\kappa$-module.

(iii) *For the critical level κ_c, we have*

$$\mathbb{M}_{\kappa_c,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha)) \simeq \mathbb{W}_{\kappa_c,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha))$$

if λ satisfies $\langle \lambda+\rho, \gamma^\vee\rangle \notin -\mathbb{N}$ for all $\gamma \in \Delta_+$, i.e. λ is a dominant weight.

We see that generically the relaxed Verma module $\mathbb{M}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha))$ and the relaxed Wakimoto module $\mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha))$ are isomorphic. The generic condition is given by the simplicity of the Verma $\widehat{\mathfrak{g}}_\kappa$-module with highest weight λ and the non-criticality of the level κ. On the other hand, if the level κ is critical and λ is dominant, then these modules are always isomorphic, which is an analogue of the corresponding result of Frenkel [Fre07] for Wakimoto modules.

Further on, in Sect. 6 we describe families of positive energy representations of the simple affine vertex algebra $\mathcal{L}_\kappa(\mathfrak{g})$ of admissible level $\kappa = k\kappa_0$ associated to a simple Lie algebra $\mathfrak{g}$, where κ_0 is the normalized $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$ and $k \in \mathbb{C}$.

By a result of Zhu [Zhu96] there is a one-to-one correspondence between the set of simple positive energy $\mathcal{L}_{k\kappa_0}(\mathfrak{g})$-modules and the set of simple $U(\mathfrak{g})/I_k$-modules, where I_k is a two-sided ideal of $U(\mathfrak{g})$. The algebra $U(\mathfrak{g})/I_k$ is the Zhu's algebra $A(\mathcal{L}_{k\kappa_0}(\mathfrak{g}))$ of the vertex algebra $\mathcal{L}_{k\kappa_0}(\mathfrak{g})$. We say that a $\mathfrak{g}$-module E is *admissible of level k* if E is an $A(\mathcal{L}_{k\kappa_0}(\mathfrak{g}))$-module.

Admissible highest weight $\mathfrak{g}$-modules of an admissible level $k \in \mathbb{Q}$ were classified by Arakawa in [Ara16]. For $\lambda \in \widehat{\mathfrak{h}}^*$, we denote by $\widehat{\Delta}(\lambda)$ its integral root system. Then Pr_k stands for the set of admissible weights $\lambda \in \widehat{\mathfrak{h}}^*$ such that $\widehat{\Delta}(\lambda) = y(\widehat{\Delta}(k\Lambda_0))$ for some element y of the extended affine Weyl group $\widetilde{W}$ of $\mathfrak{g}$, and $\overline{\mathrm{Pr}}_k$ for its canonical projection to $\mathfrak{h}^*$. Further, let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{g}$ and let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. Let us denote by $\Omega_k(\mathfrak{p})$ the set of weights $\lambda \in \overline{\mathrm{Pr}}_k \cap \Lambda^+(\mathfrak{p})$ such that $\langle \lambda+\rho, \alpha^\vee\rangle \notin \mathbb{N}$ for all $\alpha \in \Delta_+^{\mathfrak{u}}$. For these weights the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is a simple $\mathfrak{g}$-module, and we have the following result.

Theorem C. *Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. Furthermore, let $\lambda \in \Omega_k(\mathfrak{p})$ and $\alpha \in \Delta_+^{\mathfrak{u}}$. Then the $\mathfrak{g}$-module $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda,\alpha)$ is admissible of level k.*

The nilpotent cone of $\mathfrak{g}$ is an irreducible closed algebraic subvariety of $\mathfrak{g}$. It decomposes into the finite union of adjoint orbits. If $\mathcal{O}$ is such an adjoint orbit, then $\mathcal{O}^*$ denotes the corresponding coadjoint orbit of $\mathfrak{g}^*$ through the Cartan–Killing $\kappa_{\mathfrak{g}}$ on $\mathfrak{g}$. We say that a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$ is the orbit of a simple $\mathfrak{g}$-module E provided $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}E) = \overline{\mathcal{O}^*}$, where $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}E)$ denotes the zero locus in $\mathfrak{g}^*$ of the associated graded ideal $\mathrm{gr}\,\mathrm{Ann}_{U(\mathfrak{g})}E$ of $S(\mathfrak{g})$.

Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. Nilpotent orbits of admissible $\mathfrak{g}$-modules of level k were described in [Ara15a]. For a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$, we define

$$\overline{\mathrm{Pr}}_k^{\mathcal{O}} = \{\lambda \in \overline{\mathrm{Pr}}_k;\ \mathcal{V}(J_\lambda) = \overline{\mathcal{O}^*}\},$$

where J_λ is the annihilator of the simple $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$ and $\mathcal{V}(J_\lambda)$ is the associated variety. Then for the standard Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ we have that $\Omega_k(\mathfrak{b}) = \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{reg}}}$, where $\mathcal{O}_{\mathrm{reg}}$ is the regular nilpotent orbit (cf. Proposition 6.6). Moreover, in the case $\mathfrak{g} = \mathfrak{sl}_n$ we give a more convenient description of the set $\Omega_k(\mathfrak{p})$ for any standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ (cf. Theorem 6.9).

We denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}_0$ and $\mathbb{N}$ the set of complex numbers, real numbers, integers, non-negative integers and positive integers, respectively. All algebras and modules are considered over the field of complex numbers.

1. Preliminaries

Let $\mathfrak{g}$ be a complex semisimple finite-dimensional Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, by Δ_+ a positive root system in Δ and by $\Pi \subset \Delta_+$ the set of simple roots. For $\alpha \in \Delta_+$, let $h_\alpha \in \mathfrak{h}$ be the corresponding coroot and let e_α and f_α be basis of root subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, respectively, defined by the requirement $[e_\alpha, f_\alpha] = h_\alpha$. We also set

$$Q = \sum_{\alpha\in\Pi} \mathbb{Z}\alpha \qquad \text{and} \qquad Q_+ = \sum_{\alpha\in\Pi} \mathbb{N}_0\alpha$$

together with

$$P = \sum_{\alpha\in\Pi} \mathbb{Z}\omega_\alpha \qquad \text{and} \qquad P_+ = \sum_{\alpha\in\Pi} \mathbb{N}_0\omega_\alpha,$$

where $\omega_\alpha \in \mathfrak{h}^*$ for $\alpha \in \Pi$ is the fundamental weight determined by $\omega_\alpha(h_\gamma) = \delta_{\alpha,\gamma}$ for all $\gamma \in \Pi$. We call Q the root lattice and P the weight lattice. Further, we define the Weyl vector $\rho \in \mathfrak{h}^*$ by

$$\rho = \frac{1}{2} \sum_{\alpha\in\Delta_+} \alpha.$$

The standard Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is defined through $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ with the nilradical $\mathfrak{n}$ and the opposite nilradical $\overline{\mathfrak{n}}$ given by

$$\mathfrak{n} = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha \qquad \text{and} \qquad \overline{\mathfrak{n}} = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha}.$$

Besides, we have the corresponding triangular decomposition

$$\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$$

of the Lie algebra $\mathfrak{g}$.

Let $\kappa_\mathfrak{g}$ be the Cartan–Killing form on $\mathfrak{g}$ and $(\cdot\,,\cdot)_\mathfrak{g}$ the corresponding induced bilinear form on $\mathfrak{g}^*$. Whenever $\alpha \in \mathfrak{h}^*$ satisfies $(\alpha,\alpha)_\mathfrak{g} \neq 0$, we define $s_\alpha \in \mathrm{GL}(\mathfrak{h}^*)$ by

$$s_\alpha(\gamma) = \gamma - \frac{2(\alpha,\gamma)_\mathfrak{g}}{(\alpha,\alpha)_\mathfrak{g}}\,\alpha$$

for $\gamma \in \mathfrak{h}^*$. The subgroup W of $\mathrm{GL}(\mathfrak{h}^*)$ given by

$$W = \langle s_\alpha;\ \alpha \in \Pi \rangle$$

is called the Weyl group of $\mathfrak{g}$. Let us note that W is a finite Coxeter group.

Moreover, if $\mathfrak{g}$ is a simple Lie algebra, we denote by κ_0 the $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$ normalized in such a way that $(\theta, \theta) = 2$, where $\theta \in \Delta_+$ is the highest root of $\mathfrak{g}$ (by definition the highest weight of the adjoint representation $\mathfrak{g}$) and $(\cdot\,,\cdot)$ is the corresponding induced bilinear form on $\mathfrak{g}^*$. Further, we denote by θ_s the highest short root of $\mathfrak{g}$. Then we have $(\theta_s, \theta_s) = 2/r^\vee$, where $r^\vee$ is the lacing number of $\mathfrak{g}$, i.e. the maximal number of edges in the Dynkin diagram of $\mathfrak{g}$. We also define

$$P^\vee = \bigoplus_{\alpha\in\Pi} \mathbb{Z}\omega_\alpha^\vee \qquad \text{and} \qquad P_+^\vee = \bigoplus_{\alpha\in\Pi} \mathbb{N}_0\omega_\alpha^\vee,$$

where $\omega_\alpha^\vee \in \mathfrak{h}^*$ for $\alpha \in \Pi$ is the fundamental coweight defined by $(\omega_\alpha^\vee, \gamma) = \delta_{\alpha,\gamma}$ for all $\gamma \in \Pi$. We call $P^\vee$ the coweight lattice.

The category of all $\mathfrak{g}$-modules we denote by $\mathcal{M}(\mathfrak{g})$. We say that a $\mathfrak{g}$-module M is a *generalized weight* (with respect to $\mathfrak{h}$) $\mathfrak{g}$-module, if the action of $\mathfrak{h}$ on M is locally finite. If the action of $\mathfrak{h}$ is semisimple on M, then M is called a *weight* (with respect to $\mathfrak{h}$) $\mathfrak{g}$-module. In particular, any simple generalized weight $\mathfrak{g}$-module is a weight $\mathfrak{g}$-module. Further, for a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ we denote by $\mathcal{I}(\mathfrak{g}, \mathfrak{a})$ and $\mathcal{I}_f(\mathfrak{g}, \mathfrak{a})$ the full subcategories of $\mathcal{M}(\mathfrak{g})$ consisting of locally $\mathfrak{a}$-finite weight $\mathfrak{g}$-modules and finitely generated locally $\mathfrak{a}$-finite weight $\mathfrak{g}$-modules, respectively.

For a commutative algebra Γ we denote by $\mathrm{Hom}(\Gamma, \mathbb{C})$ the set of all characters of Γ, i.e. algebra homomorphisms from Γ to $\mathbb{C}$. Let M be a Γ-module. For each character $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ we set

$$M_\chi = \{v \in M;\ (\exists k \in \mathbb{N})\,(\forall a \in \Gamma)\,(a - \chi(a))^k v = 0\}. \tag{1.1}$$

When $M_\chi \neq \{0\}$, we say that χ is a Γ-weight of M, the vector space M_χ is called the Γ-weight subspace of M with weight χ and the elements of M_χ are Γ-weight vectors with weight χ. Moreover, if a Γ-module M satisfies

$$M = \bigoplus_{\chi\in\mathrm{Hom}(\Gamma,\mathbb{C})} M_\chi, \tag{1.2}$$

then we call M a Γ-*weight module*. The dimension of the Γ-weight subspace M_χ will be called the Γ-*multiplicity* of χ in M.

Let $\mathfrak{g}$ be a semisimple Lie algebra and let Γ be a commutative subalgebra of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Then we denote by $\mathcal{H}(\mathfrak{g}, \Gamma)$ the full subcategory of $\mathcal{M}(\mathfrak{g})$ consisting of Γ-weight $\mathfrak{g}$-modules. Let us note that $\mathcal{H}(\mathfrak{g}, \Gamma)$ is closed with respect to the operations of taking submodules and quotients. Besides, if Γ contains the Cartan subalgebra $\mathfrak{h}$, a Γ-weight $\mathfrak{g}$-module M is called a Γ-*Gelfand–Tsetlin* $\mathfrak{g}$-*module*.

2. Affine Kac–Moody Algebras and Weyl Algebras

In this section we define smooth and induced modules for affine Kac–Moody algebras and introduce a formalism for infinite-dimensional Weyl algebras.

2.1. Affine Kac–Moody algebras. Let $\mathfrak{g}$ be a semisimple (reductive) finite-dimensional Lie algebra and κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$. The affine Kac–Moody algebra $\widehat{\mathfrak{g}}_\kappa$ associated to $\mathfrak{g}$ of level κ is the 1-dimensional central extension $\widehat{\mathfrak{g}}_\kappa = \mathfrak{g}((t)) \oplus \mathbb{C}c$ of the formal loop algebra $\mathfrak{g}((t)) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t))$, with the commutation relations

$$[a \otimes f(t), b \otimes g(t)] = [a, b] \otimes f(t)g(t) - \kappa(a, b)\mathrm{Res}_{t=0}(f(t)dg(t))c, \qquad (2.1)$$

where c is the central element of $\widehat{\mathfrak{g}}_\kappa$, $a, b \in \mathfrak{g}$ and $f(t), g(t) \in \mathbb{C}((t))$. Let us note that Lie algebras $\widehat{\mathfrak{g}}_\kappa$ and $\widehat{\mathfrak{g}}_{\kappa'}$ for $\mathfrak{g}$-invariant symmetric bilinear forms κ and κ' on $\mathfrak{g}$ are isomorphic if $\kappa' = k\kappa$ for some $k \in \mathbb{C}^\times$. By introducing the notation $a_n = a \otimes t^n$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$, the commutation relations (2.1) can be simplified into the form

$$[a_m, b_n] = [a, b]_{m+n} + m\kappa(a, b)\delta_{m,-n}c \qquad (2.2)$$

for $m, n \in \mathbb{Z}$ and $a, b \in \mathfrak{g}$.

As $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, we introduce a Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}_\kappa$ by

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}1 \oplus \mathbb{C}c.$$

While any two Borel subalgebras of $\mathfrak{g}$ are conjugate by an automorphism of $\mathfrak{g}$, any two Borel subalgebras of $\widehat{\mathfrak{g}}_\kappa$ may not be conjugate by an automorphism of $\widehat{\mathfrak{g}}_\kappa$, see [Fut97].

Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{g}$ with the nilradical $\mathfrak{u}$, the opposite nilradical $\overline{\mathfrak{u}}$ and the Levi subalgebra $\mathfrak{l}$. Then the *standard parabolic subalgebra* $\widehat{\mathfrak{p}}_{\mathrm{st}}$ of $\widehat{\mathfrak{g}}_\kappa$ associated to $\mathfrak{p}$ is given through

$$\widehat{\mathfrak{p}}_{\mathrm{st}} = \widehat{\mathfrak{l}}_{\mathrm{st}} \oplus \widehat{\mathfrak{u}}_{\mathrm{st}},$$

where the Levi subalgebra $\widehat{\mathfrak{l}}_{\mathrm{st}}$ is defined by

$$\widehat{\mathfrak{l}}_{\mathrm{st}} = \mathfrak{l} \otimes_{\mathbb{C}} \mathbb{C}1 \oplus \mathbb{C}c$$

and the nilradical $\widehat{\mathfrak{u}}_{\mathrm{st}}$ and the opposite nilradical $\widehat{\overline{\mathfrak{u}}}_{\mathrm{st}}$ by

$$\widehat{\mathfrak{u}}_{\mathrm{st}} = \mathfrak{u} \otimes_{\mathbb{C}} \mathbb{C}1 \oplus \mathfrak{g} \otimes_{\mathbb{C}} t\mathbb{C}[[t]] \quad \text{and} \quad \widehat{\overline{\mathfrak{u}}}_{\mathrm{st}} = \overline{\mathfrak{u}} \otimes_{\mathbb{C}} \mathbb{C}1 \oplus \mathfrak{g} \otimes_{\mathbb{C}} t^{-1}\mathbb{C}[t^{-1}].$$

Moreover, we have the corresponding triangular decomposition

$$\widehat{\mathfrak{g}}_\kappa = \widehat{\overline{\mathfrak{u}}}_{\mathrm{st}} \oplus \widehat{\mathfrak{l}}_{\mathrm{st}} \oplus \widehat{\mathfrak{u}}_{\mathrm{st}}$$

of the Lie algebra $\widehat{\mathfrak{g}}_\kappa$. If $\mathfrak{p} = \mathfrak{b}$ then $\widehat{\mathfrak{b}}_{\mathrm{st}}$ is the standard Borel subalgebra, and if $\mathfrak{p} = \mathfrak{g}$ then $\widehat{\mathfrak{g}}_{\mathrm{st}}$ is the maximal standard parabolic subalgebra.

On the other hand, the *natural parabolic subalgebra* $\widehat{\mathfrak{p}}_{\mathrm{nat}}$ of $\widehat{\mathfrak{g}}_\kappa$ associated to $\mathfrak{p}$ is given through

$$\widehat{\mathfrak{p}}_{\mathrm{nat}} = \widehat{\mathfrak{l}}_{\mathrm{nat}} \oplus \widehat{\mathfrak{u}}_{\mathrm{nat}},$$

where the Levi subalgebra $\widehat{\mathfrak{l}}_{\mathrm{nat}}$ is defined by

$$\widehat{\mathfrak{l}}_{\mathrm{nat}} = \mathfrak{l} \otimes_{\mathbb{C}} \mathbb{C}((t)) \oplus \mathbb{C}c$$

and the nilradical $\widehat{\mathfrak{u}}_{\mathrm{nat}}$ and the opposite nilradical $\widehat{\overline{\mathfrak{u}}}_{\mathrm{nat}}$ by

$$\widehat{\mathfrak{u}}_{\mathrm{nat}} = \mathfrak{u} \otimes_{\mathbb{C}} \mathbb{C}((t)) \quad \text{and} \quad \widehat{\overline{\mathfrak{u}}}_{\mathrm{nat}} = \overline{\mathfrak{u}} \otimes_{\mathbb{C}} \mathbb{C}((t)).$$

We have again the corresponding triangular decomposition

$$\widehat{\mathfrak{g}}_\kappa = \widehat{\overline{\mathfrak{u}}}_{\mathrm{nat}} \oplus \widehat{\mathfrak{l}}_{\mathrm{nat}} \oplus \widehat{\mathfrak{u}}_{\mathrm{nat}}$$

of the Lie algebra $\widehat{\mathfrak{g}}_\kappa$. If $\mathfrak{p} = \mathfrak{b}$ then $\widehat{\mathfrak{b}}_{\mathrm{nat}}$ is the natural Borel subalgebra, and if $\mathfrak{p} = \mathfrak{g}$ then we have $\widehat{\mathfrak{g}}_{\mathrm{nat}} = \widehat{\mathfrak{g}}_\kappa$.

For more general parabolic subalgebras of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}_\kappa$ see [FK18].

To describe the root structure of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}_\kappa$ we define the extended affine Kac–Moody algebra $\widetilde{\mathfrak{g}}_\kappa$ as

$$\widetilde{\mathfrak{g}}_\kappa = \mathfrak{g}((t)) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where we extend the Lie algebra structure of $\widehat{\mathfrak{g}}_\kappa$ on $\widetilde{\mathfrak{g}}_\kappa$ by the commutation relations

$$[d, a \otimes f(t)] = a \otimes t\partial_t f(t), \qquad [d, c] = 0 \tag{2.3}$$

for $a \in \mathfrak{g}$ and $f(t) \in \mathbb{C}((t))$. The corresponding Cartan subalgebra $\widetilde{\mathfrak{h}}$ of $\widetilde{\mathfrak{g}}_\kappa$ is defined by

$$\widetilde{\mathfrak{h}} = \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}1 \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

The dual space to $\widetilde{\mathfrak{h}}$ we identify with $\mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$, where $\Lambda_0|_{\mathfrak{h}\otimes_{\mathbb{C}}\mathbb{C}1} = 0$, $\Lambda_0(c) = 1$, $\Lambda_0(d) = 0$ and $\delta|_{\mathfrak{h}\otimes_{\mathbb{C}}\mathbb{C}1} = 0$, $\delta(c) = 0$, $\delta(d) = 1$. The set of roots of $\widehat{\mathfrak{g}}_\kappa$ is then naturally a subset of $\mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ given by

$$\widehat{\Delta} = \widehat{\Delta}^{\mathrm{re}} \cup \widehat{\Delta}^{\mathrm{im}}, \tag{2.4}$$

where

$$\widehat{\Delta}^{\mathrm{re}} = \{\alpha + n\delta;\ \alpha \in \Delta,\ n \in \mathbb{Z}\} \qquad \text{and} \qquad \widehat{\Delta}^{\mathrm{im}} = \{n\delta;\ n \in \mathbb{Z}\backslash\{0\}\} \tag{2.5}$$

are the sets of real and imaginary roots, respectively. Moreover, we have the following decomposition $\widehat{\Delta} = \widehat{\Delta}_+ \cup \widehat{\Delta}_-$ into positive and negative roots, where

$$\widehat{\Delta}_+ = \{\alpha + n\delta;\ \alpha \in \Delta_+,\ n \in \mathbb{N}_0\} \cup \{-\alpha + n\delta;\ \alpha \in \Delta_+,\ n \in \mathbb{N}\} \cup \{n\delta;\ n \in \mathbb{N}\}$$

and $\widehat{\Delta}_- = -\widehat{\Delta}_+$. We also set $\widehat{\Delta}^{\mathrm{re}}_\pm = \widehat{\Delta}_\pm \cap \widehat{\Delta}^{\mathrm{re}}$ and $\widehat{\Delta}^{\mathrm{im}}_\pm = \widehat{\Delta}_\pm \cap \widehat{\Delta}^{\mathrm{im}}$.

Let us note that the grading element $d \in \widetilde{\mathfrak{g}}_\kappa$ gives $\widetilde{\mathfrak{g}}_\kappa$ and also $\widehat{\mathfrak{g}}_\kappa$ the structure of $\mathbb{Z}$-graded topological Lie algebras with the gradation defined by $-d$, i.e. we have $\deg c = 0$, $\deg d = 0$ and $\deg a_n = -n$ for $a \in \mathfrak{g}$, $n \in \mathbb{Z}$.

2.2. Smooth modules over affine Kac–Moody algebras. We denote by $\mathcal{M}(\widehat{\mathfrak{g}}_\kappa)$ the category of $\widehat{\mathfrak{g}}_\kappa$-modules. However, since the objects of this category may be very complicated, we focus our attention to some nice full subcategories of $\mathcal{M}(\widehat{\mathfrak{g}}_\kappa)$.

Definition 2.1. Let M be a $\widehat{\mathfrak{g}}_\kappa$-module. We say that M is a *smooth* $\widehat{\mathfrak{g}}_\kappa$-module if for each vector $v \in M$ there exists a positive integer $N_v \in \mathbb{N}$ such that

$$(\mathfrak{g} \otimes_{\mathbb{C}} t^{N_v}\mathbb{C}[[t]])v = 0,$$

or in other words that the Lie subalgebra $\mathfrak{g} \otimes_{\mathbb{C}} t^{N_v}\mathbb{C}[[t]]$ annihilates v. The category of smooth $\widehat{\mathfrak{g}}_\kappa$-modules on which the central element c acts as the identity we will denote by $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$.

Let us recall that by a graded $\widehat{\mathfrak{g}}_\kappa$-module M we mean a $\mathbb{C}$-graded vector space M having the structure of a $\widehat{\mathfrak{g}}_\kappa$-module compatible with the gradation of $\widehat{\mathfrak{g}}_\kappa$. Let us note that by shifting a given gradation on M by a complex number we obtain a new gradation on M.

Definition 2.2. Let M be a graded $\widehat{\mathfrak{g}}_\kappa$-module. We say that M is a *positive energy* $\widehat{\mathfrak{g}}_\kappa$-module if $M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$ and $M_\lambda \neq 0$, where $\lambda \in \mathbb{C}$. The category of positive energy $\widehat{\mathfrak{g}}_\kappa$-modules on which the central element c acts as the identity we will denote by $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$.

Let us note that if M is a positive energy $\widehat{\mathfrak{g}}_\kappa$-module, then it follows immediately that M is also a smooth $\widehat{\mathfrak{g}}_\kappa$-module. Therefore, the category $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$ is a full subcategory of $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$.

Let us recall that the category $\mathcal{M}(\widehat{\mathfrak{g}}_\kappa)$ coincides with the category of modules over the universal enveloping algebra $U(\widehat{\mathfrak{g}}_\kappa)$. There exists an analogous associative algebra for the category $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$ which is constructed as follows, see [Fre07]. Since the central element c acts as the identity on all $\widehat{\mathfrak{g}}_\kappa$-modules from the category $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$, the action of $U(\widehat{\mathfrak{g}}_\kappa)$ factors through the quotient algebra

$$U_c(\widehat{\mathfrak{g}}_\kappa) = U(\widehat{\mathfrak{g}}_\kappa)/(c-1).$$

Further, let us introduce a linear topology on $U_c(\widehat{\mathfrak{g}}_\kappa)$ in which the basis of open neighbourhoods of 0 are the left ideals I_N defined by

$$I_N = U_c(\widehat{\mathfrak{g}}_\kappa)(\mathfrak{g} \otimes_{\mathbb{C}} t^N \mathbb{C}[[t]])$$

for $N \in \mathbb{N}_0$. Let $\widetilde{U}_c(\widehat{\mathfrak{g}}_\kappa)$ be the completion of $U_c(\widehat{\mathfrak{g}}_\kappa)$ with respect to this linear topology, i.e. we get

$$\widetilde{U}_c(\widehat{\mathfrak{g}}_\kappa) = \varprojlim U_c(\widehat{\mathfrak{g}}_\kappa)/I_N.$$

Then the structure of an associative algebra on $U_c(\widehat{\mathfrak{g}}_\kappa)$ extends to the structure of an associative algebra on $\widetilde{U}_c(\widehat{\mathfrak{g}}_\kappa)$ by continuity. Hence, we obtain that $\widetilde{U}_c(\widehat{\mathfrak{g}}_\kappa)$ is a complete topological associative algebra, which we will call the *completed universal enveloping algebra* of $\widehat{\mathfrak{g}}_\kappa$. Moreover, the category $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$ coincides with the category of discrete modules over the associative algebra $U_c(\widehat{\mathfrak{g}}_\kappa)$ on which the action of $\widetilde{U}_c(\widehat{\mathfrak{g}}_\kappa)$ is pointwise continuous.

Now, we construct a class of $\widehat{\mathfrak{g}}_\kappa$-modules, the so called induced modules, which belong to the category $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$. Let $\widehat{\mathfrak{p}}_{\mathrm{st}} = \widehat{\mathfrak{l}}_{\mathrm{st}} \oplus \widehat{\mathfrak{u}}_{\mathrm{st}}$ be the standard parabolic subalgebra of $\widehat{\mathfrak{g}}_\kappa$ associated to a standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and let E be an $\mathfrak{l}$-module. Then the induced $\widehat{\mathfrak{g}}_\kappa$-module

$$\mathbb{M}_{\kappa,\mathfrak{p}}(E) = U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\widehat{\mathfrak{p}}_{\mathrm{st}})} E,$$

where E is considered as the $\widehat{\mathfrak{p}}_{\mathrm{st}}$-module on which $\widehat{\mathfrak{u}}_{\mathrm{st}}$ acts trivially and c acts as the identity, has a unique maximal $\widehat{\mathfrak{g}}_\kappa$-submodule $\mathbb{K}_{\kappa,\mathfrak{p}}(E)$ having zero intersection with the $\mathfrak{l}$-submodule E of $\mathbb{M}_{\kappa,\mathfrak{p}}(E)$. Therefore, we may set

$$\mathbb{L}_{\kappa,\mathfrak{p}}(E) = \mathbb{M}_{\kappa,\mathfrak{p}}(E)/\mathbb{K}_{\kappa,\mathfrak{p}}(E)$$

for an $\mathfrak{l}$-module E. Moreover, it is easy to see that if E is a simple $\mathfrak{l}$-module, then $\mathbb{L}_{\kappa,\mathfrak{p}}(E)$ is also a simple $\widehat{\mathfrak{g}}_\kappa$-module. The $\widehat{\mathfrak{g}}_\kappa$-module $\mathbb{M}_{\kappa,\mathfrak{p}}(E)$ is called the *generalized Verma module* induced from E for the standard parabolic subalgebra $\widehat{\mathfrak{p}}_{\mathrm{st}}$.

Further, if we set $F = U(\overline{\mathfrak{u}})E \subset \mathbb{M}_{\kappa,\mathfrak{p}}(E)$, then F is a $\widehat{\mathfrak{g}}_{\mathrm{st}}$-module such that $\mathfrak{g} \otimes_{\mathbb{C}} t\mathbb{C}[[t]]$ acts trivially and c acts as the identity. The induced $\widehat{\mathfrak{g}}_\kappa$-module $\mathbb{M}_{\kappa,\mathfrak{g}}(F)$ is isomorphic to $\mathbb{M}_{\kappa,\mathfrak{p}}(E)$ and $\mathbb{L}_{\kappa,\mathfrak{g}}(F) \simeq \mathbb{L}_{\kappa,\mathfrak{p}}(E)$ if F is a simple $\mathfrak{g}$-module. We will always consider induced $\widehat{\mathfrak{g}}_\kappa$-modules as generalized Verma modules for the maximal standard parabolic subalgebra $\widehat{\mathfrak{g}}_{\mathrm{st}}$.

Therefore, we have the induction functor

$$\mathbb{M}_{\kappa,\mathfrak{g}} \colon \mathcal{M}(\mathfrak{g}) \to \mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa) \tag{2.6}$$

and the functor

$$\mathbb{L}_{\kappa,\mathfrak{g}} \colon \mathcal{M}(\mathfrak{g}) \to \mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa). \tag{2.7}$$

Let us recall that if E is a simple finite-dimensional $\mathfrak{g}$-module, then $\mathbb{M}_{\kappa,\mathfrak{g}}(E)$ is usually called the *Weyl module*.

In the same way as $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$ we may define the category $\mathcal{E}(\widetilde{\mathfrak{g}}_\kappa)$. Moreover, as $\widehat{\mathfrak{g}}_\kappa$ is a Lie subalgebra of $\widetilde{\mathfrak{g}}_\kappa$, we have also a natural forgetful functor $\mathcal{E}(\widetilde{\mathfrak{g}}_\kappa) \to \mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$. On the other hand, for a non-critical level κ we can view on $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$ as a full subcategory of $\mathcal{E}(\widetilde{\mathfrak{g}}_\kappa)$. Let M be a smooth $\widehat{\mathfrak{g}}_\kappa$-module on which the central element c acts as the identity. Since κ is a non-critical level, any smooth $\widehat{\mathfrak{g}}_\kappa$-module carries an action of the Virasoro algebra obtained by the Segal–Sugawara construction, and so in particular an action of L_0 (the nontrivial semisimple generator). Hence, the action of the grading element d of $\widetilde{\mathfrak{g}}_\kappa$ is then defined as the action of $-L_0$. However, if κ is not a non-critical level, then general smooth $\widehat{\mathfrak{g}}_\kappa$-modules do not necessarily carry an action of L_0.

In the rest of this section we described non-critical levels for a semisimple Lie algebra $\mathfrak{g}$. Let us consider a $\mathfrak{b}$-invariant symmetric bilinear form on $\mathfrak{b}$ defined by

$$\kappa_c^{\mathfrak{b}}(a, b) = -\mathrm{tr}_{\mathfrak{g}/\mathfrak{b}}(\mathrm{ad}(a)\mathrm{ad}(b)) \tag{2.8}$$

for $a, b \in \mathfrak{b}$.

Definition 2.3. Let κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$. We say that κ is *non-critical* if $\kappa - \kappa_c^{\mathfrak{b}}$ is non-degenerate on $\mathfrak{h}$, *partially critical* if $\kappa - \kappa_c^{\mathfrak{b}}$ is degenerate on $\mathfrak{h}$, and *critical* if $\kappa - \kappa_c^{\mathfrak{b}}$ is zero on $\mathfrak{h}$. The critical $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$ we will denote by κ_c.

Since $\mathfrak{g}$ is a semisimple Lie algebra, we have the direct sum decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$$

of $\mathfrak{g}$ into the direct sum of simple Lie algebras $\mathfrak{g}_i$ for $i = 1, 2, \ldots, r$ such that these direct summands are mutually orthogonal with respect to the Cartan–Killing form $\kappa_{\mathfrak{g}}$ on $\mathfrak{g}$. We denote by $\kappa_0^{\mathfrak{g}_i}$ the normalized $\mathfrak{g}_i$-invariant symmetric bilinear form on $\mathfrak{g}_i$, i.e. we have $\kappa_{\mathfrak{g}_i} = 2h_i^\vee \kappa_0^{\mathfrak{g}_i}$, where $h_i^\vee$ is the dual Coxeter number of $\mathfrak{g}_i$, for $i = 1, 2, \ldots, r$. We have the following criterion.

Lemma 2.4. *Let κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$. Then κ is partially critical if $\kappa_{|\mathfrak{g}_i} = -h_i^\vee \kappa_0^{\mathfrak{g}_i}$ for some $i = 1, 2, \ldots, r$, and κ is critical if $\kappa_{|\mathfrak{g}_i} = -h_i^\vee \kappa_0^{\mathfrak{g}_i}$ for all $i = 1, 2, \ldots, r$.*

Proof. For $a, b \in \mathfrak{h}$, we have

$$\kappa_c^{\mathfrak{b}}(a,b) = -\mathrm{tr}_{\mathfrak{g}/\mathfrak{b}}(\mathrm{ad}(a)\mathrm{ad}(b)) = -\sum_{\alpha\in\Delta_+}\alpha(a)\alpha(b).$$

Further, for each $i = 1, 2, \ldots, r$ there exists $k_i \in \mathbb{C}$ such that $\kappa_{|\mathfrak{g}_i} = k_i\kappa_0^{\mathfrak{g}_i}$, since $\mathfrak{g}_i$ is a simple Lie algebra. Hence, we may write

$$\begin{aligned}(\kappa - \kappa_c^{\mathfrak{b}})(a,b) = (k_i\kappa_0^{\mathfrak{g}_i} - \kappa_c^{\mathfrak{b}})(a,b) &= \frac{k_i}{2h_i^\vee}\kappa_{\mathfrak{g}_i}(a,b) - \kappa_c^{\mathfrak{b}}(a,b)\\ &= \frac{k_i + h_i^\vee}{h_i^\vee}\sum_{\alpha\in\Delta_+^{\mathfrak{g}_i}}\alpha(a)\alpha(b)\end{aligned}$$

for $a, b \in \mathfrak{h} \cap \mathfrak{g}_i$, where $\Delta_+^{\mathfrak{g}_i} \subset \Delta_+$ is the set of positive roots of $\mathfrak{g}_i$. The required statement then follows immediately. □

2.3. Weyl algebras. Let us consider the commutative algebra $\mathcal{K} = \mathbb{C}((t))$ with the subalgebra $\mathcal{O} = \mathbb{C}[[t]]$. Further, let $\Omega_{\mathcal{K}} = \mathbb{C}((t))dt$ and $\Omega_{\mathcal{O}} = \mathbb{C}[[t]]dt$ be the corresponding modules of Kähler differentials. Then for a finite-dimensional complex vector space V we introduce the infinite-dimensional vector spaces $\mathcal{K}(V) = V \otimes_{\mathbb{C}} \mathcal{K}$ and $\Omega_{\mathcal{K}}(V) = V \otimes_{\mathbb{C}} \Omega_{\mathcal{K}}$. Using the pairing $(\cdot,\cdot)\colon \Omega_{\mathcal{K}}(V^*) \otimes \mathcal{K}(V) \to \mathbb{C}$ defined by

$$(\alpha \otimes f(t)dt, v \otimes g(t)) = \alpha(v)\mathrm{Res}_{t=0}(g(t)f(t)dt), \tag{2.9}$$

where $\alpha \in V^*$, $v \in V$ and $f(t), g(t) \in \mathcal{K}$, we identify the restricted dual space to $\mathcal{K}(V)$ with the vector space $\Omega_{\mathcal{K}}(V^*)$, and vice versa. Moreover, the pairing (2.9) gives us a skew-symmetric non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on $\Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V)$ defined by

$$\langle\alpha \otimes f(t)dt, v \otimes g(t)\rangle = -\langle v \otimes g(t), \alpha \otimes f(t)dt\rangle = (\alpha \otimes f(t)dt, v \otimes g(t))$$

for $\alpha \in V^*$, $v \in V$, $f(t), g(t) \in \mathcal{K}$, and

$$\langle v \otimes f(t), w \otimes g(t)\rangle = \langle\alpha \otimes f(t)dt, \beta \otimes g(t)dt\rangle = 0$$

for $\alpha, \beta \in V^*$, $v, w \in V$, $f(t), g(t) \in \mathcal{K}$. The Weyl algebras $\mathcal{A}_{\mathcal{K}(V)}$ and $\mathcal{A}_{\Omega_{\mathcal{K}}(V^*)}$ are given by

$$\begin{aligned}\mathcal{A}_{\mathcal{K}(V)} &= T(\Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V))/I_{\mathcal{K}(V)},\\ \mathcal{A}_{\Omega_{\mathcal{K}}(V^*)} &= T(\Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V))/I_{\Omega_{\mathcal{K}}(V^*)},\end{aligned}$$

where $I_{\mathcal{K}(V)}$ and $I_{\Omega_{\mathcal{K}}(V^*)}$ denote the two-sided ideals of the tensor algebra $T(\Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V))$ generated by $a \otimes b - b \otimes a + \langle a, b\rangle \cdot 1$ for $a, b \in \Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V)$ and by $a \otimes b - b \otimes a - \langle a, b\rangle \cdot 1$ for $a, b \in \Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V)$, respectively. The algebras of polynomials on $\mathcal{K}(V)$ and $\Omega_{\mathcal{K}}(V^*)$ are defined by $\mathrm{Pol}\,\mathcal{K}(V) = S(\Omega_{\mathcal{K}}(V^*))$ and $\mathrm{Pol}\,\Omega_{\mathcal{K}}(V^*) = S(\mathcal{K}(V))$.

Let $\mathcal{L}$ and $\mathcal{L}^{\mathrm{c}}$ be complementary Lagrangian (maximal isotropic) subspaces of $\Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V)$, i.e. we have $\Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V) = \mathcal{L} \oplus \mathcal{L}^{\mathrm{c}}$. Then the symmetric algebra $S(\mathcal{L})$ is a subalgebra of $\mathcal{A}_{\mathcal{K}(V)}$ since the elements of $\mathcal{L}$ commute in $\mathcal{A}_{\mathcal{K}(V)}$. Moreover, it

is a maximal commutative subalgebra of $\mathcal{A}_{\mathcal{K}(V)}$. Further, let us consider the induced $\mathcal{A}_{\mathcal{K}(V)}$-module

$$\operatorname{Ind}_{S(\mathcal{L})}^{\mathcal{A}_{\mathcal{K}(V)}}\mathbb{C} \simeq S(\mathcal{L}^{\mathrm{c}}),$$

where $\mathbb{C}$ is the trivial 1-dimensional $S(\mathcal{L})$-module. It follows immediately that the induced $\mathcal{A}_{\mathcal{K}(V)}$-module has the natural structure of a commutative algebra. We denote by $M_{\mathcal{L}}$ the commutative algebra which is the completion of $S(\mathcal{L}^{\mathrm{c}})$ with respect to the linear topology on $S(\mathcal{L}^{\mathrm{c}})$ in which the basis of open neighbourhoods of 0 are the subspaces $\mathcal{I}_{n,m}$ for $n, m \in \mathbb{Z}$, where $\mathcal{I}_{n,m}$ is the ideal of $S(\mathcal{L}^{\mathrm{c}})$ generated by $\mathcal{L}^{\mathrm{c}} \cap (V^* \otimes_{\mathbb{C}} t^n \Omega_{\mathcal{O}} \oplus V \otimes_{\mathbb{C}} t^m \mathcal{O})$. In addition, we may extend the action of the Weyl algebra $\mathcal{A}_{\mathcal{K}(V)}$ to $M_{\mathcal{L}}$.

Our next step is to pass to a completion of the Weyl algebra $\mathcal{A}_{\mathcal{K}(V)}$, because $\mathcal{A}_{\mathcal{K}(V)}$ is not sufficiently large for our considerations. Let us denote by $\operatorname{Fun}\mathcal{K}(V)$ the completion of the commutative algebra $\operatorname{Pol}\mathcal{K}(V)$ with respect to the linear topology on $\operatorname{Pol}\mathcal{K}(V)$ in which the basis of open neighbourhoods of 0 are the subspaces $\mathcal{J}_n$ for $n \in \mathbb{Z}$, where $\mathcal{J}_n$ is the ideal of $\operatorname{Pol}\mathcal{K}(V)$ generated by $V^* \otimes_{\mathbb{C}} t^n \Omega_{\mathcal{O}}$. Consequently, we have $\operatorname{Fun}\mathcal{K}(V) = M_{\mathcal{K}(V)}$. Then a vector field on $\mathcal{K}(V)$ is by definition a continuous linear endomorphism ξ of $\operatorname{Fun}\mathcal{K}(V)$ which satisfies the Leibniz rule

$$\xi(fg) = \xi(f)g + f\xi(g)$$

for all $f, g \in \operatorname{Fun}\mathcal{K}(V)$. In other words, a vector field on $\mathcal{K}(V)$ is a linear endomorphism ξ of $\operatorname{Fun}\mathcal{K}(V)$ such that for any $m \in \mathbb{Z}$ there exists $n \in \mathbb{Z}$, $m \leq n$ and a derivation

$$\xi_{n,m}\colon \operatorname{Pol}\mathcal{K}(V)/\mathcal{J}_n \to \operatorname{Pol}\mathcal{K}(V)/\mathcal{J}_m$$

satisfying

$$s_m(\xi(f)) = \xi_{n,m}(s_n(f))$$

for all $f \in \operatorname{Fun}\mathcal{K}(V)$, where

$$s_n\colon \operatorname{Fun}\mathcal{K}(V) \to \operatorname{Pol}\mathcal{K}(V)/\mathcal{J}_n$$

is the canonical homomorphism of algebras. The vector space of all vector fields is naturally a topological Lie algebra, which we denote by $\operatorname{Vect}\mathcal{K}(V)$. There is a non-split short exact sequence

$$0 \to \operatorname{Fun}\mathcal{K}(V) \to \widetilde{\mathcal{A}}_{\mathcal{K}(V),\leq 1} \to \operatorname{Vect}\mathcal{K}(V) \to 0$$

of topological Lie algebras, see [FBZ04] for more details. This extension of the topological Lie algebra $\operatorname{Vect}\mathcal{K}(V)$ by its module $\operatorname{Fun}\mathcal{K}(V)$ is however different from the standard split extension

$$0 \to \operatorname{Fun}\mathcal{K}(V) \to \mathcal{A}^{\sharp}_{\mathcal{K}(V),\leq 1} \to \operatorname{Vect}\mathcal{K}(V) \to 0$$

of $\operatorname{Vect}\mathcal{K}(V)$ by $\operatorname{Fun}\mathcal{K}(V)$. The completed Weyl algebra $\widetilde{\mathcal{A}}_{\mathcal{K}(V)}$ is then defined as the associative algebra over $\operatorname{Fun}\mathcal{K}(V)$ generated by the images of the homomorphisms $i\colon \operatorname{Fun}\mathcal{K}(V) \to \widetilde{\mathcal{A}}_{\mathcal{K}(V)}$ (as associative algebras) and $j\colon \widetilde{\mathcal{A}}_{\mathcal{K}(V),\leq 1} \to \widetilde{\mathcal{A}}_{\mathcal{K}(V)}$ (as Lie algebras), with the relation

$$[j(P), i(f)] = i(a(P)(f))$$

for $f \in \operatorname{Fun}\mathcal{K}(V)$ and $P \in \widetilde{\mathcal{A}}_{\mathcal{K}(V),\leq 1}$, where $a\colon \widetilde{\mathcal{A}}_{\mathcal{K}(V),\leq 1} \to \operatorname{Vect}\mathcal{K}(V)$ is the homomorphism of Lie algebras originated from the corresponding short exact sequence.

Now, we define a class of $\mathcal{A}_{\mathcal{K}(V)}$-modules called induced modules. Let us consider the vector subspaces $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_0$ of $\Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V)$ given by

$$\begin{aligned}
\mathcal{L}_+ &= V^* \otimes_{\mathbb{C}} \mathbb{C}[[t]]dt \oplus V \otimes_{\mathbb{C}} t\mathbb{C}[[t]],\\
\mathcal{L}_- &= V^* \otimes_{\mathbb{C}} t^{-2}\mathbb{C}[t^{-1}]dt \oplus V \otimes_{\mathbb{C}} t^{-1}\mathbb{C}[t^{-1}],\\
\mathcal{L}_0 &= V^* \otimes_{\mathbb{C}} \mathbb{C}t^{-1}dt \oplus V \otimes_{\mathbb{C}} \mathbb{C}1.
\end{aligned}$$

Then we have the direct sum decomposition

$$\Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V) = \mathcal{L}_- \oplus \mathcal{L}_0 \oplus \mathcal{L}_+$$

of $\Omega_{\mathcal{K}}(V^*) \oplus \mathcal{K}(V)$, which induces the triangular decomposition

$$\mathcal{A}_{\mathcal{K}(V)} \simeq \mathcal{A}_{\mathcal{K}(V),-} \otimes_{\mathbb{C}} \mathcal{A}_{\mathcal{K}(V),0} \otimes_{\mathbb{C}} \mathcal{A}_{\mathcal{K}(V),+}$$

of the Weyl algebra $\mathcal{A}_{\mathcal{K}(V)}$, where

$$\mathcal{A}_{\mathcal{K}(V),-} \simeq S(\mathcal{L}_-), \qquad \mathcal{A}_{\mathcal{K}(V),0} \simeq \mathcal{A}_V, \qquad \mathcal{A}_{\mathcal{K}(V),+} \simeq S(\mathcal{L}_+),$$

and $\mathcal{A}_V$ is the Weyl algebra of the finite-dimensional vector space V. Moreover, the Weyl algebra $\mathcal{A}_{\mathcal{K}(V)}$ is a $\mathbb{Z}$-graded algebra with the gradation determined by

$$\deg(v \otimes t^n) = -n, \qquad \deg 1 = 0, \qquad \deg(\alpha \otimes t^{-n-1}dt) = n$$

for $v \in V$, $\alpha \in V^*$ and $n \in \mathbb{Z}$.

Definition 2.5. Let M be an $\mathcal{A}_{\mathcal{K}(V)}$-module. Then we say that M is a *smooth* $\mathcal{A}_{\mathcal{K}(V)}$-module if for each vector $v \in M$ there exists a positive integer $N_v \in \mathbb{N}$ such that

$$(V^* \otimes_{\mathbb{C}} t^{N_v}\mathbb{C}[[t]]dt \oplus V \otimes_{\mathbb{C}} t^{N_v}\mathbb{C}[[t]])v = 0.$$

The category of smooth $\mathcal{A}_{\mathcal{K}(V)}$-modules we will denote by $\mathcal{E}(\mathcal{A}_{\mathcal{K}(V)})$.

Completely analogously as for $\widehat{\mathfrak{g}}_\kappa$-modules, we may introduce graded $\mathcal{A}_{\mathcal{K}(V)}$-modules and positive energy $\mathcal{A}_{\mathcal{K}(V)}$-modules. The category of positive energy $\mathcal{A}_{\mathcal{K}(V)}$-modules we will denote by $\mathcal{E}_+(\mathcal{A}_{\mathcal{K}(V)})$.

Let E be an $\mathcal{A}_V$-module. Then the induced $\mathcal{A}_{\mathcal{K}(V)}$-module

$$\mathbb{M}_{\mathcal{K}(V)}(E) = \mathcal{A}_{\mathcal{K}(V)} \otimes_{\mathcal{A}_{\mathcal{K}(V),0} \otimes_{\mathbb{C}} \mathcal{A}_{\mathcal{K}(V),+}} E,$$

where E is considered as the $\mathcal{A}_{\mathcal{K}(V),0} \otimes_{\mathbb{C}} \mathcal{A}_{\mathcal{K}(V),+}$-module on which the Weyl algebra $\mathcal{A}_{\mathcal{K}(V),0}$ acts via the canonical isomorphism $\mathcal{A}_{\mathcal{K}(V),0} \simeq \mathcal{A}_V$ and $\mathcal{A}_{\mathcal{K}(V),+}$ acts trivially, has a unique maximal $\mathcal{A}_{\mathcal{K}(V)}$-submodule $\mathbb{K}_{\mathcal{K}(V)}(E)$ having zero intersection with the $\mathcal{A}_V$-submodule E of $\mathbb{M}_{\mathcal{K}(V)}(E)$. Therefore, we may set

$$\mathbb{L}_{\mathcal{K}(V)}(E) = \mathbb{M}_{\mathcal{K}(V)}(E)/\mathbb{K}_{\mathcal{K}(V)}(E)$$

for an $\mathcal{A}_V$-module E. Moreover, it is easy to see that if E is a simple $\mathcal{A}_V$-module, then $\mathbb{L}_{\mathcal{K}(V)}(E)$ is also a simple $\mathcal{A}_{\mathcal{K}(V)}$-module.

Therefore, we have the induction functor

$$\mathbb{M}_{\mathcal{K}(V)}\colon \mathcal{M}(\mathcal{A}_V) \to \mathcal{E}_+(\mathcal{A}_{\mathcal{K}(V)}) \tag{2.10}$$

and the functor

$$\mathbb{L}_{\mathcal{K}(V)}\colon \mathcal{M}(\mathcal{A}_V) \to \mathcal{E}_+(\mathcal{A}_{\mathcal{K}(V)}). \tag{2.11}$$

Besides, it follows immediately that $\mathbb{M}_{\mathcal{K}(V)}(E)$ and $\mathbb{L}_{\mathcal{K}(V)}(E)$ are also $\widetilde{\mathcal{A}}_{\mathcal{K}(V)}$-modules.

Let us consider a Borel subalgebra $\mathfrak{b}$ of a semisimple Lie algebra $\mathfrak{g}$ with the nilradical $\mathfrak{n}$, the opposite nilradical $\overline{\mathfrak{n}}$ and the Cartan subalgebra $\mathfrak{h}$. Further, let $\{f_\alpha;\ \alpha \in \Delta_+\}$ be a root basis of the opposite nilradical $\overline{\mathfrak{n}}$. We denote by $\{x_\alpha;\ \alpha \in \Delta_+\}$ the linear coordinate functions on $\overline{\mathfrak{n}}$ with respect to the given basis of $\overline{\mathfrak{n}}$. Then the set $\{f_\alpha \otimes t^n;\ \alpha \in \Delta_+,\ n \in \mathbb{Z}\}$ forms a topological basis of $\mathcal{K}(\overline{\mathfrak{n}}) = \overline{\mathfrak{n}}_{\mathrm{nat}}$, and the set $\{x_\alpha \otimes t^{-n-1}dt;\ \alpha \in \Delta_+,\ n \in \mathbb{Z}\}$ forms the dual topological basis of $\Omega_{\mathcal{K}}(\overline{\mathfrak{n}}^*) \simeq (\overline{\mathfrak{n}}_{\mathrm{nat}})^*$ with respect to the pairing (2.9), i.e. we have

$$(x_\alpha \otimes t^{-n-1}dt,\, f_\beta \otimes t^m) = x_\alpha(f_\beta)\mathrm{Res}_{t=0}t^{m-n-1}dt = \delta_{\alpha,\beta}\delta_{n,m}$$

for $\alpha, \beta \in \Delta_+$ and $m, n \in \mathbb{Z}$. If we denote $x_{\alpha,n} = x_\alpha \otimes t^{-n-1}dt$ and $\partial_{x_{\alpha,n}} = f_\alpha \otimes t^n$ for $\alpha \in \Delta_+$ and $n \in \mathbb{Z}$, then the two-sided ideal $I_{\mathcal{K}(\overline{\mathfrak{n}})}$ is generated by elements

$$\Big(\sum_{n\in\mathbb{Z}} a_n x_{\alpha,n}\Big)\otimes\Big(\sum_{m\in\mathbb{Z}} b_m \partial_{x_{\beta,m}}\Big) - \Big(\sum_{m\in\mathbb{Z}} b_m \partial_{x_{\beta,m}}\Big)\otimes\Big(\sum_{n\in\mathbb{Z}} a_n x_{\alpha,n}\Big) + \delta_{\alpha,\beta}\Big(\sum_{n\in\mathbb{Z}} a_n b_n\Big)\cdot 1$$

and it coincides with the canonical commutation relations

$$[x_{\alpha,n}, \partial_{x_{\beta,m}}] = -\delta_{\alpha,\beta}\delta_{n,m}$$

for $\alpha, \beta \in \Delta_+$ and $m, n \in \mathbb{Z}$. Therefore, we obtain that the Weyl algebra $\mathcal{A}_{\mathcal{K}(\overline{\mathfrak{n}})}$ is topologically generated by the set $\{x_{\alpha,n}, \partial_{x_{\alpha,n}};\ \alpha \in \Delta_+,\ n \in \mathbb{Z}\}$ with the canonical commutation relations.

3. Representations of Vertex Algebras

3.1. Vertex algebras. In this section we recall some notions and basic facts on vertex algebra, for more details see [Bor86, Kac98, DLM98, FBZ04, Fre07].

Let R be an algebra over $\mathbb{C}$. Then an R-valued formal power series (or formal distribution) in the variables $z_1, \ldots, z_n$ is a series

$$a(z_1, \ldots, z_n) = \sum_{m_1,\ldots,m_n\in\mathbb{Z}} a_{m_1,\ldots,m_n} z_1^{m_1} \cdots z_n^{m_n},$$

where $a_{m_1,\ldots,m_n} \in R$. The complex vector space of all R-valued formal power series is denoted by $R[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$. For a formal power series $a(z) = \sum_{m\in\mathbb{Z}} a_m z^m$, the residue is defined by

$$\mathrm{Res}_{z=0}a(z) = a_{-1}.$$

A particulary important example of a $\mathbb{C}$-valued formal power series in two variables z, w is the formal delta function $\delta(z-w)$ given by

$$\delta(z-w) = \sum_{m\in\mathbb{Z}} z^m w^{-m-1}.$$

Let V be a complex vector space, so $\mathrm{End}\,V$ is an algebra over $\mathbb{C}$. We say that a formal power series $a(z) \in \mathrm{End}\,V[[z^{\pm 1}]]$ is a field, if $a(z)v \in V((z))$ for all $v \in V$. We shall write the field $a(z)$ as

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}. \tag{3.1}$$

The complex vector space of all fields on V in the variable z we will be denote by $\mathcal{F}(V)(z)$.

Definition 3.1. A vertex algebra consists of the following data:

(1) a complex vector space $\mathcal{V}$ (the space of states);
(2) a vector $|0\rangle \in \mathcal{V}$ (the vacuum vector);
(3) an endomorphism $T : \mathcal{V} \to \mathcal{V}$ (the translation operator);
(4) a linear mapping $Y(\cdot, z) : \mathcal{V} \to \mathrm{End}\mathcal{V}[[z^{\pm 1}]]$ sending

$$a \in \mathcal{V} \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in \mathcal{F}(\mathcal{V})(z)$$

(the state-field correspondence)

satisfying the subsequent axioms:

(1) $Y(|0\rangle, z) = \mathrm{id}_{\mathcal{V}}$, $Y(a, z)|0\rangle|_{z=0} = a$ (the vacuum axiom);
(2) $T|0\rangle = 0$, $[T, Y(a, z)] = \partial_z Y(a, z)$ (the translation axiom);
(3) for all $a, b \in \mathcal{V}$, there is a non-negative integer $N_{a,b} \in \mathbb{N}_0$ such that

$$(z - w)^{N_{a,b}} [Y(a, z), Y(b, w)] = 0$$

(the locality axiom).

A vertex algebra $\mathcal{V}$ is called $\mathbb{Z}$-graded if $\mathcal{V}$ is a $\mathbb{Z}$-graded vector space, $|0\rangle$ is a vector of degree 0, T is an endomorphism of degree 1, and for $a \in \mathcal{V}_m$ the field $Y(a, z)$ has conformal dimension m, i.e. we have

$$\deg a_{(n)} = -n + m - 1$$

for all $n \in \mathbb{Z}$.

Let us note that according to the translation axiom, the action of T on the space of states $\mathcal{V}$ is completely determined by Y, since we have $T(a) = a_{(-2)}|0\rangle$. Moreover, we have $a = a_{(-1)}|0\rangle$.

3.2. Positive energy representations. Let us consider a $\mathbb{Z}$-graded vertex algebra $\mathcal{V}$. Then a $\mathcal{V}$-module M is called graded if M is a $\mathbb{C}$-graded vector space and for $a \in \mathcal{V}_m$ the field $Y_M(a, z)$ has conformal dimension m, i.e. we have

$$\deg a^M_{(n)} = -n + m - 1$$

for all $n \in \mathbb{Z}$. Let us note that by shifting a given gradation on M by a complex number we obtain a new gradation on M.

Definition 3.2. Let $\mathcal{V}$ be a $\mathbb{Z}$-graded vertex algebra. We say that a graded $\mathcal{V}$-module M is a *positive energy* $\mathcal{V}$*-module* if $M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$ and $M_\lambda \neq 0$, where $\lambda \in \mathbb{C}$. Moreover, we denote by M_{top} the top degree component M_λ of M. The category of positive energy $\mathcal{V}$-modules we will denote by $\mathcal{E}_+(\mathcal{V})$.

In [Zhu96], Zhu introduced a functorial construction which assigns to a $\mathbb{Z}$-graded vertex algebra an associative algebra known as the Zhu's algebra. Let $\mathcal{V}$ be a $\mathbb{Z}$-graded vertex algebra. We define a bilinear mapping on $\mathcal{V}$ by

$$a * b = \text{Res}_{z=0}\left(Y(a,z)\frac{(1+z)^{\deg a}}{z} b \right) = \sum_{i=0}^{\deg a} \binom{\deg a}{i} a_{(i-1)}b \tag{3.2}$$

for homogeneous elements $a, b \in \mathcal{V}$ and extend linearly. The Zhu's algebra $A(\mathcal{V})$ is defined as

$$A(\mathcal{V}) = \mathcal{V}/O(\mathcal{V}), \tag{3.3}$$

where $O(\mathcal{V})$ is the vector subspace of $\mathcal{V}$ spanned by

$$\text{Res}_{z=0}\left(Y(a,z)\frac{(1+z)^{\deg a}}{z^2} b \right) = \sum_{i=0}^{\deg a} \binom{\deg a}{i} a_{(i-2)}b \tag{3.4}$$

for homogeneous elements $a, b \in \mathcal{V}$. We denote by π_{Zhu} the canonical projection from $\mathcal{V}$ to $A(\mathcal{V})$. The bilinear mapping (3.2) induces an associative multiplication on the quotient $A(\mathcal{V})$. Further, we define

$$o(a) = a_{(\deg a - 1)}$$

for a homogeneous element $a \in \mathcal{V}$. Then it easily follows that for a homogeneous element $a \in \mathcal{V}$ the operator $o_M(a) = a^M_{(\deg a - 1)}$ preserves the homogeneous components of any graded $\mathcal{V}$-module M.

As the following theorem proved in [Zhu96] shows, the Zhu's algebra $A(\mathcal{V})$ plays a prominent role in the representation theory of vertex algebras.

Theorem 3.3. *Let $\mathcal{V}$ be a $\mathbb{Z}$-graded vertex algebra and let M be a positive energy $\mathcal{V}$-module. Then the top degree component M_{top} is an $A(\mathcal{V})$-module, where the action of $\pi_{\text{Zhu}}(a) \in A(\mathcal{V})$ for $a \in \mathcal{V}$ is given by $o_M(a)$. In addition, the correspondence $M \mapsto M_{\text{top}}$ gives a bijection between the set of simple positive energy $\mathcal{V}$-modules and that of simple $A(\mathcal{V})$-modules.*

To a $\mathbb{Z}$-graded vertex algebra $\mathcal{V}$ we may associate a complete topological Lie algebra $U(\mathcal{V})$, first introduced by Borcherds [Bor86], by

$$U(\mathcal{V}) = (\mathcal{V} \otimes_{\mathbb{C}} \mathbb{C}((t)))/\text{im}\partial,$$

where

$$\partial = T \otimes \text{id} + \text{id} \otimes \partial_t.$$

If we denote by $a_{[n]}$ for $a \in \mathcal{V}$ and $n \in \mathbb{Z}$ the projection of $a \otimes t^n \in \mathcal{V} \otimes_{\mathbb{C}} \mathbb{C}((t))$ onto $U(\mathcal{V})$, then the Lie bracket on $U(\mathcal{V})$ is determined by

$$[a_{[m]}, b_{[n]}] = \sum_{k=0}^{\infty} \binom{m}{k} (a_{(k)}b)_{[m+n-k]}$$

for $a, b \in \mathcal{V}$ and $m, n \in \mathbb{Z}$.

Further, for a homogeneous element $a \in \mathcal{V}$, we set $\deg a_{[n]} = -n + \deg a - 1$. Then the degree assignment to elements of $U(\mathcal{V})$ gives us a triangular decomposition

$$U(\mathcal{V}) = U(\mathcal{V})_- \oplus U(\mathcal{V})_0 \oplus U(\mathcal{V})_+ \tag{3.5}$$

of the Lie algebra $U(\mathcal{V})$ together with a canonical surjective homomorphism

$$U(\mathcal{V})_0 \to A(\mathcal{V})$$

of Lie algebras defined by

$$a_{[\deg a - 1]} \mapsto \pi_{\mathrm{Zhu}}(a)$$

for a homogeneous element $a \in \mathcal{V}$.

Let us consider a $\mathcal{V}$-module M. Then it has also a natural structure of a $U(\mathcal{V})$-module since we have a canonical homomorphism

$$U(\mathcal{V}) \to \mathrm{End}\mathcal{V}$$

of Lie algebras defined through

$$a_{[n]} \mapsto a_{(n)}$$

for $a \in \mathcal{V}$ and $n \in \mathbb{Z}$. We denote by $\Omega_{\mathcal{V}}(M)$ the vector subspace of M consisting of lowest weight vectors, i.e. we have

$$\Omega_{\mathcal{V}}(M) = \{v \in M;\ U(\mathcal{V})_- v = 0\}. \tag{3.6}$$

It follows immediately using the triangular decomposition of $U(\mathcal{V})$ that $\Omega_{\mathcal{V}}(M)$ is a $U(\mathcal{V})_0$-module. Moreover, by [DLM98] we have that $\Omega_{\mathcal{V}}(M)$ is an $A(\mathcal{V})$-module, where the action of $\pi_{\mathrm{Zhu}}(a) \in A(\mathcal{V})$ for $a \in \mathcal{V}$ is given by $o_M(a)$. It is clear that

$$\Omega_{\mathcal{V}} \colon \mathcal{E}(\mathcal{V}) \to \mathcal{M}(A(\mathcal{V})) \tag{3.7}$$

is a functor. Let us note that if M is a positive energy $\mathcal{V}$-module, then we have $\Omega_{\mathcal{V}}(M) \supset M_{\mathrm{top}}$ and $\Omega_{\mathcal{V}}(M) = M_{\mathrm{top}}$ provided M is a simple $\mathcal{V}$-module.

Therefore, we may consider a functor $\Omega_{\mathcal{V}} \colon \mathcal{E}_+(\mathcal{V}) \to \mathcal{M}(A(\mathcal{V}))$. On the other hand, there exists also an induction functor

$$\mathbb{M}_{\mathcal{V}} \colon \mathcal{M}(A(\mathcal{V})) \to \mathcal{E}_+(\mathcal{V}) \tag{3.8}$$

which is a left adjoint functor to $\Omega_{\mathcal{V}}$ and has the following universal property. For a $\mathcal{V}$-module M and a morphism $\varphi \colon E \to \Omega_{\mathcal{V}}(M)$ of $A(\mathcal{V})$-modules, there is a unique morphism $\widetilde{\varphi} \colon \mathbb{M}_{\mathcal{V}}(E) \to M$ of $\mathcal{V}$-modules which extends φ, see [DLM98]. Moreover, for an $A(\mathcal{V})$-module E we have $\mathbb{M}_{\mathcal{V}}(E)_{\mathrm{top}} \simeq E$ as modules over $A(\mathcal{V})$. Besides, since the $\mathcal{V}$-module $\mathbb{M}_{\mathcal{V}}(E)$ has a unique maximal $\mathcal{V}$-submodule $\mathbb{K}_{\mathcal{V}}(E)$ having zero intersection with the $A(\mathcal{V})$-submodule E of $\mathbb{M}_{\mathcal{V}}(E)$, we may set

$$\mathbb{L}_{\mathcal{V}}(E) = \mathbb{M}_{\mathcal{V}}(E)/\mathbb{K}_{\mathcal{V}}(E) \tag{3.9}$$

for an $A(\mathcal{V})$-module E.

3.3. Affine vertex algebras. Let $\mathfrak{g}$ be a semisimple (reductive) finite-dimensional Lie algebra and κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$. The induced $\widehat{\mathfrak{g}}_\kappa$-module $\mathbb{M}_{\kappa,\mathfrak{g}}(\mathbb{C})$, where $\mathbb{C}$ is the trivial 1-dimensional $\mathfrak{g}$-module, is of a special importance in the theory of vertex algebras since it is equipped with the natural structure of an $\mathbb{N}_0$-graded vertex algebra, called the *universal affine vertex algebra*, see [Kac98], which we will denote by $\mathcal{V}_\kappa(\mathfrak{g})$. For an element $a \in \mathfrak{g}$, we denote by $a(z) \in \widehat{\mathfrak{g}}_\kappa[[z^{\pm 1}]]$ the formal distribution defined by

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}. \tag{3.10}$$

By using this formal power series we may rewrite the commutation relations (2.2) for $\widehat{\mathfrak{g}}_\kappa$ into the form

$$[a(z), b(w)] = [a, b](w)\delta(z - w) + \kappa(a, b)c\partial_w\delta(z - w) \tag{3.11}$$

for $a, b \in \mathfrak{g}$. The state field correspondence $Y : \mathcal{V}_\kappa(\mathfrak{g}) \to \mathrm{End}\mathcal{V}_\kappa(\mathfrak{g})[[z^{\pm 1}]]$ is given by

$$Y(a_{1,-n_1-1} \dots a_{k,-n_k-1}|0\rangle, z) = \frac{1}{n_1! \dots n_k!} :\partial_z^{n_1} a_1(z) \dots \partial_z^{n_k} a_k(z):$$

for $k \in \mathbb{N}, n_1, n_2, \dots, n_k \in \mathbb{N}_0$ and $a_1, a_2, \dots, a_k \in \mathfrak{g}$, where $|0\rangle \in \mathcal{V}_\kappa(\mathfrak{g})$ is the vacuum vector (a highest weight vector of $\mathbb{M}_{\kappa,\mathfrak{g}}(\mathbb{C})$). The translation operator $T : \mathcal{V}_\kappa(\mathfrak{g}) \to \mathcal{V}_\kappa(\mathfrak{g})$ is defined by $T|0\rangle = 0$ and $[T, a_n] = -na_{n-1}$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$.

To describe positive energy representations of $\mathcal{V}_\kappa(\mathfrak{g})$, we need to know its Zhu's algebra. It is easy to see that for $\mathcal{V}_\kappa(\mathfrak{g})$ we have a canonical isomorphism

$$A(\mathcal{V}_\kappa(\mathfrak{g})) \simeq U(\mathfrak{g}) \tag{3.12}$$

of associative algebras determined by

$$a_{1,-n_1-1}a_{2,-n_2-1} \dots a_{k,-n_k-1}|0\rangle \mapsto (-1)^{n_1+n_2+\dots+n_k} a_k \dots a_2 a_1 \tag{3.13}$$

for $k \in \mathbb{N}, n_1, n_2, \dots, n_k \in \mathbb{N}_0$ and $a_1, a_2, \dots, a_k \in \mathfrak{g}$.

Let us note that for the universal affine vertex algebra $\mathcal{V}_\kappa(\mathfrak{g})$ the functors $\mathbb{M}_{\mathcal{V}_\kappa(\mathfrak{g})}$ and $\mathbb{L}_{\mathcal{V}_\kappa(\mathfrak{g})}$ coincide with the functors $\mathbb{M}_{\kappa,\mathfrak{g}}$ and $\mathbb{L}_{\kappa,\mathfrak{g}}$, respectively. Also both categories $\mathcal{E}(\mathcal{V}_\kappa(\mathfrak{g}))$ and $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$ coincide.

According to a theorem of Zhu the assignment $E \mapsto \mathbb{L}_{\kappa,\mathfrak{g}}(E)$ gives a one-to-one correspondence between the isomorphism classes of simple $\mathfrak{g}$-modules and simple positive energy $\mathcal{V}_\kappa(\mathfrak{g})$-modules. Therefore, the study of positive energy $\mathcal{V}_\kappa(\mathfrak{g})$-modules reduces to the study of $\mathfrak{g}$-modules.

In addition, the unique simple quotient $\mathbb{L}_{\kappa,\mathfrak{g}}(\mathbb{C})$ of $\mathbb{M}_{\kappa,\mathfrak{g}}(\mathbb{C})$ has also the natural structure of an $\mathbb{N}_0$-graded vertex algebra, called the *simple affine vertex algebra*, which we will denote by $\mathcal{L}_\kappa(\mathfrak{g})$. The Zhu's algebra $A(\mathcal{L}_\kappa(\mathfrak{g}))$ is a homomorphic image of $U(\mathfrak{g})$, hence we have

$$A(\mathcal{L}_\kappa(\mathfrak{g})) \simeq U(\mathfrak{g})/I_\kappa \tag{3.14}$$

for some two-sided ideal I_κ of $U(\mathfrak{g})$. Moreover, the assignment $E \mapsto \mathbb{L}_{\kappa,\mathfrak{g}}(E)$ gives a one-to-one correspondence between isomorphism classes of simple modules over $U(\mathfrak{g})/I_\kappa$ and simple positive energy $\mathcal{L}_\kappa(\mathfrak{g})$-modules.

3.4. Weyl vertex algebras. Let V be a finite-dimensional vector space. Then the induced $\mathcal{A}_{\mathcal{K}(V)}$-module $\mathbb{M}_{\mathcal{K}(V)}(S(V^*))$ carries the natural structure of an $\mathbb{N}_0$-graded vertex algebra, called the *Weyl vertex algebra*, which we will denote by $\mathcal{M}_V$. Let $\{x_1, x_2, \dots, x_m\}$, where $\dim V = m$, be linear coordinate functions on V and let $a_i(z), a_i^*(z) \in \mathcal{A}_{\mathcal{K}(V)}[[z^{\pm 1}]]$ for $i = 1, 2, \dots, m$ be the formal distributions defined by

$$a_i(z) = \sum_{n\in\mathbb{Z}} a_{i,n} z^{-n-1} \qquad \text{and} \qquad a_i^*(z) = \sum_{n\in\mathbb{Z}} a_{i,n}^* z^{-n}, \tag{3.15}$$

where $a_{i,n} = \partial_{x_{i,n}}$ and $a_{i,n}^* = x_{i,-n}$ for $n \in \mathbb{Z}$ and $i = 1, 2, \dots, m$. By using these formal power series, the canonical commutation relations for $\mathcal{A}_{\mathcal{K}(V)}$ we may write in the form

$$[a_i(z), a_j(w)] = 0, \quad [a_i(z), a_j^*(w)] = \delta_{i,j}\delta(z-w), \quad [a_i^*(z), a_j^*(w)] = 0 \tag{3.16}$$

for $i, j = 1, 2, \dots, m$. The state field correspondence $Y\colon \mathcal{M}_V \to \mathrm{End}\mathcal{M}_V[[z^{\pm 1}]]$ is given by

$$Y(a_{i_1,-n_1-1}\dots a_{i_r,-n_r-1}a^*_{j_1,-m_1}\dots a^*_{j_s,-m_s}|0\rangle, z) = \\ \frac{1}{n_1!\dots n_r!}\frac{1}{m_1!\dots m_s!} :\partial_z^{n_1}a_{i_1}(z)\dots\partial_z^{n_r}a_{i_r}(z)\partial_z^{m_1}a^*_{j_1}(z)\dots\partial_z^{m_s}a^*_{j_s}(z):$$

for $r, s \in \mathbb{N}$, $n_1, \dots, n_r, m_1, \dots, m_s \in \mathbb{N}_0$, where $|0\rangle \in \mathcal{M}_V$ is the vacuum vector (a highest weight vector of $\mathbb{M}_{\mathcal{K}(V)}(S(V^*))$). The translation operator $T\colon \mathcal{M}_V \to \mathcal{M}_V$ is defined by $T|0\rangle = 0$, $[T, a_{i,n}] = -na_{i,n-1}$ and $[T, a^*_{i,n}] = -(n-1)a^*_{i,n-1}$ for $n \in \mathbb{Z}$ and $i = 1, 2, \dots, m$. Moreover, we have a canonical isomorphism

$$A(\mathcal{M}_V) \simeq \mathcal{A}_V \tag{3.17}$$

of associative algebras determined by

$$a_{i_1,-n_1-1}\dots a_{i_r,-n_r-1}a^*_{j_1,-m_1}\dots a^*_{j_s,-m_s}|0\rangle \mapsto \\ \delta_{m_1,0}\dots\delta_{m_s,0}(-1)^{n_1+\dots+n_r}x_{j_1}\dots x_{j_s}\partial_{x_{i_1}}\dots\partial_{x_{i_r}} \tag{3.18}$$

for $r, s \in \mathbb{N}$, $n_1, \dots, n_r, m_1, \dots, m_s \in \mathbb{N}_0$.

Let us note that for the Weyl vertex algebra $\mathcal{M}_V$ the functors $\mathbb{M}_{\mathcal{M}_V}$ and $\mathbb{L}_{\mathcal{M}_V}$ coincide with the functors $\mathbb{M}_{\mathcal{K}(V)}$ and $\mathbb{L}_{\mathcal{K}(V)}$, respectively. Also both categories $\mathcal{E}(\mathcal{M}_V)$ and $\mathcal{E}(\mathcal{A}_{\mathcal{K}(V)})$ coincide.

4. Twisting Functors and Gelfand–Tsetlin Modules

We recall the definition of Gelfand–Tsetlin modules for a complex semisimple finite-dimensional Lie algebra $\mathfrak{g}$ and describe the construction of the twisting functor T_α assigned to a positive root α of $\mathfrak{g}$.

4.1. Twisting functors for semisimple Lie algebras. Let us consider a complex semisimple Lie algebra $\mathfrak{g}$. For a positive root $\alpha \in \Delta_+$ of $\mathfrak{g}$ we denote by $\mathfrak{s}_\alpha$ the Lie subalgebra of $\mathfrak{g}$ generated by the $\mathfrak{sl}_2$-triple $(e_\alpha, h_\alpha, f_\alpha)$, where $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ satisfy $[e_\alpha, f_\alpha] = h_\alpha$. Further, we define the Lie subalgebras $\mathfrak{s}_\alpha^+ = \mathfrak{s}_\alpha \cap \mathfrak{n} = \mathbb{C}e_\alpha$ and $\mathfrak{s}_\alpha^- = \mathfrak{s}_\alpha \cap \overline{\mathfrak{n}} = \mathbb{C}f_\alpha$ of $\mathfrak{s}_\alpha$. Let c_α be the quadratic Casimir element given through

$$c_\alpha = e_\alpha f_\alpha + f_\alpha e_\alpha + \tfrac{1}{2}h_\alpha^2,$$

which is a free generator of the center $Z(\mathfrak{s}_\alpha)$ of $U(\mathfrak{s}_\alpha)$. Then we denote by Γ_α the commutative subalgebra of $U(\mathfrak{g})$ generated by the Cartan subalgebra $\mathfrak{h}$ and by the center $Z(\mathfrak{s}_\alpha)$. Therefore, we have the category $\mathcal{H}(\mathfrak{g}, \Gamma_\alpha)$ of Γ_α-Gelfand–Tsetlin modules, or simply α-Gelfand–Tsetlin modules, which was studied in [FK19a,FK19b]. Let us note that $\mathcal{H}(\mathfrak{g}, \Gamma_\alpha)$ contains $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^+)$ and $\mathcal{I}(\mathfrak{g}, \mathfrak{s}_\alpha^-)$ as full subcategories.

Since the multiplicative set $\{f_\alpha^n;\ n \in \mathbb{N}_0\}$ in the universal enveloping algebra $U(\mathfrak{g})$ is a left (right) denominator set, thanks to the fact that f_α is a locally ad-nilpotent regular element in $U(\mathfrak{g})$, we define the *twisting functor*

$$T_\alpha = T_{f_\alpha} : \mathcal{M}(\mathfrak{g}) \to \mathcal{M}(\mathfrak{g})$$

by

$$T_\alpha(M) = (U(\mathfrak{g})_{(f_\alpha)}/U(\mathfrak{g})) \otimes_{U(\mathfrak{g})} M \simeq M_{(f_\alpha)}/M$$

for $M \in \mathcal{M}(\mathfrak{g})$. Let us note that the quotient $M_{(f_\alpha)}/M$ means the quotient of $M_{(f_\alpha)}$ by the image of the canonical homomorphism $M \to M_{(f_\alpha)}$ of $\mathfrak{g}$-modules. As f_α is a locally ad-nilpotent regular element in $U(\mathfrak{g})$, the functor T_α is right exact. The twisting functor T_α for a simple root $\alpha \in \Pi$ is well studied (see e.g. [AS03]). In this case the functor T_α preserves the category $\mathcal{O}(\mathfrak{g})$ up to a conjugation of the action of $\mathfrak{g}$.

Theorem 4.1 [FK19a, Theorem 3.3]. *For $\alpha \in \Delta_+$ the functor T_α induces the restricted functor*

$$T_\alpha : \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^+) \to \mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^-),$$

where $\mathcal{I}_f(\mathfrak{g}, \mathfrak{s}_\alpha^\pm)$ is the category of finitely generated locally $\mathfrak{s}_\alpha^\pm$-finite weight $\mathfrak{g}$-modules.

It was shown in [FK19b] how the twisting functor T_α can be used to construct $\mathfrak{g}$-modules in $\mathcal{H}(\mathfrak{g}, \Gamma_\alpha)$ with finite Γ_α-multiplicities. Namely, we have the following theorem.

Theorem 4.2 [FK19a, Theorem 3.4]. *Let $\alpha \in \Delta_+$.*

(i) *For a weight $\mathfrak{g}$-module M, the $\mathfrak{g}$-module $T_\alpha(M)$ belongs to $\mathcal{H}(\mathfrak{g}, \Gamma_\alpha)$. In addition, $T_\alpha(M)$ has finite Γ_α-multiplicities if and only if the first cohomology group $H^1(\mathfrak{s}_\alpha^-; M)$ is a weight $\mathfrak{h}$-module with finite-dimensional weight spaces.*

(ii) *If M is a highest weight $\mathfrak{g}$-module, then $T_\alpha(M)$ is a cyclic weight $\mathfrak{g}$-module with finite Γ_α-multiplicities.*

For a $\mathfrak{g}$-module M, we denote by $\mathrm{Ann}_{U(\mathfrak{g})}M$ the annihilator of M in $U(\mathfrak{g})$. Then we get the following important proposition.

Proposition 4.3. *Let M be a $U(\mathfrak{s}_\alpha^-)$-free $\mathfrak{g}$-module for $\alpha \in \Delta_+$. We have* $\mathrm{Ann}_{U(\mathfrak{g})} T_\alpha(M) = \mathrm{Ann}_{U(\mathfrak{g})} M$.

Proof. First, we observe that $\mathrm{Ann}_{U(\mathfrak{g})} M_{(f_\alpha)} = \mathrm{Ann}_{U(\mathfrak{g})} M$. Indeed, the inclusion $M \subset M_{(f_\alpha)}$ gives us

$$\mathrm{Ann}_{U(\mathfrak{g})} M_{(f_\alpha)} \subset \mathrm{Ann}_{U(\mathfrak{g})} M.$$

The opposite embedding is obvious since f_α is a locally ad-nilpotent element in $U(\mathfrak{g})$. Further, from $T_\alpha(M) = M_{(f_\alpha)}/M$ we get that $\mathrm{Ann}_{U(\mathfrak{g})} M \subset \mathrm{Ann}_{U(\mathfrak{g})} T_\alpha(M)$. Now, we prove the opposite inclusion. Let us consider a vector $v \in M$ and let us assume that $a \in \mathrm{Ann}_{U(\mathfrak{g})} T_\alpha(M)$. Then we have $a f_\alpha^{-n} v \in M$ for all $n \in \mathbb{N}_0$. By using the fact $f_\alpha^n a f_\alpha^{-n-m} v \in M$ and the formula

$$f_\alpha^n a f_\alpha^{-n-m} v = f_\alpha^{-m} \sum_{k=0}^{\infty} \binom{n+m+k-1}{k} f_\alpha^{-k} \mathrm{ad}(f_\alpha)^k(a) v$$

for all $n, m \in \mathbb{N}_0$, we obtain immediately $f_\alpha^{-k-m} \mathrm{ad}(f_\alpha)^k(a) v \in M$ for all $k, m \in \mathbb{N}_0$ which gives us $\mathrm{ad}(f_\alpha)^k(a) v = 0$ for all $k \in \mathbb{N}_0$. We may write

$$a f_\alpha^{-n} v = f_\alpha^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} f_\alpha^{-k} \mathrm{ad}(f_\alpha)^k(a) v = 0$$

for all $n \in \mathbb{N}_0$. Hence, we have $a \in \mathrm{Ann}_{U(\mathfrak{g})} M_{(f_\alpha)} = \mathrm{Ann}_{U(\mathfrak{g})} M$ and we are done. □

If M is a $\mathfrak{g}$-module, then $T_\alpha(M)$ is a locally $\mathfrak{s}_\alpha^-$-finite $\mathfrak{g}$-module for any $\alpha \in \Delta_+$. On the other hand, if M is a locally $\mathfrak{s}_\alpha^-$-finite $\mathfrak{g}$-module, then $T_\alpha(M) = 0$. Let us also note that for any $\alpha \in \Delta_+$ the twisting functor T_α commutes with the translation functors by [FK19b, Theorem 3.6].

Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be the standard parabolic subalgebra of $\mathfrak{g}$ associated to a subset Σ of Π with the nilradical $\mathfrak{u}$, the opposite nilradical $\overline{\mathfrak{u}}$ and the Levi subalgebra $\mathfrak{l}$. We define the subsets

$$\Delta_+^{\mathfrak{u}} = \{\alpha \in \Delta_+;\ \mathfrak{g}_\alpha \subset \mathfrak{u}\}, \qquad \Delta_+^{\mathfrak{l}} = \{\alpha \in \Delta_+;\ \mathfrak{g}_\alpha \subset \mathfrak{l}\}$$

of Δ_+ and we set

$$\Lambda^+(\mathfrak{p}) = \{\lambda \in \mathfrak{h}^*;\ (\forall \alpha \in \Sigma)\, \lambda(h_\alpha) \in \mathbb{N}_0\}.$$

For a weight $\lambda \in \Lambda^+(\mathfrak{p})$, we denote by $\mathbb{F}_\lambda$ the simple finite-dimensional $\mathfrak{p}$-module with highest weight λ and by $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ the generalized Verma $\mathfrak{g}$-module with highest weight λ defined by

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{F}_\lambda.$$

Then for $\alpha \in \Delta_+^{\mathfrak{u}}$ we define the α-Gelfand–Tsetlin $\mathfrak{g}$-module $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ by

$$W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha) = T_\alpha(M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)).$$

As a consequence of the Poincaré–Birkhoff–Witt theorem we have isomorphisms

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \simeq U(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} \mathbb{F}_\lambda \qquad \text{and} \qquad W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha) \simeq (U(\overline{\mathfrak{u}})_{(f_\alpha)}/U(\overline{\mathfrak{u}})) \otimes_{\mathbb{C}} \mathbb{F}_\lambda$$

of $U(\overline{\mathfrak{u}})$-modules. Moreover, since the twisting functor T_α is right exact, by applying of T_α on a surjective homomorphism

$$M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda) \to M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$$

of generalized Verma modules, we get a surjective homomorphism

$$W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda, \alpha) \to W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$$

of α-Gelfand–Tsetlin modules.

By applying Theorem 4.2 and Proposition 4.3 on $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_+^{\mathfrak{u}}$ we obtain the following immediate corollaries.

Corollary 4.4. *Let $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_+^{\mathfrak{u}}$. Then the $\mathfrak{g}$-module $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha) \in \mathcal{H}(\mathfrak{g}, \Gamma_\alpha)$ is cyclic weight with central character and finite Γ_α-multiplicities. Moreover, $W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha)$ belongs to $\mathcal{H}(\mathfrak{g}, \Gamma)$ and has finite Γ-multiplicities for any commutative subalgebra Γ of $U(\mathfrak{g})$ containing Γ_α.*

Corollary 4.5. *Let $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta_+^{\mathfrak{u}}$. Then we have* $\mathrm{Ann}_{U(\mathfrak{g})} W_{\mathfrak{p}}^{\mathfrak{g}}(\lambda, \alpha) = \mathrm{Ann}_{U(\mathfrak{g})} M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$.

4.2. Affine α-Gelfand–Tsetlin modules. Let $\widetilde{\mathfrak{g}}_\kappa$ be the extended affine Kac–Moody algebra associated to a complex semisimple Lie algebra $\mathfrak{g}$ of level κ. Then for a commutative subalgebra Γ of the completed universal enveloping algebra $\widetilde{U}_c(\widetilde{\mathfrak{g}}_\kappa)$ of $\widetilde{\mathfrak{g}}_\kappa$, we denote by $\mathcal{H}(\widetilde{\mathfrak{g}}_\kappa, \Gamma)$ the full subcategory of $\mathcal{E}(\widetilde{\mathfrak{g}}_\kappa)$ consisting of smooth Γ-weight $\widetilde{\mathfrak{g}}_\kappa$-modules on which the central element c acts as the identity. Let us note that $\mathcal{H}(\widetilde{\mathfrak{g}}_\kappa, \Gamma)$ is closed with respect to the operations of taking submodules and quotients. In addition, if Γ contains the Cartan subalgebra $\widetilde{\mathfrak{h}}$, the $\widetilde{\mathfrak{g}}_\kappa$-modules from $\mathcal{H}(\widetilde{\mathfrak{g}}_\kappa, \Gamma)$ are called Γ-*Gelfand–Tsetlin $\widetilde{\mathfrak{g}}_\kappa$-modules*.

For a real positive root $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$, we denote by $\mathfrak{s}_\alpha$ the Lie subalgebra of $\widetilde{\mathfrak{g}}_\kappa$ generated by the $\mathfrak{sl}_2$-triple $(e_\alpha, h_\alpha, f_\alpha)$, where

$$e_\alpha = e_{\gamma,n}, \qquad h_\alpha = h_{\gamma,0} + n\kappa(e_\gamma, f_\gamma)c, \qquad f_\alpha = f_{\gamma,-n}$$

if $\alpha = \gamma + n\delta$ with $\gamma \in \Delta_+$, $n \in \mathbb{N}_0$ and

$$e_\alpha = f_{\gamma,n}, \qquad h_\alpha = -h_{\gamma,0} + n\kappa(e_\gamma, f_\gamma)c, \qquad f_\alpha = e_{\gamma,-n}$$

provided $\alpha = -\gamma + n\delta$ with $\gamma \in \Delta_+$, $n \in \mathbb{N}$. The quadratic Casimir element c_α given by

$$c_\alpha = e_\alpha f_\alpha + f_\alpha e_\alpha + \tfrac{1}{2} h_\alpha^2$$

is a free generator of the center $Z(\mathfrak{s}_\alpha)$ of $U(\mathfrak{s}_\alpha)$. Furthermore, the commutative subalgebra of $\widetilde{U}_c(\widetilde{\mathfrak{g}}_\kappa)$ generated by the Cartan subalgebra $\widetilde{\mathfrak{h}}$ and by the center $Z(\mathfrak{s}_\alpha)$ we denote by Γ_α. We also define the Lie subalgebras $\mathfrak{s}_\alpha^+ = \mathfrak{s}_\alpha \cap \widehat{\mathfrak{n}}_{\mathrm{st}} = \mathbb{C}e_\alpha$ and $\mathfrak{s}_\alpha^- = \mathfrak{s}_\alpha \cap \widehat{\overline{\mathfrak{n}}}_{\mathrm{st}} = \mathbb{C}f_\alpha$ of $\widetilde{\mathfrak{g}}_\kappa$. The objects of $\mathcal{H}(\widetilde{\mathfrak{g}}_\kappa, \Gamma_\alpha)$ will be simply called α-*Gelfand–Tsetlin $\widetilde{\mathfrak{g}}_\kappa$-modules*.

Let Γ be a commutative subalgebra of $\widetilde{U}_c(\widetilde{\mathfrak{g}}_\kappa)$ containing the Cartan subalgebra $\widetilde{\mathfrak{h}}$. Besides, let us assume that $[a, \mathfrak{s}_\alpha^\pm] = \pm\chi(a)\mathfrak{s}_\alpha^\pm$ for all $a \in \Gamma$ and some character $\chi \in \Gamma$. Then for a Γ-Gelfand–Tsetlin $\widetilde{\mathfrak{g}}_\kappa$-module M, the Lie algebra cohomology groups $H^n(\mathfrak{s}_\alpha^\pm; M)$ are Γ-weight modules for all $n \in \mathbb{N}_0$. By general results we get

$H^0(\mathfrak{s}_\alpha^\pm; M) \simeq M^{\mathfrak{s}_\alpha^\pm}$, $H^1(\mathfrak{s}_\alpha^\pm; M) \simeq M/\mathfrak{s}_\alpha^\pm M$ and $H^n(\mathfrak{s}_\alpha^\pm; M) = 0$ for $n > 1$. Moreover, we have that $Z(\mathfrak{s}_\alpha)$ commutes with Γ, which enables us to extend Γ by $Z(\mathfrak{s}_\alpha)$. Therefore, we denote by $\Gamma_{\alpha,\mathrm{ext}}$ the commutative subalgebra of $\widetilde{U}_c(\widetilde{\mathfrak{g}}_\kappa)$ generated by Γ and by $Z(\mathfrak{s}_\alpha)$. As a mild and straightforward generalization of Theorem 2.3 in [FK19b] we obtain the following important statement.

Theorem 4.6. *Let $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$ and let M be a Γ-Gelfand–Tsetlin $\widetilde{\mathfrak{g}}_\kappa$-module which is locally $\mathfrak{s}_\alpha^-$-finite. Then M is a $\Gamma_{\alpha,\mathrm{ext}}$-Gelfand–Tsetlin $\widetilde{\mathfrak{g}}_\kappa$-module with finite $\Gamma_{\alpha,\mathrm{ext}}$-multiplicities if and only if the zeroth cohomology group $H^0(\mathfrak{s}_\alpha^-; M)$ is a Γ-weight module with finite Γ-multiplicities.*

The following result is a consequence of Theorem 4.6 and the fact that the extension of $U_c(\widetilde{\mathfrak{h}})$ by $Z(\mathfrak{s}_\alpha)$ is the commutative algebra Γ_α.

Corollary 4.7. *Let $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$ and let M be a locally $\mathfrak{s}_\alpha^-$-finite smooth weight $\widetilde{\mathfrak{g}}_\kappa$-module on which the central element c acts as the identity. Then M is a Γ_α-Gelfand–Tsetlin $\widetilde{\mathfrak{g}}_\kappa$-module with finite Γ_α-multiplicities if and only if the zeroth cohomology group $H^0(\mathfrak{s}_\alpha^-; M)$ is a weight $\widetilde{\mathfrak{h}}$-module with finite-dimensional weight spaces.*

By duality we can analogously prove a similar statement if we replace $\mathfrak{s}_\alpha^-$ by $\mathfrak{s}_\alpha^+$ and $H^0(\mathfrak{s}_\alpha^-; M)$ by $H^0(\mathfrak{s}_\alpha^+; M)$.

For a Lie subalgebra $\mathfrak{a}$ of $\widetilde{\mathfrak{g}}_\kappa$ we denote by $\mathcal{I}(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{a})$ and $\mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{a})$ the full subcategories of $\mathcal{E}(\widetilde{\mathfrak{g}}_\kappa)$ consisting of locally $\mathfrak{a}$-finite and finitely generated locally $\mathfrak{a}$-finite, respectively, smooth weight $\widetilde{\mathfrak{g}}_\kappa$-modules on which the central element c acts as the identity. Therefore, for a real positive root $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$ we have the full subcategories $\mathcal{I}(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^+)$ and $\mathcal{I}(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^-)$ of $\mathcal{E}(\widetilde{\mathfrak{g}}_\kappa)$ assigned to the Lie subalgebras $\mathfrak{s}_\alpha^+$ and $\mathfrak{s}_\alpha^-$ of $\widetilde{\mathfrak{g}}_\kappa$.

The following statement is obvious, see Proposition 1.3 in [FK19a] for details.

Proposition 4.8. *The categories $\mathcal{I}(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^+)$ and $\mathcal{I}(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^-)$ are full subcategories of $\mathcal{H}(\widetilde{\mathfrak{g}}_\kappa, \Gamma_\alpha)$ for $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$.*

4.3. Twisting functors for affine Kac–Moody algebras. For a real positive root $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$ the multiplicative sets $\{f_\alpha^n;\ n \in \mathbb{N}_0\}$ and $\{e_\alpha^n;\ n \in \mathbb{N}_0\}$ in the universal enveloping algebra $U(\widehat{\mathfrak{g}}_\kappa)$ of $\widehat{\mathfrak{g}}_\kappa$ are left (right) denominator sets, since f_α and e_α are locally adnilpotent regular elements. Therefore, based on the general construction we define the *twisting functor*

$$T_\alpha = T_{f_\alpha} : \mathcal{M}(\widehat{\mathfrak{g}}_\kappa) \to \mathcal{M}(\widehat{\mathfrak{g}}_\kappa) \qquad \text{and} \qquad T_{-\alpha} = T_{e_\alpha} : \mathcal{M}(\widehat{\mathfrak{g}}_\kappa) \to \mathcal{M}(\widehat{\mathfrak{g}}_\kappa)$$

by

$$T_\alpha(M) = U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}/U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\widehat{\mathfrak{g}}_\kappa)} M, \qquad T_{-\alpha}(M) = U(\widehat{\mathfrak{g}}_\kappa)_{(e_\alpha)}/U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\widehat{\mathfrak{g}}_\kappa)} M$$

for $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$ and $M \in \mathcal{M}(\widehat{\mathfrak{g}}_\kappa)$. In addition, it is easy to see that if M is a $\widetilde{\mathfrak{g}}_\kappa$-module, then $T_\alpha(M)$ has also a natural structure of a $\widetilde{\mathfrak{g}}_\kappa$-module for $\alpha \in \widehat{\Delta}^{\mathrm{re}}$.

In the next, we prove some basic characteristics of T_α for $\alpha \in \widehat{\Delta}^{\mathrm{re}}$. Let us note that the twisting functor T_α for $\alpha \in \Delta \subset \widehat{\Delta}^{\mathrm{re}}$ is of a special importance and will be discussed later in details.

Let us consider the subset

$$\Phi_\alpha = \{\gamma \in \widehat{\Delta} \backslash \{\pm\alpha\};\ \widehat{\mathfrak{g}}_{\kappa,\gamma,\alpha} \subset \widehat{\mathfrak{n}}_{\mathrm{st}}\}$$

of $\widehat{\Delta}$ for $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$, where $\widehat{\mathfrak{g}}_{\kappa,\gamma,\alpha}$ for $\gamma \in \widehat{\Delta}$ is the finite-dimensional $\mathfrak{s}_\alpha$-module given by

$$\widehat{\mathfrak{g}}_{\kappa,\gamma,\alpha} = \bigoplus_{j\in\mathbb{Z}} \widehat{\mathfrak{g}}_{\kappa,\gamma+j\alpha}.$$

It easily follows that $\Phi_\alpha \subset \widehat{\Delta}_+$ and that Φ_α is closed, i.e. if $\gamma_1, \gamma_2 \in \Phi_\alpha$ and $\gamma_1+\gamma_2 \in \widehat{\Delta}$, then we have $\gamma_1 + \gamma_2 \in \Phi_\alpha$. Therefore, the subset Φ_α of $\widehat{\Delta}_+$ gives rise to the Lie subalgebras $\mathfrak{t}^+_\alpha$, $\mathfrak{t}^-_\alpha$ and $\mathfrak{t}_\alpha$ of $\widehat{\mathfrak{g}}_\kappa$ defined through

$$\mathfrak{t}^+_\alpha = \widehat{\mathfrak{g}}_{\kappa,\alpha} \oplus \bigoplus_{\gamma\in\Phi_\alpha} \widehat{\mathfrak{g}}_{\kappa,\gamma}, \qquad \mathfrak{t}_\alpha = \bigoplus_{\gamma\in\Phi_\alpha} \widehat{\mathfrak{g}}_{\kappa,\gamma}, \qquad \mathfrak{t}^-_\alpha = \widehat{\mathfrak{g}}_{\kappa,-\alpha} \oplus \bigoplus_{\gamma\in\Phi_\alpha} \widehat{\mathfrak{g}}_{\kappa,\gamma},$$

where the direct sum over the set Φ_α is the topological direct sum. Moreover, since we have the inclusions $\mathfrak{s}^+_\alpha \subset \mathfrak{t}^+_\alpha \subset \widehat{\mathfrak{n}}_{\mathrm{st}}$ of Lie algebras, we get immediately the embeddings of categories

$$\mathcal{O}(\widetilde{\mathfrak{g}}_\kappa) \subset \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{t}^+_\alpha) \subset \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}^+_\alpha)$$

for $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$. There is also an automorphism $\mathrm{Ad}(\dot{r}_\alpha)$ of $\widetilde{\mathfrak{g}}_\kappa$ given by

$$\mathrm{Ad}(\dot{r}_\alpha) = \exp(\mathrm{ad}\, f_\alpha)\exp(-\mathrm{ad} e_\alpha)\exp(\mathrm{ad}\, f_\alpha),$$

which satisfies

$$\mathrm{Ad}(\dot{r}_\alpha)(\mathfrak{s}^\pm_\alpha) = \mathfrak{s}^\mp_\alpha \qquad \text{and} \qquad \mathrm{Ad}(\dot{r}_\alpha)(\mathfrak{t}^\pm_\alpha) = \mathfrak{t}^\mp_\alpha$$

for $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$.

Theorem 4.9. *For $\alpha \in \widehat{\Delta}^{\mathrm{re}}$, the twisting functor T_α preserves the category $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$. Moreover, if $\alpha \in \Delta \subset \widehat{\Delta}^{\mathrm{re}}$, then the twisting functor T_α preserves also the category $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$.*

Proof. Let M be a smooth $\widehat{\mathfrak{g}}_\kappa$-module on which the central element c acts as the identity. Then for each $v \in M$ there exists $N_v \in \mathbb{N}$ such that $(\mathfrak{g}\otimes_{\mathbb{C}} t^{N_v}\mathbb{C}[[t]])v = 0$. Further, for $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$ there exists $k_\alpha \in \mathbb{N}$ satisfying $\mathrm{ad}(f_\alpha)^{k_\alpha}(a) = 0$ for all $a \in \widehat{\mathfrak{g}}_\kappa$. Hence, we may write

$$a f_\alpha^{-n} v = \sum_{k=0}^{k_\alpha-1} \binom{n+k-1}{k} f_\alpha^{-n-k} \mathrm{ad}(f_\alpha)^k(a) v$$

for $a \in \widehat{\mathfrak{g}}_\kappa$. Moreover, since $\mathrm{ad}(f_\alpha)^k(a_m) \in \mathfrak{g}\otimes_{\mathbb{C}} t^{m-k\alpha(d)}\mathbb{C}[[t]]$ for $a \in \mathfrak{g}$ and $m \in \mathbb{Z}$, we get that $(\mathfrak{g}\otimes_{\mathbb{C}} t^{N_{n,v}}\mathbb{C}[[t]])\, f_\alpha^{-n} v = 0$ for $N_{n,v} = \max\{N_v + k\alpha(d);\ k \in \{0,1,\dots,k_\alpha - 1\}\}$. Hence, we have that $T_\alpha(M)$ for $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$ is a smooth $\widehat{\mathfrak{g}}_\kappa$-module on which the central element c acts as the identity. For $\alpha \in \widehat{\Delta}^{\mathrm{re}}_-$ the proof goes along the same lines.

Further, let M belong to the category $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$. Then from the previous part we immediately get that $T_\alpha(M)$ for $\alpha \in \Delta$ is from the category $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$. As M is a positive energy $\widehat{\mathfrak{g}}_\kappa$-module we have $M = \bigoplus_{n=0}^\infty M_{\lambda+n}$ with $M_\lambda \neq 0$ for some $\lambda \in \mathbb{C}$. Since f_α preserves $M_{\lambda+n}$ for $n \in \mathbb{N}_0$, or in other words $\deg f_\alpha = 0$ by definition of the gradation on $\widehat{\mathfrak{g}}_\kappa$, we may set $\deg f_\alpha^{-n} v = \deg v$ for $n \in \mathbb{N}_0$ and a homogeneous vector $v \in M$. Then it follows immediately that $T_\alpha(M)$ is a positive energy $\widehat{\mathfrak{g}}_\kappa$-module. □

Let us note that for $\alpha \in \widehat{\Delta}^{\mathrm{re}}_-\backslash\Delta_-$ the twisting functor T_α also preserves the category $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$. However, we have $T_\alpha(M) = 0$ for $\alpha \in \widehat{\Delta}^{\mathrm{re}}_-\backslash\Delta_-$ and $M \in \mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$ which follows from Proposition 4.10 and the fact that M is a locally $\mathfrak{s}^+_\alpha$-finite $\widehat{\mathfrak{g}}_\kappa$-module.

Proposition 4.10. *Let M be a $\widehat{\mathfrak{g}}_\kappa$-module. Then $T_\alpha(M)$ is a locally $\mathfrak{s}_\alpha^-$-finite $\widehat{\mathfrak{g}}_\kappa$-module for any $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$. Moreover, if M is a locally $\mathfrak{s}_\alpha^-$-finite $\widehat{\mathfrak{g}}_\kappa$-module, then we obtain $T_\alpha(M) = 0$.*

Proof. Let M be a $\widehat{\mathfrak{g}}_\kappa$-module. Then by definition we have

$$T_\alpha(M) = (U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}/U(\widehat{\mathfrak{g}}_\kappa)) \otimes_{U(\widehat{\mathfrak{g}}_\kappa)} M \simeq M_{(f_\alpha)}/M$$

for $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$. Since every element of $M_{(f_\alpha)}$ can be written in the form $f_\alpha^{-n} v$ for $n \in \mathbb{N}_0$ and $v \in M$, we obtain immediately that $T_\alpha(M)$ is locally $\mathfrak{s}_\alpha^-$-finite. Further, let us assume that M is a locally $\mathfrak{s}_\alpha^-$-finite $\widehat{\mathfrak{g}}_\kappa$-module. Since for each $v \in M$ there exists $n_v \in \mathbb{N}_0$ such that $f_\alpha^{n_v} v = 0$, we may write $f_\alpha^{-n} v = f_\alpha^{-n-n_v} f_\alpha^{n_v} v = 0$ for $n \in \mathbb{N}_0$. This implies the required statement. □

Lemma 4.11. *Let $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$. Then we have*

$$\begin{aligned} e_\alpha f_\alpha^{-n} &= f_\alpha^{-n} e_\alpha - n f_\alpha^{-n-1} h_\alpha - n(n+1) f_\alpha^{-n-1}, \\ h f_\alpha^{-n} &= f_\alpha^{-n} h + n\alpha(h) f_\alpha^{-n}, \\ f_\alpha f_\alpha^{-n} &= f_\alpha^{-n} f_\alpha \end{aligned}$$

in $U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}$ for $n \in \mathbb{Z}$ and $h \in \widehat{\mathfrak{h}}$.

Proof. It follows immediately from the formula

$$a f_\alpha^{-n} = f_\alpha^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} f_\alpha^{-k} \mathrm{ad}(f_\alpha)^k(a)$$

in $U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}$ for all $a \in U(\widehat{\mathfrak{g}}_\kappa)$ and $n \in \mathbb{N}_0$. □

Theorem 4.12. *We have*

(i) *if $M \in \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^+)$, then $T_\alpha(M) \in \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^-)$;*
(ii) *if $M \in \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{t}_\alpha^+)$, then $T_\alpha(M) \in \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{t}_\alpha^-)$*

for $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$. Therefore, we have the restricted functors

$$T_\alpha : \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^+) \to \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^-) \quad \text{and} \quad T_\alpha : \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{t}_\alpha^+) \to \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{t}_\alpha^-)$$

for $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$.

Proof. (i) If M is a weight $\widetilde{\mathfrak{g}}_\kappa$-module, then it easily follows from definition that $T_\alpha(M)$ is also a weight $\widetilde{\mathfrak{g}}_\kappa$-module. Moreover, by Theorem 4.9 and Proposition 4.10 we have that $T_\alpha(M)$ belongs to the category $\mathcal{I}(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^-)$. Hence, the rest of the proof is to show that $T_\alpha(M)$ is finitely generated provided M is finitely generated and locally $\mathfrak{s}_\alpha^+$-finite.

Let $R \subset M$ be a finite set of generators of M. Then the vector subspace $V = U(\mathfrak{s}_\alpha^+)\langle R \rangle$ of M is finite dimensional. Further, let us introduce a filtration $\{F_k V\}_{k \in \mathbb{N}_0}$ on V by

$$F_k V = \{v \in V;\ e_\alpha^k v = 0\}$$

for $k \in \mathbb{N}_0$ and let $n_0 \in \mathbb{N}$ be the smallest positive integer satisfying

$$n_0 \geq \max\{-\mu(h_\alpha);\ \mu \in \widetilde{\mathfrak{h}}^*,\ V_\mu \neq \{0\},\ \mu(h_\alpha) \in \mathbb{R}\}.$$

Let us consider a vector $v \in F_k V \cap V_\mu$ for $\mu \in \widetilde{\mathfrak{h}}^*$ and $k \in \mathbb{N}_0$. Then by Lemma 4.11 we get

$$e_\alpha f_\alpha^{-(n_0+n)} v = f_\alpha^{-(n_0+n)} e_\alpha v - (n_0+n)(\mu(h_\alpha) + n_0 + n + 1) f_\alpha^{-(n_0+n+1)} v$$

for $n \in \mathbb{N}_0$, which together with $f_\alpha f_\alpha^{-n-1} v = f_\alpha^{-n} v$ for $n \in \mathbb{N}_0$ gives us

$$U(\mathfrak{s}_\alpha) f_\alpha^{-n_0} F_k V / \mathbb{C}[f_\alpha^{-1}] F_{k-1} V = \mathbb{C}[f_\alpha^{-1}] F_k V / \mathbb{C}[f_\alpha^{-1}] F_{k-1} V$$

for all $k \in \mathbb{N}$. As $F_0 V = \{0\}$ and V is a finite-dimensional vector space, we immediately obtain $U(\mathfrak{s}_\alpha) f_\alpha^{-n_0} V = \mathbb{C}[f_\alpha^{-1}] V$.

Further, since for any $a \in U(\widehat{\mathfrak{g}}_\kappa)$ and $n \in \mathbb{N}_0$ there exist $b \in U(\widehat{\mathfrak{g}}_\kappa)$ and $m \in \mathbb{N}_0$ satisfying $f_\alpha^{-n} a = b f_\alpha^{-m}$, we obtain $f_\alpha^{-n} U(\widehat{\mathfrak{g}}_\kappa) \subset U(\widehat{\mathfrak{g}}_\kappa)\mathbb{C}[f_\alpha^{-1}] \subset U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}$ for $n \in \mathbb{N}_0$. Hence, we may write

$$f_\alpha^{-n} M = f_\alpha^{-n} U(\widehat{\mathfrak{g}}_\kappa) V \subset U(\widehat{\mathfrak{g}}_\kappa)\mathbb{C}[f_\alpha^{-1}] V,$$

which implies $M_{(f_\alpha)} = U(\widehat{\mathfrak{g}}_\kappa)\mathbb{C}[f_\alpha^{-1}]V = U(\widehat{\mathfrak{g}}_\kappa) f_\alpha^{-n_0} V$. In other words, this means that $T_\alpha(M) \simeq M_{(f_\alpha)}/M$ is finitely generated and the number of generators is bounded by $\dim V$.

ii) Since $\mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{t}_\alpha^+)$ is a full subcategory of $\mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^+)$, by item (i) we only need to show that $T_\alpha(M)$ is locally $\mathfrak{t}_\alpha^-$-finite if M is locally $\mathfrak{t}_\alpha^+$-finite. As we have $U(\mathfrak{t}_\alpha^-) = U(\mathfrak{s}_\alpha^-)U(\mathfrak{t}_\alpha)$ and $T_\alpha(M)$ is locally $\mathfrak{s}_\alpha^-$-finite, the condition that $T_\alpha(M)$ is locally $\mathfrak{t}_\alpha^-$-finite is equivalent to saying that $T_\alpha(M)$ is locally $\mathfrak{t}_\alpha$-finite. Further, since $T_\alpha(M)$ is a smooth $\widehat{\mathfrak{g}}_\kappa$-module and the quotient $\mathfrak{t}_\alpha/(\mathfrak{t}_\alpha \cap (\mathfrak{g} \otimes_{\mathbb{C}} t^N \mathbb{C}[[t]]))$ for $N \in \mathbb{N}$ is a finite-dimensional nilpotent Lie algebra, we get that $T_\alpha(M)$ is locally $\mathfrak{t}_\alpha$-finite if the commutative Lie algebra $\widehat{\mathfrak{g}}_{\kappa,\gamma}$ acts locally nilpotently on $T_\alpha(M)$ for all $\gamma \in \Phi_\alpha$.

Let $\mathrm{ht}\colon \mathbb{Z}\widehat{\Delta} \to \mathbb{Z}$ be the $\mathbb{Z}$-linear height function with $\mathrm{ht}(\alpha) = 1$ if α is a simple root. Then we get an $\mathbb{N}_0$-grading $U(\mathfrak{t}_\alpha) = \bigoplus_{n\in\mathbb{N}_0} U(\mathfrak{t}_\alpha)_n$, where

$$U(\mathfrak{t}_\alpha)_n = \bigoplus_{\mu \in \widetilde{\mathfrak{h}}^*,\, \mathrm{ht}(\mu)=n} U(\mathfrak{t}_\alpha)_\mu.$$

Let $\gamma \in \Phi_\alpha$ and let $r \in \mathbb{N}_0$ be the smallest nonnegative integer such that $\gamma - (r+1)\alpha \notin \Phi_\alpha$. Let us recall that if $\gamma - k\alpha \in \widehat{\Delta}$ for some $k \in \mathbb{Z}$ then $\gamma - k\alpha \in \Phi_\alpha$. Let us consider a vector $v \in M$. Then from the formula

$$a f_\alpha^{-n} = f_\alpha^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} f_\alpha^{-k} \mathrm{ad}(f_\alpha)^k(a)$$

in $U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}$ for $a \in U(\widehat{\mathfrak{g}}_\kappa)$ and $n \in \mathbb{N}_0$, we obtain

$$e_\gamma^t f_\alpha^{-n} v = f_\alpha^{-n} \sum_{k=0}^{tr} \binom{n+k-1}{k} f_\alpha^{-k} \mathrm{ad}(f_\alpha)^k(e_\gamma^t) v$$

for $e_\gamma \in \widehat{\mathfrak{g}}_{\kappa,\gamma}$, $t \in \mathbb{N}$ and $n \in \mathbb{N}_0$, where we used the fact that $\mathrm{ad}(f_\alpha)^k(e_\gamma) \neq 0$ only for $k = 0, 1, \ldots, r$. Moreover, we have $\mathrm{ad}(f_\alpha)^k(e_\gamma^t) \in U(\mathfrak{t}_\alpha)_{t\gamma - k\alpha} \subset U(\mathfrak{t}_\alpha)_{\mathrm{ht}(t\gamma - k\alpha)}$ for $k = 0, 1, \ldots, tr$. Since M is locally $\mathfrak{t}_\alpha^+$-finite, there exists an integer $n_v \in \mathbb{N}_0$ such that $U(\mathfrak{t}_\alpha)_n v = \{0\}$ for $n > n_v$. Therefore, it is enough to show that $\mathrm{ht}(t\gamma - k\alpha) > n_v$ for $k = 0, 1, \ldots, tr$.

As we may write

$$\mathrm{ht}(t\gamma - k\alpha) \geq \mathrm{ht}(t\gamma - tr\alpha) = t\mathrm{ht}(\gamma - r\alpha) \geq t$$

for $k = 0, 1, \ldots, tr$, since $\gamma - r\alpha \in \Phi_\alpha$ and hence $\mathrm{ht}(\gamma - r\alpha) \geq 1$, we obtain that $e_\gamma^t f_\alpha^{-n} v = 0$ for $n \in \mathbb{N}_0$ provided $t > n_v$. Hence, the element e_γ acts locally nilpotently on $T_\alpha(M)$ for $\gamma \in \Phi_\alpha$. □

The following shows how the twisting functor T_α can be used to construct α-Gelfand–Tsetlin modules with finite Γ_α-multiplicities for $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$.

Theorem 4.13. *Let $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$ and let M be a smooth weight $\widetilde{\mathfrak{g}}_\kappa$-module on which the central element c acts as the identity.*

(i) *The $\widetilde{\mathfrak{g}}_\kappa$-module $T_\alpha(M)$ is an α-Gelfand–Tsetlin module with finite Γ_α-multiplicities if and only if the first cohomology group $H^1(\mathfrak{s}_\alpha^-; M)$ is a weight $\widetilde{\mathfrak{h}}$-module with finite-dimensional weight spaces.*

(ii) *If M is a highest weight $\widetilde{\mathfrak{g}}_\kappa$-module, then $T_\alpha(M)$ is a locally $\mathfrak{t}_\alpha^-$-finite smooth cyclic weight $\widetilde{\mathfrak{g}}_\kappa$-module with finite Γ_α-multiplicities.*

Proof. (i) Let M be a smooth weight $\widetilde{\mathfrak{g}}_\kappa$-module. Then by Proposition 4.10 and Theorem 4.9 we obtain that $T_\alpha(M)$ is a locally $\mathfrak{s}_\alpha^-$-finite smooth weight $\widetilde{\mathfrak{g}}_\kappa$-module for $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$. Hence, we may apply Corollary 4.7 on $T_\alpha(M)$ and we get that $T_\alpha(M)$ is an α-Gelfand–Tsetlin module with finite Γ_α-multiplicities if and only if $H^0(s_\alpha^-; T_\alpha(M))$ is a weight $\widetilde{\mathfrak{h}}$-module with finite-dimensional weight spaces. Further, the linear mapping $\varphi_\alpha \colon M \to T_\alpha(M)$ defined by

$$\varphi_\alpha(v) = f_\alpha^{-1} v$$

for $v \in M$ gives rise to the linear mapping

$$\widetilde{\varphi}_\alpha \colon H^1(\mathfrak{s}_\alpha^-; M) \to H^0(\mathfrak{s}_\alpha^-; T_\alpha(M))$$

for $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$, which is in fact an isomorphism. Therefore, we obtain an isomorphism $H^1(\mathfrak{s}_\alpha^-; M) \simeq H^0(\mathfrak{s}_\alpha^-; T_\alpha(M)) \otimes_{\mathbb{C}} \mathbb{C}_{-\alpha}$ of $\widetilde{\mathfrak{h}}$-modules, where $\mathbb{C}_{-\alpha}$ is the 1-dimensional $\widetilde{\mathfrak{h}}$-module determined by the character $-\alpha$ of $\widetilde{\mathfrak{h}}$, which implies the first statement.

(ii) If M is a highest weight $\widetilde{\mathfrak{g}}_\kappa$-module, then it belongs to the category $\mathcal{O}(\widetilde{\mathfrak{g}}_\kappa)$. Hence, using Theorem 4.12 we obtain that $T_\alpha(M)$ is a locally $\mathfrak{t}_\alpha^-$-finite smooth weight $\widetilde{\mathfrak{g}}_\kappa$-module. In fact, from the proof of Theorem 4.12 (i) it follows that $T_\alpha(M)$ is not only finitely generated by also cyclic. To finish the proof, we need to show by Theorem 4.13 that $H^1(\mathfrak{s}_\alpha^-; M)$ is a weight $\widetilde{\mathfrak{h}}$-module with finite-dimensional weight spaces. Since $H^1(\mathfrak{s}_\alpha^-; M) \simeq M/\mathfrak{s}_\alpha^- M$ and M is a weight $\widetilde{\mathfrak{h}}$-module with finite-dimensional weight spaces, we immediately obtain that also $M/\mathfrak{s}_\alpha^- M$ is a weight $\widetilde{\mathfrak{h}}$-module with finite-dimensional weight spaces. This gives us the required statement. □

If we denote by $\Theta_\alpha \colon \mathcal{M}(\widetilde{\mathfrak{g}}_\kappa) \to \mathcal{M}(\widetilde{\mathfrak{g}}_\kappa)$ the functor sending a $\widetilde{\mathfrak{g}}_\kappa$-module to the same $\widetilde{\mathfrak{g}}_\kappa$-module with the action twisted by the automorphism $\mathrm{Ad}(\dot{r}_\alpha) \colon \widetilde{\mathfrak{g}}_\kappa \to \widetilde{\mathfrak{g}}_\kappa$, then we obtain the endofunctor

$$\Theta_\alpha \circ T_\alpha \colon \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{t}_\alpha^+) \to \mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{t}_\alpha^+).$$

Besides, for $\alpha \in \widehat{\Pi}$ we have $\mathfrak{t}_\alpha^+ = \widehat{\mathfrak{n}}_{\mathrm{st}}$ which implies $\mathcal{I}_f(\widetilde{\mathfrak{g}}_\kappa, \mathfrak{t}_\alpha^+) = \mathcal{O}(\widetilde{\mathfrak{g}}_\kappa)$ and in this case the functor coincides with the Arkhipov's twisting functor, see [Ark97, Ark04].

4.4. Tensoring with Weyl modules. In this subsection we show that the twisting functors behave well with respect to tensoring with certain $\widehat{\mathfrak{g}}_\kappa$-modules. In the finite-dimensional setting this was considered in [FK19b, AS03].

Let us recall that the universal enveloping algebra $U(\widehat{\mathfrak{g}}_\kappa)$ is a Hopf algebra with the comultiplication $\Delta\colon U(\widehat{\mathfrak{g}}_\kappa) \to U(\widehat{\mathfrak{g}}_\kappa) \otimes_{\mathbb{C}} U(\widehat{\mathfrak{g}}_\kappa)$, the counit $\varepsilon\colon U(\widehat{\mathfrak{g}}_\kappa) \to \mathbb{C}$ and the antipode $S\colon U(\widehat{\mathfrak{g}}_\kappa) \to U(\widehat{\mathfrak{g}}_\kappa)$ given by

$$\Delta(a) = a \otimes 1 + 1 \otimes a, \qquad \varepsilon(a) = 0, \qquad S(a) = -a$$

for $a \in \widehat{\mathfrak{g}}_\kappa$. For $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$, the localization $U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}$ has the structure of a left $\mathbb{C}[f_\alpha^{-1}]$-module, hence also $U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \otimes_{\mathbb{C}} U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}$ is a left $\mathbb{C}[f_\alpha^{-1}]$-module and we denote by $U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \mathbin{\widehat{\otimes}_{\mathbb{C}}} U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}$ its extension to a left $\mathbb{C}[[f_\alpha^{-1}]]$-module, i.e. we set

$$U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \mathbin{\widehat{\otimes}_{\mathbb{C}}} U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} = \mathbb{C}[[f_\alpha^{-1}]] \otimes_{\mathbb{C}[f_\alpha^{-1}]} U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \otimes_{\mathbb{C}} U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}.$$

There is an obvious extension of the algebra structure on $U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \otimes_{\mathbb{C}} U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}$ to the completion $U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \mathbin{\widehat{\otimes}_{\mathbb{C}}} U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}$. Then the linear mapping

$$\widetilde{\Delta}\colon U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \to U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \mathbin{\widehat{\otimes}_{\mathbb{C}}} U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)}$$

given through

$$\widetilde{\Delta}(f_\alpha^{-n} u) = \left(\sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} f_\alpha^{-n-k} \otimes f_\alpha^k \right) \Delta(u)$$

for $n \in \mathbb{N}_0$ and $u \in U(\widehat{\mathfrak{g}}_\kappa)$ defines an algebra homomorphism, see [AS03]. The following theorem is analogous to [AS03, Theorem 3.2].

Theorem 4.14. *Let $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$. Then there exists a family $\{\eta_E\}_{E \in \mathcal{M}(\widehat{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^-)}$ of natural isomorphisms*

$$\eta_E\colon T_\alpha \circ (\bullet \otimes_{\mathbb{C}} E) \to (\bullet \otimes_{\mathbb{C}} E) \circ T_\alpha$$

of functors, where $\mathcal{M}(\widehat{\mathfrak{g}}_\kappa, \mathfrak{s}_\alpha^-)$ is the category of locally $\mathfrak{s}_\alpha^-$-finite $\widehat{\mathfrak{g}}_\kappa$-modules. In other words, the twisting functor T_α for $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$ commutes with tensoring by a locally $\mathfrak{s}_\alpha^-$-finite $\widehat{\mathfrak{g}}_\kappa$-module E.

Remark 4.15. The previous theorem has a natural justification following from the fact that $T_\alpha(E) = 0$ for $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$ and a locally $\mathfrak{s}_\alpha^-$-finite $\widehat{\mathfrak{g}}_\kappa$-module E by Proposition 4.10.

By duality we have a similar statement if we replace $\widehat{\Delta}^{\mathrm{re}}_+$ by $\widehat{\Delta}^{\mathrm{re}}_-$ and $\mathfrak{s}_\alpha^-$ by $\mathfrak{s}_\alpha^+$. Besides, since Weyl modules are locally $\mathfrak{s}_\alpha^+$-finite and $\mathfrak{s}_\alpha^-$-finite $\widehat{\mathfrak{g}}_\kappa$-modules, Theorem 4.14 implies that for a root $\alpha \in \Delta \subset \widehat{\Delta}^{\mathrm{re}}$ the twisting functor T_α commutes with tensoring by Weyl modules.

4.5. Relaxed Verma modules. Let us recall that we have the induction functor

$$\mathbb{M}_{\kappa,\mathfrak{g}}\colon \mathcal{M}(\mathfrak{g}) \to \mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa) \tag{4.1}$$

introduced in Sect. 2.2 by $\mathbb{M}_{\kappa,\mathfrak{g}}(E) = U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\widehat{\mathfrak{g}}_{\mathrm{st}})} E$ for a $\mathfrak{g}$-module E, where E is considered as $\widehat{\mathfrak{g}}_{\mathrm{st}}$-module on which $\mathfrak{g} \otimes_{\mathbb{C}} t\mathbb{C}[[t]]$ acts trivially and c acts as the identity.

The link between the twisting functor T_α for $\alpha \in \Delta \subset \widehat{\Delta}^{\mathrm{re}}$ and the induction functor $\mathbb{M}_{\kappa,\mathfrak{g}}$ is given in the following theorem.

Theorem 4.16. *Let $\alpha \in \Delta \subset \widehat{\Delta}^{\mathrm{re}}$. Then there exists a natural isomorphism*

$$\eta_\alpha\colon T_\alpha \circ \mathbb{M}_{\kappa,\mathfrak{g}} \to \mathbb{M}_{\kappa,\mathfrak{g}} \circ T^{\mathfrak{g}}_\alpha$$

of functors, where $T^{\mathfrak{g}}_\alpha\colon \mathcal{M}(\mathfrak{g}) \to \mathcal{M}(\mathfrak{g})$ is the twisting functor for $\mathfrak{g}$ assigned to α. In particular, we have

$$T_\alpha(\mathbb{M}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda))) \simeq \mathbb{M}_{\kappa,\mathfrak{g}}(W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda,\alpha))$$

for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^{\mathfrak{u}}_+$.

Proof. We prove the statement only for $\alpha \in \Delta_+$, since for $\alpha \in \Delta_-$ the proof goes along the same lines. By using the triangular decomposition $\widehat{\mathfrak{g}}_\kappa = \widehat{\mathfrak{g}}_{\kappa,-} \oplus \widehat{\mathfrak{g}}_{\kappa,0} \oplus \widehat{\mathfrak{g}}_{\kappa,+}$, where

$$\widehat{\mathfrak{g}}_{\kappa,-} = \mathfrak{g} \otimes_{\mathbb{C}} t^{-1}\mathbb{C}[t^{-1}], \qquad \widehat{\mathfrak{g}}_{\kappa,0} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}1 \oplus \mathbb{C}c, \qquad \widehat{\mathfrak{g}}_{\kappa,+} = \mathfrak{g} \otimes_{\mathbb{C}} t\mathbb{C}[[t]],$$

and the Poincaré–Birkhoff–Witt theorem we get an isomorphism

$$U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \simeq U(\widehat{\mathfrak{g}}_{\kappa,-})_{(f_\alpha)} \otimes_{\mathbb{C}} U(\widehat{\mathfrak{g}}_{\mathrm{st}})$$

of $U(\widehat{\mathfrak{g}}_{\kappa,-})$-modules for $\alpha \in \Delta_+$. Hence, we may write

$$U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \otimes_{U(\widehat{\mathfrak{g}}_\kappa)} \mathbb{M}_{\kappa,\mathfrak{g}}(E) \simeq U(\widehat{\mathfrak{g}}_\kappa)_{(f_\alpha)} \otimes_{U(\widehat{\mathfrak{g}}_\kappa)} U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\widehat{\mathfrak{g}}_{\mathrm{st}})} E \simeq U(\widehat{\mathfrak{g}}_{\kappa,-})_{(f_\alpha)} \otimes_{\mathbb{C}} E$$

for a $\mathfrak{g}$-module E and $\alpha \in \Delta_+$, which gives us an isomorphism

$$(T_\alpha \circ \mathbb{M}_{\kappa,\mathfrak{g}})(E) \simeq (U(\widehat{\mathfrak{g}}_{\kappa,-})_{(f_\alpha)} / U(\widehat{\mathfrak{g}}_{\kappa,-})) \otimes_{\mathbb{C}} E$$

of $U(\widehat{\mathfrak{g}}_{\kappa,-})$-modules. On the other hand, for $\alpha \in \Delta_+$ we have an isomorphism of $U(\widehat{\mathfrak{g}}_{\kappa,-})$-modules

$$(\mathbb{M}_{\kappa,\mathfrak{g}} \circ T^{\mathfrak{g}}_\alpha)(E) \simeq U(\widehat{\mathfrak{g}}_{\kappa,-}) \otimes_{\mathbb{C}} T^{\mathfrak{g}}_\alpha(E).$$

Therefore, we define the isomorphism $\eta_{\alpha,E}$ of $U(\widehat{\mathfrak{g}}_{\kappa,-})$-modules by

$$f^{-n}_\alpha u \otimes v \mapsto \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} \mathrm{ad}(f_\alpha)^k(u) \otimes f^{-n-k}_\alpha v$$

with the inverse

$$u \otimes f^{-n}_\alpha v \mapsto \sum_{k=0}^{\infty} \binom{n+k-1}{k} f^{-n-k}_\alpha \mathrm{ad}(f_\alpha)^k(u) \otimes v$$

for $u \in U(\widehat{\mathfrak{g}}_{\kappa,-})$, $v \in E$ and $n \in \mathbb{N}$. It is straightforward to check that $\eta_{\alpha,E}$ is an isomorphism of $U(\widehat{\mathfrak{g}}_\kappa)$-modules.

The rest of the statement follows from the fact that $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda,\alpha) = T^{\mathfrak{g}}_\alpha(M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda))$ for $\lambda \in \Lambda^+(\mathfrak{p})$ and $\alpha \in \Delta^{\mathfrak{u}}_+$ by definition. □

5. Relaxed Wakimoto Modules

We introduce a class of positive energy $\widehat{\mathfrak{g}}_\kappa$-modules which we will call relaxed Wakimoto modules and also give a free field realization of relaxed Verma modules.

5.1. Feigin–Frenkel homomorphism. Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra. Let us consider a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ with the nilradical $\mathfrak{n}$, the opposite nilradical $\overline{\mathfrak{n}}$ and the Cartan subalgebra $\mathfrak{h}$. Let $\{f_\alpha;\ \alpha \in \Delta_+\}$ be a root basis of the opposite nilradical $\overline{\mathfrak{n}}$. We denote by $\{x_\alpha;\ \alpha \in \Delta_+\}$ the linear coordinate functions on $\overline{\mathfrak{n}}$ with respect to the given basis of $\overline{\mathfrak{n}}$. Then the Weyl algebra $\mathcal{A}_{\overline{\mathfrak{n}}}$ of the vector space $\overline{\mathfrak{n}}$ is generated by $\{x_\alpha, \partial_{x_\alpha};\ \alpha \in \Delta_+\}$ together with the canonical commutation relations. Further, by [KS17] there exists a homomorphism

$$\pi_{\mathfrak{g}} \colon U(\mathfrak{g}) \to \mathcal{A}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} U(\mathfrak{h}) \tag{5.1}$$

of associative algebras uniquely determined by

$$\pi_{\mathfrak{g}}(a) = -\sum_{\alpha\in\Delta_+} \left[\frac{\operatorname{ad}(u(x))e^{\operatorname{ad}(u(x))}}{e^{\operatorname{ad}(u(x))} - \operatorname{id}} (e^{-\operatorname{ad}(u(x))}a)_{\overline{\mathfrak{n}}} \right]_\alpha \partial_{x_\alpha} + (e^{-\operatorname{ad}(u(x))}a)_{\mathfrak{h}} \tag{5.2}$$

for $a \in \mathfrak{g}$, where $[a]_\alpha$ denotes the α-th coordinate of $a \in \overline{\mathfrak{n}}$ with respect to the basis $\{f_\alpha;\ \alpha \in \Delta_+\}$ of $\overline{\mathfrak{n}}$, $a_{\overline{\mathfrak{n}}}$ and $a_{\mathfrak{h}}$ are the $\overline{\mathfrak{n}}$-part and $\mathfrak{h}$-part of $a \in \mathfrak{g}$ with respect to the triangular decomposition $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$, and the element $u(x) \in \mathbb{C}[\overline{\mathfrak{n}}] \otimes_{\mathbb{C}} \mathfrak{g}$ is given by

$$u(x) = \sum_{\alpha\in\Delta_+} x_\alpha f_\alpha.$$

Let us note that $\mathbb{C}[\overline{\mathfrak{n}}] \otimes_{\mathbb{C}} \mathfrak{g}$ has the natural structure of a Lie algebra. Hence, we have a well-defined linear mapping $\operatorname{ad}(u(x)) \colon \mathbb{C}[\overline{\mathfrak{n}}] \otimes_{\mathbb{C}} \mathfrak{g} \to \mathbb{C}[\overline{\mathfrak{n}}] \otimes_{\mathbb{C}} \mathfrak{g}$.

Let us recall that by Sect. 3.4 we may assign to the vector space $\overline{\mathfrak{n}}$ the Weyl vertex algebra $\mathcal{M}_{\overline{\mathfrak{n}}}$ generated by the fields

$$a_\alpha(z) = \sum_{n\in\mathbb{Z}} \partial_{x_{\alpha,n}} z^{-n-1} \qquad \text{and} \qquad a^*_\alpha(z) = \sum_{n\in\mathbb{Z}} x_{\alpha,-n} z^{-n}$$

for $\alpha \in \Delta_+$.

The following theorem can be deduced from [Fre05, Theorem 5.1].

Theorem 5.1. *Let κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$. Then there exists a homomorphism*

$$w_{\kappa,\mathfrak{g}} \colon \mathcal{V}_\kappa(\mathfrak{g}) \to \mathcal{M}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} \mathcal{V}_{\kappa-\kappa_c}(\mathfrak{h})$$

of $\mathbb{N}_0$-graded vertex algebras such that

$$\begin{aligned} w_{\kappa,\mathfrak{g}}(e_\gamma(z)) &= -\sum_{\alpha\in\Delta_+} {:}q^\gamma_\alpha(a^*_\beta(z))a_\alpha(z){:} - (c_\gamma + (\kappa-\kappa_c)(e_\gamma, f_\gamma))\partial_z a^*_\gamma(z) \\ &\quad + a^*_\gamma(z)h_\gamma(z), \\ w_{\kappa,\mathfrak{g}}(h_\gamma(z)) &= \sum_{\alpha\in\Delta_+} \alpha(h_\gamma){:}a^*_\alpha(z)a_\alpha(z){:} + h_\gamma(z), \end{aligned}$$

$$w_{\kappa,\mathfrak{g}}(f_\gamma(z)) = -a_\gamma(z) - \sum_{\alpha\in\Delta_+} :p_\alpha^\gamma(a_\beta^*(z))a_\alpha(z):$$

for $\gamma \in \Pi$, where $c_\gamma \in \mathbb{C}$ are constants and the polynomials $p_\alpha^\gamma, q_\alpha^\gamma \in \mathbb{C}[\overline{\mathfrak{n}}]$ are given by

$$p_\alpha^\gamma(x_\beta) = \left[\left(\frac{\mathrm{ad}(u(x))}{e^{\mathrm{ad}(u(x))} - \mathrm{id}} - \mathrm{id}\right) f_\gamma\right]_\alpha, \qquad q_\alpha^\gamma(x_\beta) = \left[\frac{\mathrm{ad}(u(x))e^{\mathrm{ad}(u(x))}}{e^{\mathrm{ad}(u(x))} - \mathrm{id}} (e^{-\mathrm{ad}(u(x))}e_\gamma)_{\overline{\mathfrak{n}}}\right]_\alpha$$

for $\gamma \in \Pi$ and $\alpha \in \Delta_+$.

The link between the Feigin–Frenkel homomorphism $w_{\kappa,\mathfrak{g}}$ of vertex algebras and the homomorphism $\pi_\mathfrak{g}$ of associative algebras is given by the following theorem.

Theorem 5.2. *Let κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$. Then the diagram*

$$\begin{array}{ccc} \mathcal{V}_\kappa(\mathfrak{g}) & \xrightarrow{w_{\kappa,\mathfrak{g}}} & \mathcal{M}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} \mathcal{V}_{\kappa-\kappa_c}(\mathfrak{h}) \\ \downarrow{\scriptstyle \pi_{\mathrm{Zhu}}} & & \downarrow{\scriptstyle \pi_{\mathrm{Zhu}}} \\ U(\mathfrak{g}) & \xrightarrow[\pi_\mathfrak{g}]{} & \mathcal{A}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} U(\mathfrak{h}) \end{array} \tag{5.3}$$

is commutative.

Proof. As $w_{\kappa,\mathfrak{g}} \colon \mathcal{V}_\kappa(\mathfrak{g}) \to \mathcal{M}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} \mathcal{V}_{\kappa-\kappa_c}(\mathfrak{h})$ is a homomorphism of $\mathbb{N}_0$-graded vertex algebras by Theorem 5.1, we obtain a homomorphism $\overline{w}_{\kappa,\mathfrak{g}} \colon U(\mathfrak{g}) \to \mathcal{A}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} U(\mathfrak{h})$ of the corresponding Zhu's algebras, since we have $A(\mathcal{V}_\kappa(\mathfrak{g})) \simeq U(\mathfrak{g})$ and $A(\mathcal{M}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} \mathcal{V}_{\kappa-\kappa_c}(\mathfrak{h})) \simeq \mathcal{A}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} U(\mathfrak{h})$. Moreover, we have $\pi_{\mathrm{Zhu}} \circ w_{\kappa,\mathfrak{g}} = \overline{w}_{\kappa,\mathfrak{g}} \circ \pi_{\mathrm{Zhu}}$. Therefore, we need only to show that $\overline{w}_{\kappa,\mathfrak{g}} = \pi_\mathfrak{g}$. We may write

$$\begin{aligned} \overline{w}_{\kappa,\mathfrak{g}}(e_\gamma) &= \overline{w}_{\kappa,\mathfrak{g}}(\pi_{\mathrm{Zhu}}(e_{\gamma,-1}|0\rangle)) = \pi_{\mathrm{Zhu}}(w_{\kappa,\mathfrak{g}}(e_{\gamma,-1}|0\rangle)) \\ &= \pi_{\mathrm{Zhu}}\Big(-\sum_{\alpha\in\Delta_+} q_\alpha^\gamma(x_{\beta,0})\partial_{x_{\alpha,-1}}|0\rangle - (c_\gamma + (\kappa-\kappa_c)(e_\gamma, f_\gamma))x_{\gamma,1}|0\rangle + x_{\gamma,0}h_{\gamma,-1}|0\rangle\Big) \\ &= -\sum_{\alpha\in\Delta_+} q_\alpha^\gamma(x_\beta)\partial_{x_\alpha} + x_\gamma h_\gamma = \pi_\mathfrak{g}(e_\gamma) \end{aligned}$$

and

$$\begin{aligned} \bar{w}_{\kappa,\mathfrak{g}}(f_\gamma) &= \bar{w}_{\kappa,\mathfrak{g}}(\pi_{\mathrm{Zhu}}(f_{\gamma,-1}|0\rangle)) = \pi_{\mathrm{Zhu}}(w_{\kappa,\mathfrak{g}}(f_{\gamma,-1}|0\rangle)) \\ &= \pi_{\mathrm{Zhu}}\Big(-\partial_{x_{\gamma,-1}}|0\rangle - \sum_{\alpha\in\Delta_+} p_\alpha^\gamma(x_{\beta,0})\partial_{x_{\alpha,-1}}|0\rangle\Big) \\ &= -\partial_{x_\gamma} - \sum_{\alpha\in\Delta_+} p_\alpha^\gamma(x_\beta)\partial_{x_\alpha} = \pi_\mathfrak{g}(f_\gamma) \end{aligned}$$

for $\gamma \in \Pi$, which immediately implies that $\overline{w}_{\kappa,\mathfrak{g}} = \pi_\mathfrak{g}$. □

Let us note that an explicit form of the homomorphism $w_{\kappa,\mathfrak{g}}$ in Theorem 5.1 is given only for the generators of the vertex algebra. An alternative approach based on the Hamiltonian reduction of the WZNW model was considered in [dBF97]. However, we will need explicit formulas for all elements of the opposite nilradical $\overline{\mathfrak{n}}$ which will be established in the next theorem.

Let us denote by $\mathcal{P}_{\mathrm{loc}}^{\mathfrak{g},\mathfrak{b}}(z)$ the vector space of all polynomials in $a_\alpha^*(z)$ for $\alpha \in \Delta_+$ and by $\mathcal{F}_{\mathrm{loc}}^{\mathfrak{g},\mathfrak{b}}(z)$ the vector space of all differential polynomials in $a_\alpha^*(z)$ for $\alpha \in \Delta_+$. We define a formal power series $u(z) \in \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{P}_{\mathrm{loc}}^{\mathfrak{g},\mathfrak{b}}(z)$ by

$$u(z) = \sum_{\alpha\in\Delta_+} a_\alpha^*(z) f_\alpha. \tag{5.4}$$

Besides, the vector space $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{F}_{\mathrm{loc}}^{\mathfrak{g},\mathfrak{b}}(z)$ has the natural structure of a Lie algebra.

We recall the following statement from [FKS19].

Proposition 5.3 [FKS19, Proposition 3.8]. *We have the identities*

(1)

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} e^{\mathrm{ad}(u(z)+tx(z))}\right) e^{-\mathrm{ad}(u(z))} = \mathrm{ad}\left(\frac{e^{\mathrm{ad}(u(z))}-\mathrm{id}}{\mathrm{ad}(u(z))}\, x(z)\right)$$

for $x(z) \in \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{F}_{\mathrm{loc}}^{\mathfrak{g},\mathfrak{b}}(z)$,

(2)

$$\begin{aligned}&\left[\frac{e^{\mathrm{ad}(u(z))}-\mathrm{id}}{\mathrm{ad}(u(z))}\, x(z), \frac{e^{\mathrm{ad}(u(z))}-\mathrm{id}}{\mathrm{ad}(u(z))}\, y(z)\right]\\ &\quad= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{e^{\mathrm{ad}(u(z)+tx(z))}-\mathrm{id}}{\mathrm{ad}(u(z)+tx(z))}\, y(z) - \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{e^{\mathrm{ad}(u(z)+ty(z))}-\mathrm{id}}{\mathrm{ad}(u(z)+ty(z))}\, x(z)\end{aligned}$$

for $x(z), y(z) \in \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{F}_{\mathrm{loc}}^{\mathfrak{g},\mathfrak{b}}(z)$.

Let us introduce an element of $\overline{\mathfrak{n}} \otimes \mathcal{P}_{\mathrm{loc}}^{\mathfrak{g},\mathfrak{b}}(z)$ by

$$T(a,z) = \frac{\mathrm{ad}(u(z))}{e^{\mathrm{ad}(u(z))}-\mathrm{id}}\, a$$

for $a \in \overline{\mathfrak{n}}$. Then we may write

$$T(a,z) = \sum_{\alpha\in\Delta_+} T_\alpha(a,z) f_\alpha,$$

where $T_\alpha(a,z) = [T(a,z)]_\alpha$ for $\alpha \in \Delta_+$. Further, we define the linear mapping

$$dT(a,z) \colon \overline{\mathfrak{n}} \otimes_{\mathbb{C}} \mathcal{F}_{\mathrm{loc}}^{\mathfrak{g},\mathfrak{b}}(z) \to \overline{\mathfrak{n}} \otimes_{\mathbb{C}} \mathcal{F}_{\mathrm{loc}}^{\mathfrak{g},\mathfrak{b}}(z)$$

by

$$dT(a,z)(x(z)) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{ad}(u(z)+tx(z))}{e^{\mathrm{ad}(u(z)+tx(z))}-\mathrm{id}}\, a$$

for $x(z) \in \overline{\mathfrak{n}} \otimes_{\mathbb{C}} \mathcal{F}_{\mathrm{loc}}^{\mathfrak{g},\mathfrak{b}}(z)$. We have also

$$dT(a,z) = \sum_{\alpha\in\Delta_+} dT_\alpha(a,z) f_\alpha,$$

where $dT_\alpha(a,z) = [dT(a,z)]_\alpha$ for $\alpha \in \Delta_+$.

Lemma 5.4. *We have*

$$[a_\alpha(z), u(w)] = f_\alpha \delta(z-w)$$

and

$$[a_\alpha(z), T(a,w)] = dT(a,w)(f_\alpha)\delta(z-w)$$

for $\alpha \in \Delta_+$ *and* $a \in \overline{\mathfrak{n}}$.

Proof. By definition of $u(w)$ we have

$$[a_\alpha(z), u(w)] = \sum_{\beta\in\Delta_+} [a_\alpha(z), a^*_\beta(w)] f_\beta = \sum_{\beta\in\Delta_+} \delta_{\alpha,\beta} f_\beta \delta(z-w) = f_\alpha \delta(z-w).$$

for $\alpha \in \Delta_+$. Further, we may write

$$[a_\alpha(z), T(a,w)] = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{ad}(u(w)+tf_\alpha)}{e^{\mathrm{ad}(u(w)+tf_\alpha)}-\mathrm{id}}\, a\, \delta(z-w) = dT(a,w)(f_\alpha)\delta(z-w)$$

for $\alpha \in \Delta_+$ and $a \in \overline{\mathfrak{n}}$. □

Theorem 5.5. *Let* κ *be a* $\mathfrak{g}$*-invariant symmetric bilinear form on* $\mathfrak{g}$. *Then we have*

$$w_{\kappa,\mathfrak{g}}(a(z)) = -\sum_{\alpha\in\Delta_+} :\left[\frac{\mathrm{ad}(u(z))}{e^{\mathrm{ad}(u(z))}-\mathrm{id}}\, a\right]_\alpha a_\alpha(z): \tag{5.5}$$

for $a \in \overline{\mathfrak{n}}$ *and*

$$w_{\kappa,\mathfrak{g}}(a(z)) = \sum_{\alpha\in\Delta_+} :[\mathrm{ad}(u(z))(a)]_\alpha a_\alpha(z): + a(z) \tag{5.6}$$

for $a \in \mathfrak{h}$.

Proof. From Theorem 5.1 we know that the statement holds for all $a \in \mathfrak{h}$ and for the root vectors $f_\gamma \in \overline{\mathfrak{n}}$ for $\gamma \in \Pi$. Hence, it is enough to show that $w_{\kappa,\mathfrak{g}}$ given by (5.5) gives rise to a homomorphism of vertex algebras from $\mathcal{V}_\kappa(\overline{\mathfrak{n}})$ to $\mathcal{M}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} \mathcal{V}_{\kappa-\kappa_c}(\mathfrak{h})$.

By using the relation (3.11), we have

$$\begin{aligned} w_{\kappa,\mathfrak{g}}([a(z), b(w)]) &= w_{\kappa,\mathfrak{g}}([a,b](w))\delta(z-w) + \kappa(a,b)\partial_w \delta(z-w) \\ &= w_{\kappa,\mathfrak{g}}([a,b](w))\delta(z-w) \\ &= -\sum_{\alpha\in\Delta_+} :T_\alpha([a,b],w)a_\alpha(w):\delta(z-w) \end{aligned}$$

for $a, b \in \overline{\mathfrak{n}}$. On the other hand, we get

$$\begin{aligned} [w_{\kappa,\mathfrak{g}}(a(z)), w_{\kappa,\mathfrak{g}}(b(w))] &= \sum_{\alpha,\beta\in\Delta_+} [:T_\alpha(a,z)a_\alpha(z):, :T_\beta(b,w)a_\beta(w):] \\ &= \sum_{\alpha,\beta\in\Delta_+} :dT_\beta(b,w)(f_\alpha)T_\alpha(a,z)a_\beta(w):\delta(z-w) \\ &\quad - \sum_{\alpha,\beta\in\Delta_+} :dT_\alpha(a,z)(f_\beta)T_\beta(b,w)a_\alpha(z):\delta(z-w) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\alpha,\beta\in\Delta_+} dT_\beta(b,w)(f_\alpha)dT_\alpha(a,z)(f_\beta)\partial_w\delta(z-w)\\
&= \sum_{\alpha\in\Delta_+} :dT_\alpha(b,w)(T(a,w))a_\alpha(w):\delta(z-w)\\
&\quad - \sum_{\alpha\in\Delta_+} :dT_\alpha(a,w)(T(b,w))a_\alpha(w):\delta(z-w)\\
&\quad - \sum_{\alpha,\beta\in\Delta_+} dT_\beta(b,w)(f_\alpha)dT_\alpha(a,w)(f_\beta)\partial_w\delta(z-w)\\
&\quad - \sum_{\alpha,\beta\in\Delta_+} dT_\beta(b,w)(f_\alpha)\partial_w dT_\alpha(a,w)(f_\beta)\delta(z-w),
\end{aligned}
$$

where we used the Wick theorem and Lemma 5.4. Further, for $a, b \in \overline{\mathfrak{n}}$ we may write

$$
\begin{aligned}
\sum_{\alpha,\beta\in\Delta_+} dT_\beta(b,w)(f_\alpha)dT_\alpha(a,w)(f_\beta) &= \sum_{\beta\in\Delta_+} dT_\beta(b,w)(dT(a,w)(f_\beta))\\
&= \mathrm{tr}_{\overline{\mathfrak{n}}}(dT(b,w)\circ dT(a,w))
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{\alpha,\beta\in\Delta_+} dT_\beta(b,w)(f_\alpha)\partial_w dT_\alpha(a,w)(f_\beta) &= \sum_{\beta\in\Delta_+} dT_\beta(b,w)(\partial_w dT(a,w)(f_\beta))\\
&= \mathrm{tr}_{\overline{\mathfrak{n}}}(dT(b,w)\circ \partial_w dT(a,w)).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
[w_{\kappa,\mathfrak{g}}(a(z)), w_{\kappa,\mathfrak{g}}(b(w))] &= \sum_{\alpha\in\Delta_+} :dT_\alpha(b,w)(T(a,w))a_\alpha(w):\delta(z-w)\\
&\quad - \sum_{\alpha\in\Delta_+} :dT_\alpha(a,w)(T(b,w))a_\alpha(w):\delta(z-w)\\
&\quad - \mathrm{tr}_{\overline{\mathfrak{n}}}(dT(b,w)\circ dT(a,w))\partial_w\delta(z-w)\\
&\quad - \mathrm{tr}_{\overline{\mathfrak{n}}}(dT(b,w)\circ \partial_w dT(a,w))\delta(z-w)
\end{aligned}
$$

for $a, b \in \overline{\mathfrak{n}}$. Therefore, it is enough to show that

$$
T_\alpha([a,b],w) = dT_\alpha(a,w)(T(b,w)) - dT_\alpha(b,w)(T(a,w))
$$

for $\alpha \in \Delta_+$ and $a, b \in \overline{\mathfrak{n}}$, or equivalently

$$
T([a,b],w) = dT(a,w)(T(b,w)) - dT(b,w)(T(a,w))
$$

for $a, b \in \overline{\mathfrak{n}}$, and that

$$
\mathrm{tr}_{\overline{\mathfrak{n}}}(dT(b,w)\circ dT(a,w)) = 0, \qquad \mathrm{tr}_{\overline{\mathfrak{n}}}(dT(b,w)\circ \partial_w dT(a,w)) = 0
$$

for $a, b \in \overline{\mathfrak{n}}$.

A proof of the previous system of equations is a subject of the following lemmas, which then completes the proof of the present theorem. □

Lemma 5.6. *We have*

$$\mathrm{tr}_{\overline{\mathfrak{n}}}(dT(b,z)\circ dT(a,z))=0 \quad \textit{and} \quad \mathrm{tr}_{\overline{\mathfrak{n}}}(dT(b,z)\circ \partial_z dT(a,z))=0$$

for $a,b\in\overline{\mathfrak{n}}$.

Proof. Since $\overline{\mathfrak{n}}$ is a nilpotent Lie algebra, we have a canonical filtration

$$\overline{\mathfrak{n}}=\overline{\mathfrak{n}}_0\supset\overline{\mathfrak{n}}_1\supset\cdots\supset\overline{\mathfrak{n}}_n\supset\overline{\mathfrak{n}}_{n+1}=0$$

on $\overline{\mathfrak{n}}$ given by the lower central series of $\overline{\mathfrak{n}}$, i.e. $\overline{\mathfrak{n}}_k=[\overline{\mathfrak{n}},\overline{\mathfrak{n}}_{k-1}]$ for $k\in\mathbb{N}$ with $\overline{\mathfrak{n}}_0=\overline{\mathfrak{n}}$. Moreover, we have

$$dT(a,z)\colon \overline{\mathfrak{n}}_k\otimes_{\mathbb{C}}\mathcal{F}^{\mathfrak{g},\mathfrak{b}}_{\mathrm{loc}}(z)\to\overline{\mathfrak{n}}_{k+1}\otimes_{\mathbb{C}}\mathcal{F}^{\mathfrak{g},\mathfrak{b}}_{\mathrm{loc}}(z)$$

and

$$\partial_z dT(a,z)\colon \overline{\mathfrak{n}}_k\otimes_{\mathbb{C}}\mathcal{F}^{\mathfrak{g},\mathfrak{b}}_{\mathrm{loc}}(z)\to\overline{\mathfrak{n}}_{k+1}\otimes_{\mathbb{C}}\mathcal{F}^{\mathfrak{g},\mathfrak{b}}_{\mathrm{loc}}(z)$$

for $a\in\overline{\mathfrak{n}}$, which implies immediately that

$$\mathrm{tr}_{\overline{\mathfrak{n}}}(dT(b,z)\circ dT(a,z))=0 \quad \text{and} \quad \mathrm{tr}_{\overline{\mathfrak{n}}}(dT(b,z)\circ \partial_z dT(a,z))=0$$

for all $a,b\in\overline{\mathfrak{n}}$. □

Lemma 5.7. *We have*

$$T([a,b],z)=dT(a,z)(T(b,z))-dT(b,z)(T(a,z))$$

for $a,b\in\overline{\mathfrak{n}}$.

Proof. For $a,b\in\overline{\mathfrak{n}}$, we may write

$$\begin{aligned}
dT(a,z)(T(b,z)) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\frac{\mathrm{ad}(u(z)+tT(b,z))}{e^{\mathrm{ad}(u(z)+tT(b,z))}-\mathrm{id}}\,a\\
&= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\frac{\mathrm{ad}(u(z)+tT(b,z))}{e^{\mathrm{ad}(u(z)+tT(b,z))}-\mathrm{id}}\frac{e^{\mathrm{ad}(u(z))}-\mathrm{id}}{\mathrm{ad}(u(z))}\frac{\mathrm{ad}(u(z))}{e^{\mathrm{ad}(u(z))}-\mathrm{id}}\,a\\
&= -\frac{\mathrm{ad}(u(z))}{e^{\mathrm{ad}(u(z))}-\mathrm{id}}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\frac{e^{\mathrm{ad}(u(z)+tT(b,z))}-\mathrm{id}}{\mathrm{ad}(u(z)+tT(b,z))}\,T(a,z),
\end{aligned}$$

which gives us

$$\begin{aligned}
&dT(a,z)(T(b,z))-dT(b,z)(T(a,z))\\
&\quad= -\frac{\mathrm{ad}(u(z))}{e^{\mathrm{ad}(u(z))}-\mathrm{id}}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\frac{e^{\mathrm{ad}(u(z)+tT(b,z))}-\mathrm{id}}{\mathrm{ad}(u(z)+tT(b,z))}\,T(a,z)\\
&\qquad+\frac{\mathrm{ad}(u(z))}{e^{\mathrm{ad}(u(z))}-\mathrm{id}}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\frac{e^{\mathrm{ad}(u(z)+tT(a,z))}-\mathrm{id}}{\mathrm{ad}(u(z)+tT(b,z))}\,T(b,z).
\end{aligned}$$

By using Proposition 5.3, we obtain immediately

$$\begin{aligned}
&dT(a,z)(T(b,z))-dT(b,z)(T(a,z))\\
&\quad=\frac{\mathrm{ad}(u(z))}{e^{\mathrm{ad}(u(z))}-\mathrm{id}}\left[\frac{e^{\mathrm{ad}(u(z))}-\mathrm{id}}{\mathrm{ad}(u(z))}\,T(a,z),\frac{e^{\mathrm{ad}(u(z))}-\mathrm{id}}{\mathrm{ad}(u(z))}\,T(b,z)\right]\\
&\quad=\frac{\mathrm{ad}(u(z))}{e^{\mathrm{ad}(u(z))}-\mathrm{id}}\,[a,b]=T([a,b],z)
\end{aligned}$$

for $a,b\in\overline{\mathfrak{n}}$. □

5.2. Wakimoto functor. We introduce a functor $\mathbb{W}_{\kappa,\mathfrak{g}}$ from a certain subcategory of $\mathcal{M}(\mathfrak{g})$ to the category $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$. For a generic level κ this functor provides a free field realization of the induction functor $\mathbb{M}_{\kappa,\mathfrak{g}}$.

A straightforward reformulation of Theorems 5.1 and 5.5 by using the completed Weyl algebra $\widetilde{\mathcal{A}}_{\mathcal{K}(\overline{\mathfrak{n}})}$ and the universal enveloping algebras $U(\widehat{\mathfrak{g}}_\kappa)$, $U(\widehat{\mathfrak{h}}_{\kappa-\kappa_c})$ gives us the following statement.

Theorem 5.8. *Let κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$. Then there exists a homomorphism*

$$\pi_{\kappa,\mathfrak{g}}\colon U(\widehat{\mathfrak{g}}_\kappa) \to \widetilde{\mathcal{A}}_{\mathcal{K}(\overline{\mathfrak{n}})}\widehat{\otimes}_{\mathbb{C}}\, U(\widehat{\mathfrak{h}}_{\kappa-\kappa_c})$$

of associative $\mathbb{Z}$-graded algebras such that

$$\pi_{\kappa,\mathfrak{g}}(e_\gamma(z)) = -\sum_{\alpha\in\Delta_+} :q_\alpha^\gamma(a_\beta^*(z))a_\alpha(z): - (c_\gamma+(\kappa-\kappa_c)(e_\gamma,\, f_\gamma))\partial_z a_\gamma^*(z)+a_\gamma^*(z)h_\gamma(z) \tag{5.7}$$

for $\gamma\in\Pi$, where $c_\gamma\in\mathbb{C}$ are constants and the polynomials $q_\alpha^\gamma\in\mathbb{C}[\overline{\mathfrak{n}}]$ are given by

$$q_\alpha^\gamma(x_\beta) = \left[\frac{\mathrm{ad}(u(x))e^{\mathrm{ad}(u(x))}}{e^{\mathrm{ad}(u(x))}-\mathrm{id}}\,(e^{-\mathrm{ad}(u(x))}e_\gamma)_{\overline{\mathfrak{n}}}\right]_\alpha$$

for $\gamma\in\Pi$ and $\alpha\in\Delta_+$. Further, we have

$$\pi_{\kappa,\mathfrak{g}}(a(z)) = -\sum_{\alpha\in\Delta_+} :\left[\frac{\mathrm{ad}(u(z))}{e^{\mathrm{ad}(u(z))}-\mathrm{id}}\,a\right]_\alpha a_\alpha(z): \tag{5.8}$$

for $a\in\overline{\mathfrak{n}}$ and

$$\pi_{\kappa,\mathfrak{g}}(a(z)) = \sum_{\alpha\in\Delta_+} :[\mathrm{ad}(u(z))(a)]_\alpha a_\alpha(z): + a(z) \tag{5.9}$$

for $a\in\mathfrak{h}$.

Definition 5.9. Let N be an $\mathcal{A}_{\overline{\mathfrak{n}}}$-module and E be an $\mathfrak{h}$-module. Then the smooth $\widehat{\mathfrak{g}}_\kappa$-module

$$\mathbb{W}_{\kappa,\mathfrak{g}}(N\otimes_{\mathbb{C}} E) = \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)\otimes_{\mathbb{C}}\mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E)$$

is called the *relaxed Wakimoto module* induced from the $\mathfrak{g}$-module $N\otimes_{\mathbb{C}} E$.

Let us note that it is easy to see that $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)\simeq\mathbb{L}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)$ for any $\mathcal{A}_{\overline{\mathfrak{n}}}$-module N and that $\mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E)\simeq\mathbb{L}_{\kappa-\kappa_c,\mathfrak{h}}(E)$ for any $\mathfrak{h}$-module E provided κ is a non-critical level.

Theorem 5.10. *Let N be an $\mathcal{A}_{\overline{\mathfrak{n}}}$-module and E be an $\mathfrak{h}$-module. Then $\mathbb{W}_{\kappa,\mathfrak{g}}(N\otimes_{\mathbb{C}} E)$ is a positive energy $\widehat{\mathfrak{g}}_\kappa$-module and the top degree component of $\mathbb{W}_{\kappa,\mathfrak{g}}(N\otimes_{\mathbb{C}} E)$ is isomorphic to $N\otimes_{\mathbb{C}} E$ as a $\mathfrak{g}$-module. Moreover, there is a non-trivial homomorphism*

$$\mathbb{M}_{\kappa,\mathfrak{g}}(N\otimes_{\mathbb{C}} E)\to\mathbb{W}_{\kappa,\mathfrak{g}}(N\otimes_{\mathbb{C}} E)$$

of $\widehat{\mathfrak{g}}_\kappa$-modules. If both $\mathbb{M}_{\kappa,\mathfrak{g}}(N\otimes_{\mathbb{C}} E)$ and $\mathbb{W}_{\kappa,\mathfrak{g}}(N\otimes_{\mathbb{C}} E)$ are simple $\widehat{\mathfrak{g}}_\kappa$-modules, then the latter homomorphism is in fact an isomorphism.

Proof. By definition we have that $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)$ and $\mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E)$ are a positive energy $\mathcal{A}_{\mathcal{K}(\overline{\mathfrak{n}})}$-module and a positive energy $\widehat{\mathfrak{h}}_{\kappa-\kappa_c}$-module, respectively. Hence, we have

$$\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N) = \bigoplus_{n=0}^{\infty} \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_n \qquad \text{and} \qquad \mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E) = \bigoplus_{n=0}^{\infty} \mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E)_n$$

with $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_0 \simeq N$ as $\mathcal{A}_{\overline{\mathfrak{n}}}$-modules and $\mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E)_0 \simeq E$ as $\mathfrak{h}$-modules, which gives us the gradation on $\mathbb{W}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E)$ defined by

$$\mathbb{W}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E)_n = \bigoplus_{k=0}^{n} \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{n-k} \otimes_{\mathbb{C}} \mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E)_k$$

for $n \in \mathbb{N}_0$. Further, from Theorem 5.8 we obtain that the gradation on $\mathbb{W}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E)$ is compatible with the grading of $\widehat{\mathfrak{g}}_\kappa$, which implies that $\mathbb{W}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E)$ is a positive energy $\widehat{\mathfrak{g}}_\kappa$-module whose top degree component is isomorphic to $N \otimes_{\mathbb{C}} E$ as a $\mathfrak{g}$-module. Hence, we have a homomorphism $N \otimes_{\mathbb{C}} E \to \mathbb{W}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E)$ of $\mathfrak{g}$-modules which gives rise to a homomorphism

$$\mathbb{M}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E) \to \mathbb{W}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E)$$

of $\widehat{\mathfrak{g}}_\kappa$-modules by the universal property of the generalized Verma module $\mathbb{M}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E)$. □

The homomorphism $\pi_{\mathfrak{g}}$ of associative algebras gives rise to a bifunctor

$$\Phi_{\mathfrak{g}} \colon \mathcal{M}(\mathcal{A}_{\overline{\mathfrak{n}}}) \times \mathcal{M}(\mathfrak{h}) \to \mathcal{M}(\mathfrak{g})$$

defined by

$$\Phi_{\mathfrak{g}}(N, E) = N \otimes_{\mathbb{C}} E$$

for an $\mathcal{A}_{\overline{\mathfrak{n}}}$-module N and $\mathfrak{h}$-module E, where the $\mathfrak{g}$-module structure on $N \otimes_{\mathbb{C}} E$ is given by $\pi_{\mathfrak{g}}$. This enable us to consider $\mathbb{W}_{\kappa,\mathfrak{g}}$ as a functor from the category $\mathcal{M}(\mathcal{A}_{\overline{\mathfrak{n}}}) \times \mathcal{M}(\mathfrak{h})$ to the category $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$. We will call $\mathbb{W}_{\kappa,\mathfrak{g}}$ the *Wakimoto functor*.

Since the image of $\Phi_{\mathfrak{g}}$ need not be a subcategory of $\mathcal{M}(\mathfrak{g})$, we denote by $\mathcal{C}(\mathfrak{g})$ the full subcategory of $\mathcal{M}(\mathfrak{g})$ consisting of $\mathfrak{g}$-modules isomorphic to $N \otimes_{\mathbb{C}} E$ and their finite direct sums, where N is an $\mathcal{A}_{\overline{\mathfrak{n}}}$-module, E is a semisimple finite-dimensional $\mathfrak{h}$-module and the $\mathfrak{g}$-module structure on $N \otimes_{\mathbb{C}} E$ is given through the homomorphism $\pi_{\mathfrak{g}}$. In the next, we will consider the Wakimoto functor $\mathbb{W}_{\kappa,\mathfrak{g}}$ as a functor from the category $\mathcal{C}(\mathfrak{g})$ (there is a unique extension of $\mathbb{W}_{\kappa,\mathfrak{g}}$ from $\mathrm{im}\Phi_{\mathfrak{g}}$) to the category $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$.

Proposition 5.11. *For $\alpha \in \Delta_+$ the twisting functor $T_\alpha^{\mathfrak{g}}$ preserves the category $\mathcal{C}(\mathfrak{g})$.*

Proof. Let M be a $\mathfrak{g}$-module belonging to the category $\mathcal{C}(\mathfrak{g})$. Then we have $M \simeq \sum_{i=1}^n N_i \otimes_{\mathbb{C}} E_i$, where N_i is an $\mathcal{A}_{\overline{\mathfrak{n}}}$-module and E_i is a semisimple finite-dimensional $\mathfrak{h}$-module for $i = 1, 2, \ldots, n$, which gives us $T_\alpha(M) \simeq \sum_{i=1}^n T_\alpha(N_i \otimes_{\mathbb{C}} E_i)$. Therefore, we need to show that $T_\alpha(N \otimes_{\mathbb{C}} E)$ is also from the category $\mathcal{C}(\mathfrak{g})$ for an $\mathcal{A}_{\overline{\mathfrak{n}}}$-module N and a semisimple finite-dimensional $\mathfrak{h}$-module E.

Since f_α and $p_\alpha = \pi_{\mathfrak{g}}(f_\alpha)$ for $\alpha \in \Delta_+$ are locally ad-nilpotent regular elements in $U(\mathfrak{g})$ and $\mathcal{A}_{\overline{\mathfrak{n}}}$, respectively, we obtain that

$$(N \otimes_{\mathbb{C}} E)_{(f_\alpha)} \simeq N_{(p_\alpha)} \otimes_{\mathbb{C}} E$$

as $\mathfrak{g}$-modules, where the $\mathfrak{g}$-module structure on $N_{(p_\alpha)} \otimes_{\mathbb{C}} E$ is defined through the homomorphism $\pi_{\mathfrak{g}}$. Hence, we get $T_\alpha(N \otimes_{\mathbb{C}} E) \simeq (N_{(p_\alpha)}/N) \otimes_{\mathbb{C}} E$ for $\alpha \in \Delta_+$, which implies immediately that $T_\alpha(N \otimes_{\mathbb{C}} E)$ is an object of the category $\mathcal{C}(\mathfrak{g})$. □

Now, we may apply the Wakimoto functor $\mathbb{W}_{\kappa,\mathfrak{g}}$ on Verma modules and Gelfand–Tsetlin modules to obtain a free field realization of the corresponding relaxed Verma modules.

Let $\mathbb{C}_\lambda$ be the 1-dimensional $\mathfrak{h}$-module given by a weight $\lambda \in \mathfrak{h}^*$ and let $\alpha \in \Delta_+$ be a positive root. Then by [KS17,FK19b] there are isomorphisms

$$M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda) \simeq \mathcal{A}_{\overline{\mathfrak{n}}}/\mathcal{I}_{\mathrm{V}} \otimes_{\mathbb{C}} \mathbb{C}_{\lambda+2\rho} \qquad \text{and} \qquad W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha) \simeq \mathcal{A}_{\overline{\mathfrak{n}}}/\mathcal{I}_{\mathrm{GT},\alpha} \otimes_{\mathbb{C}} \mathbb{C}_{\lambda+2\rho} \tag{5.10}$$

of $\mathfrak{g}$-modules, where $\mathcal{I}_{\mathrm{V}}$ and $\mathcal{I}_{\mathrm{GT},\alpha}$ are left ideals of $\mathcal{A}_{\overline{\mathfrak{n}}}$ defined by

$$\mathcal{I}_{\mathrm{V}} = (x_\gamma;\ \gamma \in \Delta_+) \qquad \text{and} \qquad \mathcal{I}_{\mathrm{GT},\alpha} = (\partial_{x_\alpha}, x_\gamma;\ \gamma \in \Delta_+\backslash\{\alpha\}),$$

which implies that the Verma module $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ and α-Gelfand–Tsetlin module $W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha)$ belong to the category $\mathcal{C}(\mathfrak{g})$. Hence, we may apply the Wakimoto functor $\mathbb{W}_{\kappa,\mathfrak{g}}$ on these $\mathfrak{g}$-modules and get the corresponding relaxed Wakimoto modules together with the canonical homomorphisms

$$\varphi_{\mathrm{V}}^{\lambda} : \mathbb{M}_{\kappa,\mathfrak{g}}(M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)) \to \mathbb{W}_{\kappa,\mathfrak{g}}(M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)), \quad \varphi_{\mathrm{GT},\alpha}^{\lambda} : \mathbb{M}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha)) \to \mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha))$$

of $\widehat{\mathfrak{g}}_\kappa$-modules given by Theorem 5.10. Let us note also that $\mathbb{W}_{\kappa,\mathfrak{g}}(M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda))$ and $\mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha))$ are positive energy $\widehat{\mathfrak{g}}_\kappa$-modules with the top degree component isomorphic to $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ and $W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha)$, respectively.

Further, we introduce vector spaces

$$\mathcal{F}_{\overline{\mathfrak{n}}} = \mathbb{C}[\partial_{x_\gamma},\ \gamma \in \Delta_+] \qquad \text{and} \qquad \mathcal{F}_{\overline{\mathfrak{n}},\alpha} = \mathbb{C}[x_\alpha, \partial_{x_\gamma},\ \gamma \in \Delta_+\backslash\{\alpha\}] \tag{5.11}$$

and endow them with the structure of $\mathcal{A}_{\overline{\mathfrak{n}}}$-modules by means of the canonical isomorphisms

$$\mathcal{A}_{\overline{\mathfrak{n}}}/\mathcal{I}_{\mathrm{V}} \simeq \mathcal{F}_{\overline{\mathfrak{n}}} \qquad \text{and} \qquad \mathcal{A}_{\overline{\mathfrak{n}}}/\mathcal{I}_{\mathrm{GT},\alpha} \simeq \mathcal{F}_{\overline{\mathfrak{n}},\alpha} \tag{5.12}$$

of vector spaces. For a bilinear form κ on $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$, we define a smooth $\widehat{\mathfrak{h}}_\kappa$-module

$$\pi_\lambda^\kappa = \mathbb{C}[y_{\gamma,n},\ n \in \mathbb{N}, \gamma \in \Pi]$$

by the formula

$$h_{\gamma,n} = \begin{cases} \kappa(h,h)n\partial_{y_{\gamma,n}} & \text{if } n \in \mathbb{N}, \\ \lambda(h) & \text{if } n = 0, \\ y_{\gamma,-n} & \text{if } n \in -\mathbb{N} \end{cases}$$

for $\gamma \in \Pi$. In addition, we introduce smooth $\mathcal{A}_{\mathcal{K}(\overline{\mathfrak{n}})}$-modules

$$\mathcal{W}_{\overline{\mathfrak{n}}} = \mathbb{C}[\partial_{x_{\gamma,-n}}, x_{\gamma,n},\ n \in \mathbb{N}, \gamma \in \Delta_+;\ \partial_{x_{\gamma,0}},\ \gamma \in \Delta_+],$$
$$\mathcal{W}_{\overline{\mathfrak{n}},\alpha} = \mathbb{C}[\partial_{x_{\gamma,-n}}, x_{\gamma,n},\ n \in \mathbb{N}, \gamma \in \Delta_+;\ x_{\alpha,0}, \partial_{x_{\gamma,0}},\ \gamma \in \Delta_+\backslash\{\alpha\}]$$

for $\alpha \in \Delta_+$. Then it is easy to verify that

$$\mathbb{M}_{\kappa,\mathfrak{h}}(\mathbb{C}_\lambda) \simeq \pi_\lambda^\kappa \tag{5.13}$$

as $\widehat{\mathfrak{h}}_\kappa$-modules and that

$$\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(\mathcal{F}_{\overline{\mathfrak{n}}}) \simeq \mathcal{W}_{\overline{\mathfrak{n}}}, \qquad \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(\mathcal{F}_{\overline{\mathfrak{n}},\alpha}) \simeq \mathcal{W}_{\overline{\mathfrak{n}},\alpha} \tag{5.14}$$

as $\mathcal{A}_{\mathcal{K}(\overline{\mathfrak{n}})}$-modules.

Theorem 5.12. *Let κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$. Then we have*

$$\mathbb{W}_{\kappa,\mathfrak{g}}(M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)) \simeq \mathcal{W}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} \pi_{\lambda+2\rho}^{\kappa-\kappa_c} \quad \text{and} \quad \mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha)) \simeq \mathcal{W}_{\overline{\mathfrak{n}},\alpha} \otimes_{\mathbb{C}} \pi_{\lambda+2\rho}^{\kappa-\kappa_c}$$

for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta_+$.

Proof. By definition of the Wakimoto functor $\mathbb{W}_{\kappa,\mathfrak{g}}$ we have that

$$\mathbb{W}_{\kappa,\mathfrak{g}}(M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)) \simeq \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(\mathcal{F}_{\overline{\mathfrak{n}}}) \otimes_{\mathbb{C}} \mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(\mathbb{C}_{\lambda+2\rho}),$$
$$\mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha)) \simeq \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(\mathcal{F}_{\overline{\mathfrak{n}},\alpha}) \otimes_{\mathbb{C}} \mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(\mathbb{C}_{\lambda+2\rho})$$

for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta_+$, where we used (5.10) and (5.12). Moreover, by using the isomorphisms (5.13) and (5.14) we get

$$\mathbb{W}_{\kappa,\mathfrak{g}}(M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)) \simeq \mathcal{W}_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} \pi_{\lambda+2\rho}^{\kappa-\kappa_c} \quad \text{and} \quad \mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{b}}^{\mathfrak{g}}(\lambda,\alpha)) \simeq \mathcal{W}_{\overline{\mathfrak{n}},\alpha} \otimes_{\mathbb{C}} \pi_{\lambda+2\rho}^{\kappa-\kappa_c}$$

for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta_+$. □

The link between the twisting functor T_α for $\alpha \in \Delta_+ \subset \widehat{\Delta}_+^{\mathrm{re}}$ and the Wakimoto functor $\mathbb{W}_{\kappa,\mathfrak{g}}$ is given in the following theorem.

Let us introduce a smooth $\mathcal{A}_{\mathcal{K}(\overline{\mathfrak{n}})}$-module

$$\mathcal{E}_{\overline{\mathfrak{n}}} = \mathbb{C}[\partial_{x_{\gamma,-n}}, x_{\gamma,n},\ n \in \mathbb{N}, \gamma \in \Delta_+;\ x_{\gamma,0},\ \gamma \in \Delta_+]$$

which is isomorphic to $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(\mathbb{C}[\overline{\mathfrak{n}}])$. The commutative algebra $\mathcal{E}_{\overline{\mathfrak{n}}}$ has the natural structure of a $\widehat{Q}$-graded algebra, where $\widehat{Q}$ is the affine root lattice. The gradation is uniquely determined by

$$\deg x_{\gamma,n} = \gamma + n\delta \quad \text{and} \quad \deg \partial_{x_{\gamma,n}} = -\gamma - n\delta$$

for $\gamma \in \Delta_+$ and $n \in \mathbb{Z}$. Hence, we have a direct sum decomposition

$$\mathcal{E}_{\overline{\mathfrak{n}}} = \bigoplus_{\gamma \in \widehat{Q}} \mathcal{E}_{\overline{\mathfrak{n}},\gamma}.$$

We say that a differential operator $P \in \widetilde{\mathcal{A}}_{\mathcal{K}(\overline{\mathfrak{n}})}$ has degree $\omega \in \widehat{Q}$ if $P(\mathcal{E}_{\overline{\mathfrak{n}},\chi}) \subset \mathcal{E}_{\overline{\mathfrak{n}},\gamma+\omega}$ for all $\gamma \in \widehat{Q}$. Further, for $\alpha \in \Delta_+$ we define a differential operator $p_\alpha \in \mathcal{A}_{\mathcal{K}(\overline{\mathfrak{n}})}$ by

$$p_\alpha = \pi_{\kappa,\mathfrak{g}}(f_{\alpha,0}) = \mathrm{Res}_{z=0}\pi_{\kappa,\mathfrak{g}}(f_\alpha(z)) = -\mathrm{Res}_{z=0} \sum_{\gamma \in \Delta_+} :\left[\frac{\mathrm{ad}(u(z))}{e^{\mathrm{ad}(u(z))} - \mathrm{id}}\, f_\alpha\right]_\gamma a_\gamma(z):.$$

Then we may write

$$p_\alpha = p_{\alpha,0} + p_{\alpha,1},$$

where $p_{\alpha,0} = \pi_{\mathfrak{g}}(f_{\alpha,0})$. The differential operators $p_{\alpha,0}$, $p_{\alpha,\lrcorner}$ and p_α have degree $-\alpha$ for $\alpha \in \Delta_+$. We denote by $\mathcal{N}_\alpha$ the Lie subalgebra of $\mathcal{A}_{\mathcal{K}(\overline{\mathfrak{n}})}$ generated by the set $\{\mathrm{ad}(p_{\alpha,0})^n(p_{\alpha,1});\ n \in \mathbb{N}_0\}$ and by $\mathcal{U}_\alpha$ the subalgebra of $\widetilde{\mathcal{A}}_{\mathcal{K}(\overline{\mathfrak{n}})}$ generated by $\mathcal{N}_\alpha$. Moreover, we have the decompositions

$$\mathcal{N}_\alpha = \bigoplus_{n \in \mathbb{N}_0} \mathcal{N}_{\alpha,-n\alpha} \quad \text{and} \quad \mathcal{U}_\alpha = \bigoplus_{n \in \mathbb{N}_0} \mathcal{U}_{\alpha,-n\alpha}$$

given by the degree of differential operators, where $\mathcal{N}_{\alpha,0} = 0$ and $\mathcal{U}_{\alpha,0} = \mathbb{C}$.

Let us consider an $\mathcal{A}_{\overline{\mathfrak{n}}}$-module N. Then the smooth $\mathcal{A}_{\mathcal{K}(\overline{\mathfrak{n}})}$-module $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)$ has the canonical grading

$$\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N) = \bigoplus_{n=0}^{\infty} \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_n,$$

where $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_0 \simeq N$ as $\mathcal{A}_{\overline{\mathfrak{n}}}$-modules. Moreover, by definition of p_α we get immediately that the differential operator $p_{\alpha,0}$, $p_{\alpha,1}$ and p_α preserve the subspaces $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_n$ for $n \in \mathbb{N}_0$. Besides, we have

$$\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N) \simeq \mathcal{E}^-_{\overline{\mathfrak{n}}} \otimes_{\mathbb{C}} N,$$

where

$$\mathcal{E}^-_{\overline{\mathfrak{n}}} = \mathbb{C}[\partial_{x_{\gamma,-n}}, x_{\gamma,n},\ n \in \mathbb{N}, \gamma \in \Delta_+],$$

which gives us

$$\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_n \simeq \mathcal{E}^-_{\overline{\mathfrak{n}},n} \otimes_{\mathbb{C}} N$$

for $n \in \mathbb{N}_0$. Since the Lie algebra $\mathcal{N}_\alpha$ preserves $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_n$ for $n \in \mathbb{N}_0$, we have

$$q(\mathcal{E}^-_{\overline{\mathfrak{n}},n,\gamma} \otimes_{\mathbb{C}} N) \subset \mathcal{E}^-_{\overline{\mathfrak{n}},n,\gamma-j\alpha} \otimes_{\mathbb{C}} N$$

for $q \in \mathcal{N}_{\alpha,-j\alpha}$, where $\mathcal{E}^-_{\overline{\mathfrak{n}},n,\gamma} = \mathcal{E}^-_{\overline{\mathfrak{n}},n} \cap \mathcal{E}_{\overline{\mathfrak{n}},\gamma}$. However, the vector space $\mathcal{E}^-_{\overline{\mathfrak{n}},n}$ is finite dimensional, which implies that there exists $j_{n,\gamma} \in \mathbb{N}_0$ such that $\mathcal{E}^-_{\overline{\mathfrak{n}},n,\gamma-j\alpha} = 0$ for $j > j_{n,\gamma}$. Therefore, for each vector $v \in \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)$ there exists an integer $n_v \in \mathbb{N}_0$ such that $\mathcal{U}_{\alpha,-n\alpha} v = 0$ for all $n > n_v$.

Since p_α and $p_{\alpha,0}$ for $\alpha \in \Delta_+$ are locally ad-nilpotent regular elements of the completed Weyl algebra $\widetilde{\mathcal{A}}_{\mathcal{K}(\overline{\mathfrak{n}})}$, we may construct localized modules $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_\alpha)}$ and $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_{\alpha,0})}$. By a completely similar way as in [FK19b, Lemma 4.6], we can show that the element $p_{\alpha,1} p_{\alpha,0}^{-1} \in \widetilde{\mathcal{A}}_{\mathcal{K}(\overline{\mathfrak{n}})}$ acts locally nilpotently on $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_{\alpha,0})}$.

Proposition 5.13. *Let $\alpha \in \Delta_+$ and let N be an $\mathcal{A}_{\overline{\mathfrak{n}}}$-module. Then the linear mapping*

$$\Phi_\alpha \colon \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_\alpha)} \to \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_{\alpha,0})}$$

defined by

$$\Phi_\alpha(p_\alpha^{-n} v) = \varphi_\alpha^n(v)$$

for $n \in \mathbb{N}_0$ and $v \in \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)$, where the linear mapping $\varphi_\alpha \colon \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_{\alpha,0})} \to \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_{\alpha,0})}$ is given through

$$\varphi_\alpha(v) = p_{\alpha,0}^{-1} \sum_{k=0}^{\infty} (-1)^k (p_{\alpha,1} p_{\alpha,0}^{-1})^k v$$

for $v \in \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)$, is an isomorphism of $(\widetilde{\mathcal{A}}_{\mathcal{K}(\overline{\mathfrak{n}})})_{(p_\alpha)}$-modules.

Proof. Since the element $p_{\alpha,1}p_{\alpha,0}^{-1}$ acts locally nilpotently on $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_{\alpha,0})}$, the linear mapping φ_α is well defined. By an analogous way as in [FK19b, Lemma 4.7, Lemma 4.8], we can show that Φ_α is a homomorphism of $(\widetilde{\mathcal{A}}_{\mathcal{K}(\overline{\mathfrak{n}})})_{(p_\alpha)}$-modules. Hence, we only need to prove that the linear mapping Φ_α is injective and surjective. Since we may write

$$\Phi_\alpha(qp_\alpha^{-n}v) = q\varphi_\alpha^n(v) = qp_{\alpha,0}^{-n}v$$

for $n \in \mathbb{N}_0$, $v \in N$ and $q \in \mathcal{E}_{\overline{\mathfrak{n}}}^-$, we have the surjectivity of Φ_α. To prove the injectivity of Φ_α let us assume that $\Phi_\alpha(v) = 0$ for some $v \in \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_\alpha)}$. Then there exists an integer $n \in \mathbb{N}_0$ such that $p_\alpha^n v \in \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N) \subset \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_\alpha)}$. Hence, we have

$$0 = p_\alpha^n\Phi_\alpha(v) = \Phi_\alpha(p_\alpha^n v) = p_\alpha^n v \in \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N) \subset \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_{\alpha,0})},$$

which gives us $v = 0$. Therefore, the linear mapping Φ_α is an isomorphism. □

Theorem 5.14. *Let $\alpha \in \Delta_+ \subset \widehat{\Delta}_+^{\mathrm{re}}$. Then there exists a natural isomorphism*

$$\eta_\alpha : T_\alpha \circ \mathbb{W}_{\kappa,\mathfrak{g}} \to \mathbb{W}_{\kappa,\mathfrak{g}} \circ T_\alpha^{\mathfrak{g}}$$

of functors, where $T_\alpha^{\mathfrak{g}} : \mathcal{C}(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g})$ is the twisting functor for $\mathfrak{g}$ assigned to α. In particular, we have

$$T_\alpha(\mathbb{W}_{\kappa,\mathfrak{g}}(M_{\mathfrak{h}}^{\mathfrak{g}}(\lambda))) \simeq \mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{h}}^{\mathfrak{g}}(\lambda,\alpha))$$

for $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta_+$.

Proof. Let N be an $\mathcal{A}_{\overline{\mathfrak{n}}}$-module and E be a semisimple finite-dimensional $\mathfrak{h}$-module. Then by using the definition of the Wakimoto functor $\mathbb{W}_{\kappa,\mathfrak{g}}$ we obtain an isomorphism

$$\mathbb{W}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E)_{(f_{\alpha,0})} \simeq \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_\alpha)} \otimes_{\mathbb{C}} \mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E)$$

of $U(\widehat{\mathfrak{g}}_\kappa)$-modules, where the $U(\widehat{\mathfrak{g}}_\kappa)$-module structure on $\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_\alpha)} \otimes_{\mathbb{C}} \mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E)$ is given via the homomorphism $\pi_{\kappa,\mathfrak{g}}$. Further, from Proposition 5.13 we have that

$$\mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_\alpha)} \simeq \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N)_{(p_{\alpha,0})} \simeq \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N_{(p_{\alpha,0})})$$

as $\widetilde{\mathcal{A}}_{\mathcal{K}(\overline{\mathfrak{n}})}$-modules, which gives us

$$\begin{aligned}\mathbb{W}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E)_{(f_{\alpha,0})} &\simeq \mathbb{M}_{\mathcal{K}(\overline{\mathfrak{n}})}(N_{(p_{\alpha,0})}) \otimes_{\mathbb{C}} \mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E) \simeq \mathbb{W}_{\kappa,\mathfrak{g}}(N_{(p_{\alpha,0})} \otimes_{\mathbb{C}} E)\\ &\simeq \mathbb{W}_{\kappa,\mathfrak{g}}((N \otimes_{\mathbb{C}} E)_{(f_\alpha)})\end{aligned}$$

as $U(\widehat{\mathfrak{g}}_\kappa)$-modules. Hence, by definition of T_α and $T_\alpha^{\mathfrak{g}}$ we get

$$T_\alpha(\mathbb{W}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E)) \simeq \mathbb{W}_{\kappa,\mathfrak{g}}(T_\alpha^{\mathfrak{g}}(N \otimes_{\mathbb{C}} E))$$

for $\alpha \in \Delta_+$. Moreover, it is obvious that this isomorphism is natural. The rest of the statement follows immediately. □

With the help of Theorem 5.14 we can establish a relation between the relaxed Verma module $\mathbb{M}_{\kappa,\mathfrak{g}}(W_{\mathfrak{h}}^{\mathfrak{g}}(\lambda,\alpha))$ and the relaxed Wakimoto module $\mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{h}}^{\mathfrak{g}}(\lambda,\alpha))$.

Corollary 5.15. *Let $\lambda \in \mathfrak{h}^*$ and let us assume that the Verma module $\mathbb{M}_{\kappa,\mathfrak{g}}(M_{\mathfrak{h}}^{\mathfrak{g}}(\lambda))$, or equivalently the Wakimoto module $\mathbb{W}_{\kappa,\mathfrak{g}}(M_{\mathfrak{h}}^{\mathfrak{g}}(\lambda))$, is a simple $\widehat{\mathfrak{g}}_\kappa$-module. Then the relaxed Verma module $\mathbb{M}_{\kappa,\mathfrak{g}}(W_{\mathfrak{h}}^{\mathfrak{g}}(\lambda,\alpha))$ is isomorphic to the relaxed Wakimoto module $\mathbb{W}_{\kappa,\mathfrak{g}}(W_{\mathfrak{h}}^{\mathfrak{g}}(\lambda,\alpha))$ for $\alpha \in \Delta_+$.*

Proof. If the Verma module $\mathbb{M}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))$, or equivalently the Wakimoto module $\mathbb{W}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))$, is a simple $\widehat{\mathfrak{g}}_\kappa$-module, then $\mathbb{M}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)) \simeq \mathbb{W}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))$. By applying the twisting functor T_α for $\alpha \in \Delta_+$ on both sides of the isomorphism, we obtain

$$\mathbb{M}_{\kappa,\mathfrak{g}}(W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\alpha)) \simeq T_\alpha(\mathbb{M}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))) \simeq T_\alpha(\mathbb{W}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))) \simeq \mathbb{W}_{\kappa,\mathfrak{g}}(W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\alpha)),$$

where the first isomorphism follow from Theorem 4.16 and the last isomorphism is a consequence of Theorem 5.14. □

Let us note that Corollary 5.15 provides a free field realization of the relaxed Verma module $\mathbb{M}_{\kappa,\mathfrak{g}}(W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\alpha))$ whenever $\mathbb{M}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))$, or equivalently $\mathbb{W}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))$, is a simple $\widehat{\mathfrak{g}}_\kappa$-module. On the other hand, this does not give any information for the critical level κ_c, since $\mathbb{M}_{\kappa_c,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))$ is never a simple $\widehat{\mathfrak{g}}_{\kappa_c}$-module. However, applying [Fre07, Proposition 9.5.1] to the longest element of the Weyl group of $\mathfrak{g}$ we have that the Verma module $\mathbb{M}_{\kappa_c,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))$ and the Wakimoto module $\mathbb{W}_{\kappa_c,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))$ are isomorphic if $\langle\lambda+\rho,\alpha^\vee\rangle \notin -\mathbb{N}$ for all $\alpha \in \Delta_+$. Hence, we have the following extension of Corollary 5.15 in the case of the critical level.

Corollary 5.16. *Let $\lambda \in \mathfrak{h}^*$ satisfy $\langle\lambda+\rho,\alpha^\vee\rangle \notin -\mathbb{N}$ for all $\alpha \in \Delta_+$, i.e. λ is dominant weight. Then the relaxed Verma module $\mathbb{M}_{\kappa_c,\mathfrak{g}}(W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\alpha))$ is isomorphic to the relaxed Wakimoto module $\mathbb{W}_{\kappa_c,\mathfrak{g}}(W^{\mathfrak{g}}_{\mathfrak{b}}(\lambda,\alpha))$ for $\alpha \in \Delta_+$.*

6. Positive Energy Representations of $\mathcal{L}_\kappa(\mathfrak{g})$

In this section we describe families of positive energy representations of the simple affine vertex algebra $\mathcal{L}_\kappa(\mathfrak{g})$ of an admissible level κ associated to a simple Lie algebra $\mathfrak{g}$.

6.1. Admissible representations. Let $\mathfrak{g}$ be a complex simple Lie algebra and let κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$. Since $\mathfrak{g}$ is a simple Lie algebra, we have $\kappa = k\kappa_0$ for $k \in \mathbb{C}$, where κ_0 is the normalized $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$ satisfying

$$\kappa_{\mathfrak{g}} = 2h^\vee\kappa_0.$$

Let $\widehat{\mathfrak{g}}_\kappa$ be the affine Kac–Moody algebra associated to the Lie algebra $\mathfrak{g}$ of level κ. For $\lambda \in \mathfrak{h}^*$, we define its integral root system $\widehat{\Delta}(\lambda)$ by

$$\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{\mathrm{re}};\ \langle\lambda+\widehat{\rho},\alpha^\vee\rangle \in \mathbb{Z}\},$$

where $\widehat{\rho} = \rho + h^\vee\Lambda_0$. Further, let $\widehat{\Delta}(\lambda)_+ = \widehat{\Delta}(\lambda) \cap \widehat{\Delta}^{\mathrm{re}}_+$ be the set of positive roots of $\widehat{\Delta}(\lambda)$ and $\widehat{\Pi}(\lambda) \subset \widehat{\Delta}(\lambda)_+$ be the set of simple roots. Then we say that a weight $\lambda \in \widehat{\mathfrak{h}}^*$ is *admissible* ([KW89]) provided

i) λ is *regular dominant*, that is $\langle\lambda+\widehat{\rho},\alpha^\vee\rangle \notin -\mathbb{N}_0$ for all $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$;
ii) the $\mathbb{Q}$-span of $\widehat{\Delta}(\lambda)$ contains $\widehat{\Delta}^{\mathrm{re}}$.

In particular, if $\lambda = k\Lambda_0$ is an admissible weight for $k \in \mathbb{C}$, then k is called an *admissible number*. The admissible numbers were described in [KW89,KW08] as follows. The complex number $k \in \mathbb{C}$ is admissible if and only if

$$k + h^\vee = \frac{p}{q} \text{ with } p,q \in \mathbb{N},\ (p,q) = 1,\ p \geq \begin{cases} h^\vee & \text{if } (r^\vee,q) = 1, \\ h & \text{if } (r^\vee,q) = r^\vee, \end{cases}$$

where $r^\vee$ is the lacing number of $\mathfrak{g}$, i.e. the maximal number of edges in the Dynkin diagram of the Lie algebra $\mathfrak{g}$. Since admissibility of a number $k \in \mathbb{C}$ depends only on $\mathfrak{g}$, we shall say that k is an admissible number for $\mathfrak{g}$.

Further, let us assume that $k \in \mathbb{Q}$ is an admissible number for $\mathfrak{g}$. We say that a $\mathfrak{g}$-module E is *admissible of level* k if $\mathbb{L}_{k\kappa_0,\mathfrak{g}}(E)$ is an $\mathcal{L}_{k\kappa_0}(\mathfrak{g})$-module, or equivalently if E is an $A(\mathcal{L}_{k\kappa_0}(\mathfrak{g}))$-module. In particular, the simple highest weight $\mathfrak{g}$-module $L^{\mathfrak{g}}_{\mathfrak{b}}(0)$ with zero highest weight is an admissible $\mathfrak{g}$-module of level k. Moreover, since $A(\mathcal{L}_{k\kappa_0}(\mathfrak{g})) \simeq U(\mathfrak{g})/I_k$, where I_k is a two-sided ideal of $U(\mathfrak{g})$, we obtain that a $\mathfrak{g}$-module E is admissible of level k if and only if the ideal I_k is contained in the annihilator $\mathrm{Ann}_{U(\mathfrak{g})}E$.

Admissible highest weight $\mathfrak{g}$-modules of level k were classified in [Ara16] as follows. Let Pr_k be the set of admissible weights $\lambda \in \widehat{\mathfrak{h}}^*$ of level k such that there is an element $y \in \widetilde{W}$ of the extended affine Weyl group $\widetilde{W}$ of $\mathfrak{g}$ satisfying $\widehat{\Delta}(\lambda) = y(\widehat{\Delta}(k\Lambda_0))$. Further, let us define the subset

$$\overline{\mathrm{Pr}}_k = \{\overline{\lambda};\ \lambda \in \mathrm{Pr}_k\}$$

of $\mathfrak{h}^*$, where $\overline{\lambda} \in \mathfrak{h}^*$ denotes the canonical projection of $\lambda \in \widehat{\mathfrak{h}}^*$ to $\mathfrak{h}^*$.

Theorem 6.1 [Ara16]. *Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. Then the simple highest weight $\mathfrak{g}$-module $L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$ is admissible of level k if and only if $\lambda \in \overline{\mathrm{Pr}}_k$.*

For $\lambda \in \mathfrak{h}^*$, let $L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ be the simple $\mathfrak{g}$-module with highest weight λ and

$$J_\lambda = \mathrm{Ann}_{U(\mathfrak{g})}L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda) \tag{6.1}$$

the corresponding primitive ideal of $U(\mathfrak{g})$. A theorem of Duflo [Duf77] states that for any primitive ideal I of $U(\mathfrak{g})$ there exists $\lambda \in \mathfrak{h}^*$ such that $I = J_\lambda$. This implies that a simple weight $\mathfrak{g}$-module E is admissible of level k if and only if $\mathrm{Ann}_{U(\mathfrak{g})}E = J_\lambda$ for some $\lambda \in \overline{\mathrm{Pr}}_k$. Besides, for $\lambda, \mu \in \overline{\mathrm{Pr}}_k$ we have $J_\lambda = J_\mu$ if and only if there exists $w \in W$ such that $\mu = w \cdot \lambda$ (see Proposition 2.4 in [AFR17]). Hence, we may define an equivalence relation on $\overline{\mathrm{Pr}}_k$ by

$$\lambda \sim \mu \iff \text{there exits } w \in W \text{ such that } \mu = w \cdot \lambda$$

and set

$$[\overline{\mathrm{Pr}}_k] = \overline{\mathrm{Pr}}_k/\sim$$

for an admissible number k of $\mathfrak{g}$.

Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be a standard parabolic subalgebra of $\mathfrak{g}$, where $\mathfrak{u}$ is the nilradical of $\mathfrak{p}$ and $\mathfrak{l}$ is the Levi subalgebra of $\mathfrak{p}$, and let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. We denote by $\Omega_k(\mathfrak{p})$ the set of those weights $\lambda \in \overline{\mathrm{Pr}}_k \cap \Lambda^+(\mathfrak{p})$ for which the generalized Verma module $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)$ is simple, i.e. $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda) \simeq L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ as $\mathfrak{g}$-modules. For $\lambda \in \Omega_k(\mathfrak{p})$, we obtain immediately that λ is a regular weight, which by [Jan77] gives us that $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)$ is a simple $\mathfrak{g}$-module if and only if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N}$ for all $\alpha \in \Delta^{\mathfrak{u}}_+$.

Theorem 6.2. *Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{g}$ and let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. Then the $\mathfrak{g}$-module $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$ is admissible of level k for $\lambda \in \Omega_k(\mathfrak{p})$ and $\alpha \in \Delta^{\mathfrak{u}}_+$.*

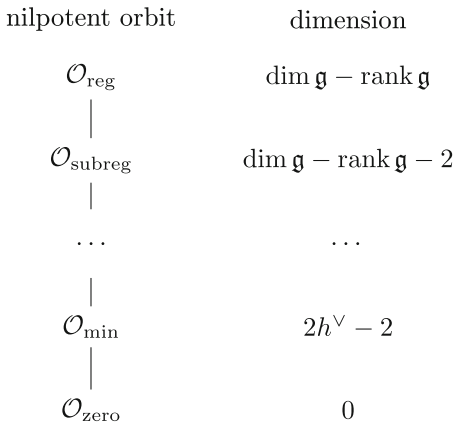


Fig. 1. Hasse diagram of nilpotent orbits

Proof. Indeed, the Zhu's algebra $A(\mathcal{L}_{k\kappa_0}(\mathfrak{g}))$ is isomorphic to $U(\mathfrak{g})/I_k$, where I_k is a two-sided ideal of $U(\mathfrak{g})$. Let us consider the simple highest weight $\mathfrak{g}$-module $L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$. For $\lambda \in \Omega_k(\mathfrak{p})$, the $\mathfrak{g}$-module $L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ is admissible of level k by Theorem 6.1. Hence, we get $I_k \subset J_\lambda$, where $J_\lambda = \operatorname{Ann}_{U(\mathfrak{g})} L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$. Further, we need to show that the $\mathfrak{g}$-module $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$ for $\alpha \in \Delta^{\mathfrak{u}}_+$ is admissible of level k, or equivalently that $I_k \subset \operatorname{Ann}_{U(\mathfrak{g})} W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$. Since $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$ is obtained from $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda) \simeq L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ by the twisting functor T_α, the statement follows from Corollary 4.5. Hence, $W^{\mathfrak{g}}_{\mathfrak{p}}(\lambda, \alpha)$ is a module over $U(\mathfrak{g})/I_k$, which implies the required statement. □

6.2. Richardson orbits and associated varieties. Let G be a complex connected semisimple algebraic group with its Lie algebra $\mathfrak{g}$. We denote by $\mathcal{N}(\mathfrak{g})$ the nilpotent cone of $\mathfrak{g}$, i.e. the set of nilpotent elements of $\mathfrak{g}$. It is an irreducible closed algebraic subvariety of $\mathfrak{g}$ and a finite union of G-orbits. There is a unique nilpotent orbit of $\mathfrak{g}$, denoted by $\mathcal{O}_{\rm reg}$ and called the *regular nilpotent orbit* of $\mathfrak{g}$, which is a dense open subset of $\mathcal{N}(\mathfrak{g})$. Next, since $\mathfrak{g}$ is simple, there exists a unique nilpotent orbit of $\mathfrak{g}$ that is a dense open subset of $\mathcal{N}(\mathfrak{g})\backslash\mathcal{O}_{\rm reg}$, denoted by $\mathcal{O}_{\rm subreg}$ and called the *subregular nilpotent orbit* of $\mathfrak{g}$. Besides, there is a unique nonzero nilpotent orbit of $\mathfrak{g}$ of minimal dimension, denoted by $\mathcal{O}_{\rm min}$ and called the *minimal nilpotent orbit* of $\mathfrak{g}$, such that it is contained in the closure of all nonzero nilpotent orbits of $\mathfrak{g}$. By $\mathcal{O}_{\rm zero}$ we denote the *zero nilpotent orbit* of $\mathfrak{g}$. For the dimension of these distinguished nilpotent orbits of $\mathfrak{g}$ see Fig. 1.

Let us consider the PBW filtration on the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ and the associated graded algebra $\operatorname{gr}U(\mathfrak{g}) \simeq S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$. The *associated variety* $\mathcal{V}(I)$ of a left ideal I of $U(\mathfrak{g})$ is defined as the zero locus in $\mathfrak{g}^*$ of the associated graded ideal $\operatorname{gr}I$ of $S(\mathfrak{g})$. Moreover, if I is a two-sided ideal of $\mathfrak{g}$, then I and $\operatorname{gr}I$ are invariant under the adjoint action of G. Consequently, the associated variety is a union of G-orbits of $\mathfrak{g}^*$. Obviously, we have

$$\mathcal{V}(I) = \operatorname{Specm}(S(\mathfrak{g})/\operatorname{gr}I) = \operatorname{Specm}(S(\mathfrak{g})/\sqrt{\operatorname{gr}I}), \tag{6.2}$$

where $\sqrt{\operatorname{gr}I}$ denotes the radical of $\operatorname{gr}I$. Since the Cartan–Killing form $\kappa_{\mathfrak{g}}$ is a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$, it provides a one-to-one correspondence between adjoint

orbits of $\mathfrak{g}$ and coadjoint orbits of $\mathfrak{g}^*$. For an adjoint orbit $\mathcal{O}$ of $\mathfrak{g}$ we denote by $\mathcal{O}^*$ the corresponding coadjoint orbit of $\mathfrak{g}^*$. In addition, for a primitive ideal I of $U(\mathfrak{g})$ the associated variety $\mathcal{V}(I)$ is the closure of $\mathcal{O}^*$ for some nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$, see [Jos85]. If E is a simple $\mathfrak{g}$-module and $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}E) = \overline{\mathcal{O}}^*$ for a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$, we say that E belongs to $\mathcal{O}$. For a description of nilpotent orbits of admissible $\mathfrak{g}$-modules we refer to [Ara15a].

Theorem 6.3 [Ara15b]. *Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$ with denominator $q \in \mathbb{N}$. Then there exists a nilpotent orbit $\mathcal{O}_q$ of $\mathfrak{g}$ such that*

$$\mathcal{V}(I_k) = \overline{\mathcal{O}}_q^*$$

and we have

$$\overline{\mathcal{O}}_q = \begin{cases} \{x \in \mathfrak{g};\ (\mathrm{ad}x)^{2q} = 0\} & \text{if } (r^\vee, q) = 1, \\ \{x \in \mathfrak{g};\ \pi_{\theta_s}(x)^{2q/r^\vee} = 0\} & \text{if } (r^\vee, q) = r^\vee, \end{cases}$$

where $\pi_{\theta_s} : \mathfrak{g} \to \mathrm{End}L_{\mathfrak{b}}^{\mathfrak{g}}(\theta_s)$ is the simple finite-dimensional $\mathfrak{g}$-module with highest weight θ_s.

Let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{g}$ and let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$ with denominator $q \in \mathbb{N}$. We define the subset

$$\overline{\mathrm{Pr}}_k^{\mathcal{O}} = \{\lambda \in \overline{\mathrm{Pr}}_k;\ \mathcal{V}(J_\lambda) = \overline{\mathcal{O}}^*\} \tag{6.3}$$

of $\mathfrak{h}^*$. Then a simple $\mathfrak{g}$-module E in the nilpotent orbit $\mathcal{O}$ is admissible of level k if and only if $\mathrm{Ann}_{U(\mathfrak{g})}E = J_\lambda$ for some $\lambda \in \overline{\mathrm{Pr}}_k^{\mathcal{O}}$. Further, as $I_k \subset J_\lambda$ for $\lambda \in \overline{\mathrm{Pr}}_k$, we have $\mathcal{V}(J_\lambda) \subset \mathcal{V}(I_k) = \overline{\mathcal{O}}_q^*$ by Theorem 6.3, which gives us

$$\overline{\mathrm{Pr}}_k = \bigsqcup_{\mathcal{O} \subset \overline{\mathcal{O}}_q} \overline{\mathrm{Pr}}_k^{\mathcal{O}}. \tag{6.4}$$

Therefore, we need to describe the subset $\overline{\mathrm{Pr}}_k^{\mathcal{O}}$ of $\overline{\mathrm{Pr}}_k$ for a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$.

For $x \in \mathfrak{g}$, we denote by $\mathfrak{g}^x$ the centralizer of x in $\mathfrak{g}$. Furthermore, for a subset X of $\mathfrak{g}$ we define the set

$$X^{\mathrm{reg}} = \{x \in X;\ \dim \mathfrak{g}^x = \min_{y \in X} \dim \mathfrak{g}^y\}$$

and call it the *set of regular elements* in X.

Theorem 6.4 [BB82]. *Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{g}$. Then for $\lambda \in \Lambda^+(\mathfrak{p})$ the associated variety $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))$ is the closure of $\mathcal{O}_{\mathfrak{p}}^*$, where*

$$\mathcal{O}_{\mathfrak{p}} = (G.\mathfrak{p}^\perp)^{\mathrm{reg}}$$

and $\mathfrak{p}^\perp$ is the orthogonal complement of $\mathfrak{p}$ with respect to the Cartan-Killing form. In particular, we have

$$\overline{\mathcal{O}}_{\mathfrak{p}} = G.\mathfrak{p}^\perp$$

and the associated variety $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))$ is irreducible. The orbit $\mathcal{O}_{\mathfrak{p}}$ is the Richardson orbit determined by $\mathfrak{p}$.

Now, we find the standard parabolic subalgebras for which the corresponding Richardson orbits are the distinguished nilpotent orbits $\mathcal{O}_{\text{zero}}$, $\mathcal{O}_{\text{min}}$, $\mathcal{O}_{\text{subreg}}$ and $\mathcal{O}_{\text{reg}}$. Let us recall that while the regular orbit $\mathcal{O}_{\text{reg}}$ and the subregular orbit $\mathcal{O}_{\text{subreg}}$ are Richardson orbits for any $\mathfrak{g}$, the minimal orbit $\mathcal{O}_{\text{min}}$ is a Richardson orbit only for $\mathfrak{sl}_n$, $n \geq 2$. Since all these nilpotent orbits are uniquely determined by their dimensions and since $\dim \mathcal{O}_{\mathfrak{p}} = 2 \dim \mathfrak{u}$ for a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ with the nilradical $\mathfrak{u}$, we obtain easily by comparison of the dimensions that

(i) $\mathcal{O}_{\text{zero}} = \mathcal{O}_{\mathfrak{g}}$,
(ii) $\mathcal{O}_{\text{min}} = \mathcal{O}_{\mathfrak{p}_\alpha^{\max}}$ for $\alpha \in \{\alpha_1, \alpha_{n-1}\}$ and $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 2$,
(iii) $\mathcal{O}_{\text{subreg}} = \mathcal{O}_{\mathfrak{p}_\alpha^{\min}}$ for $\alpha \in \Pi$,
(iv) $\mathcal{O}_{\text{reg}} = \mathcal{O}_{\mathfrak{b}}$,

where $\mathfrak{p}_\alpha^{\max}$ for $\alpha \in \Pi$ is the parabolic subalgebra of $\mathfrak{g}$ associated to the subset $\Sigma = \Pi \backslash \{\alpha\}$, and $\mathfrak{p}_\alpha^{\min}$ for $\alpha \in \Pi$ is the parabolic subalgebra of $\mathfrak{g}$ associated to the subset $\Sigma = \{\alpha\}$.

Lemma 6.5. *Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be standard parabolic subalgebras of $\mathfrak{g}$ and let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. Then we have $\Omega_k(\mathfrak{p}_1) \cap \Omega_k(\mathfrak{p}_2) \neq \emptyset$ if and only if $\mathfrak{p}_1 = \mathfrak{p}_2$.*

Proof. Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{g}$. If $\lambda \in \Omega_k(\mathfrak{p})$ we have $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N}$ for all $\alpha \in \Delta_+^{\mathfrak{u}}$ and $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{N}$ for all $\alpha \in \Delta_+^{\mathfrak{l}}$, which immediately implies that $\Omega_k(\mathfrak{p}_1) \cap \Omega_k(\mathfrak{p}_2) \neq \emptyset$ if and only if $\mathfrak{p}_1 = \mathfrak{p}_2$. □

Let $\mathcal{O}$ be a Richardson orbit of $\mathfrak{g}$ and let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. Further, let us consider a standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ satisfying $\mathcal{O}_{\mathfrak{p}} = \mathcal{O}$. Then for $\lambda \in \Omega_k(\mathfrak{p})$ we obtain $\mathcal{V}(\text{Ann}_{U(\mathfrak{g})} L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)) = \overline{\mathcal{O}^*}$ by Theorem 6.4, since $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \simeq L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ for any $\lambda \in \Omega_k(\mathfrak{p})$, which gives us

$$\Omega_k(\mathfrak{p}) \subset \overline{\text{Pr}}_k^{\mathcal{O}}.$$

Hence, by Lemma 6.5 we get

$$\bigsqcup_{\substack{\Sigma \subset \Pi \\ \mathcal{O}_{\mathfrak{p}_\Sigma} = \mathcal{O}}} \Omega_k(\mathfrak{p}_\Sigma) \subset \overline{\text{Pr}}_k^{\mathcal{O}}, \tag{6.5}$$

where $\mathfrak{p}_\Sigma$ is the standard parabolic subalgebra of $\mathfrak{g}$ associated to a subset Σ of Π.

Proposition 6.6. *Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. Then we have $\Omega_k(\mathfrak{b}) = \overline{\text{Pr}}_k^{\mathcal{O}_{\text{reg}}}$.*

Proof. Since the regular nilpotent orbit $\mathcal{O}_{\text{reg}}$ of $\mathfrak{g}$ is a Richardson orbit and $\mathcal{O}_{\text{reg}} = \mathcal{O}_{\mathfrak{b}}$, we obtain $\Omega_k(\mathfrak{b}) \subset \overline{\text{Pr}}_k^{\mathcal{O}_{\text{reg}}}$. On the other hand, for $\lambda \in \overline{\text{Pr}}_k^{\mathcal{O}_{\text{reg}}}$ we have $\mathcal{V}(J_\lambda) = \overline{\mathcal{O}_{\text{reg}}^*}$ by definition, where $J_\lambda = \text{Ann}_{U(\mathfrak{g})} L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$, and $\mathcal{V}(I_\lambda) = \overline{\mathcal{O}_{\text{reg}}^*}$ by Theorem 6.4, where $I_\lambda = \text{Ann}_{U(\mathfrak{g})} M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$. Hence, we get $\sqrt{\text{gr} J_\lambda} = \sqrt{\text{gr} I_\lambda}$. Since $\text{gr} I_\lambda$ is a prime ideal of $S(\mathfrak{g})$ by [BB82, Theorem 5.6], we have $\text{gr} J_\lambda \subset \text{gr} I_\lambda$ which gives us $J_\lambda \subset I_\lambda$. Moreover, the primitive ideal J_λ is the unique maximal two-sided ideal of $U(\mathfrak{g})$ containing I_λ by [AFR17, Proposition 2.4]. Therefore, we get $I_\lambda = J_\lambda$ for $\lambda \in \overline{\text{Pr}}_k^{\mathcal{O}_{\text{reg}}}$. As λ is regular dominant, we have $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda) \simeq M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)/I_\lambda M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ and $L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda) \simeq M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)/J_\lambda L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$, which implies $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda) \simeq L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ and thus $\lambda \in \Omega_k(\mathfrak{b})$. □

Besides, from the decomposition (6.4) is follows that the set $\overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{reg}}}$ is non-empty if and only if $\mathcal{O}_{\mathrm{reg}} = \mathcal{O}_q$, where q is the denominator of k, or equivalently if and only if

$$q \geq \begin{cases} h & \text{if } (r^\vee, q) = 1, \\ {}^L h^\vee r^\vee & \text{if } (r^\vee, q) = r^\vee, \end{cases}$$

where h is the Coxeter number of $\mathfrak{g}$ and ${}^L h^\vee$ is the dual Coxeter number of the Langlands dual Lie algebra ${}^L\mathfrak{g}$ of $\mathfrak{g}$ (see [AFR17]).

Proposition 6.7. *Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. Then we have $\Omega_k(\mathfrak{g}) = \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{zero}}}$.*

Proof. For the zero nilpotent orbit $\mathcal{O}_{\mathrm{zero}}$ of $\mathfrak{g}$, we have $\mathcal{O}_{\mathrm{zero}} = \mathcal{O}_{\mathfrak{g}}$, which gives us $\Omega_k(\mathfrak{g}) \subset \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{zero}}}$. On the other hand, for $\lambda \in \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{zero}}}$ we have $\mathcal{V}(J_\lambda) = \overline{\mathcal{O}^*_{\mathrm{zero}}}$ by definition, where $J_\lambda = \mathrm{Ann}_{U(\mathfrak{g})} L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$, which implies $\sqrt{(\mathfrak{g})} = \sqrt{\mathrm{gr} J_\lambda}$. Hence, for $\alpha \in \Delta_+$ there exists a positive integer $n_\alpha \in \mathbb{N}$ such that $f_\alpha^{n_\alpha} \in \mathrm{gr} J_\lambda$. Therefore, we get immediately that the $\mathfrak{g}$-module $L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ is finite dimensional implying $\lambda \in \Omega_k(\mathfrak{g})$. □

6.3. Admissible representations for $\mathfrak{sl}_n$. In this subsection we will consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ for $n \geq 2$. Let us note that all nilpotent orbits of $\mathfrak{g}$ are Richardson orbits.

Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$ with denominator $q \in \mathbb{N}$. Then the nilpotent orbit $\mathcal{O}_q$ from Theorem 6.3 is given by

$$\mathcal{O}_q = \mathcal{O}_{\lambda_q},$$

where λ_q is the partition of n defined through $\lambda_q = [q^r, s]$, where $n = qr + s$ with $r, s \in \mathbb{N}_0$ and $0 \leq s \leq q - 1$. Hence, we immediately get that

$$\mathcal{O}_q = \begin{cases} \mathcal{O}_{\mathrm{reg}} & \text{if } q \geq n, \\ \mathcal{O}_{\mathrm{subreg}} & \text{if } q = n - 1, \\ \mathcal{O}_{\mathrm{zero}} & \text{if } q = 1 \end{cases}$$

for $n \geq 2$. For details about nilpotent orbits in semisimple Lie algebras see [CM93] (Fig. 2).

Let us introduce the subset

$$\mathrm{Pr}_{k,\mathbb{Z}} = \{\lambda \in \mathrm{Pr}_k;\ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for } \alpha \in \Pi\}$$

of admissible weights of level k. Then by [KW89] we have

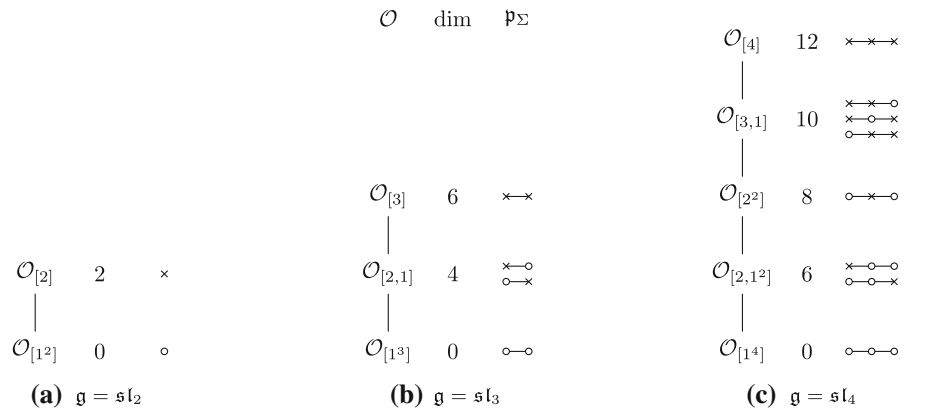
$$\mathrm{Pr}_{k,\mathbb{Z}} = \{\lambda \in \widehat{\mathfrak{h}}^*;\ \lambda(c) = k,\ \langle \lambda, \alpha^\vee \rangle \in \mathbb{N}_0 \text{ for } \alpha \in \Pi,\ \langle \lambda, \theta^\vee \rangle \leq p - h^\vee\},$$

where $p = (k + h^\vee)q$. Further, from [KW89] we obtain

$$\mathrm{Pr}_k = \bigcup_{\substack{y \in \widetilde{W} \\ y(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}_+^{\mathrm{re}}}} \mathrm{Pr}_{k,y}, \quad \mathrm{Pr}_{k,y} = \{y \cdot \lambda;\ \lambda \in \mathrm{Pr}_{k,\mathbb{Z}}\},$$

where the extended affine Weyl group $\widetilde{W}$ of $\mathfrak{g}$ is defined as

$$\widetilde{W} = W \ltimes P^\vee.$$



(a) $\mathfrak{g} = \mathfrak{sl}_2$ **(b)** $\mathfrak{g} = \mathfrak{sl}_3$ **(c)** $\mathfrak{g} = \mathfrak{sl}_4$

Fig. 2. Richardson orbits for $\mathfrak{sl}_n$

For $\mu \in P^\vee$, we denote by t_μ the corresponding element of $\widetilde{W}$. The action of t_μ on $\widetilde{\mathfrak{h}}^*$ for $\mu \in P^\vee$ is given by

$$t_\mu(\gamma) = \gamma + (\gamma, \delta)\mu - \left(\frac{(\mu, \mu)}{2}(\gamma, \delta) + (\gamma, \mu)\right)\delta$$

for $\gamma \in \widetilde{\mathfrak{h}}^*$. Moreover, for y, $y' \in \widetilde{W}$ satisfying $y(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}^{\mathrm{re}}_+$, $y'(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}^{\mathrm{re}}_+$ we have

$$\mathrm{Pr}_{k,y} \cap \mathrm{Pr}_{k,y'} \neq \emptyset \iff \mathrm{Pr}_{k,y} = \mathrm{Pr}_{k,y'} \iff y' = y t_{q\omega_j} w_j \tag{6.6}$$

for some $j \in \{1, 2, \ldots, n-1\}$, where w_j is the unique element of the Weyl group W of $\mathfrak{g}$ which preserves the set $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, -\theta\}$ and $w_j(-\theta) = \alpha_j$. The set of simple roots of $\mathfrak{g}$ is $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$. Let us recall that by [AFR17, Proposition 2.8] we have

$$[\overline{\mathrm{Pr}}_k] = \bigcup_{\substack{\eta \in P^\vee_+ \\ (\eta,\theta) \leq q-1}} [\overline{\mathrm{Pr}}_{k, t_{-\eta}}],$$

where $P^\vee_+$ is the set of dominant coweights of $\mathfrak{g}$.

Lemma 6.8. *If $\overline{y} \in W$ is not the unit element, then there exists $j \in \{0, 1, \ldots, n-1\}$ such that $\overline{y} w_j(\theta) \in \Delta_+$.*

Proof. For a proof, see [AFR17, Lemma 2.7]. □

Theorem 6.9. *Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$ with denominator $q \in \mathbb{N}$. Then we have*

$$\Omega_k(\mathfrak{p}_\Sigma) = \bigcup_{\substack{\overline{y} \in W,\, \eta \in P^\vee_+, \\ (\eta,\theta) \leq q-1,\, \overline{y}(\theta) \in \Delta_+,\, \overline{y}(\Delta_0^\eta) = \Delta_\Sigma, \\ \Delta_0^\eta \cap \Delta_+ \subset \overline{y}^{-1}(\Delta_+) \cap \Delta_+}} \overline{\mathrm{Pr}}_{k, \overline{y} t_{-\eta}}$$

for $\Sigma \subset \Pi$, where Δ_Σ is the subroot system of Δ generated by Σ and $\Delta_0^\eta = \{\alpha \in \Delta;\ (\eta, \alpha) = 0\}$ for $\eta \in P^\vee$.

Proof. Let $y = \overline{y}t_{-\eta}$ with $\overline{y} \in W$ and $\eta \in P^\vee$. Then the condition $y(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}^{\mathrm{re}}_+$ is equivalent to

$$0 \le (\eta, \alpha) \le q-1 \text{ if } \overline{y}(\alpha) \in \Delta_+ \qquad \text{and} \qquad 1 \le (\eta, \alpha) \le q \text{ if } \overline{y}(\alpha) \in \Delta_-$$

for all $\alpha \in \Delta_+$. In particular, for $\eta \in P^\vee$ and $\overline{y} \in W$ such that $\overline{y}(\theta) \in \Delta_+$ we obtain easily that $\overline{y}t_{-\eta}(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}^{\mathrm{re}}_+$ if and only if $\eta \in P^\vee_+$, $(\eta, \theta) \le q-1$ and $\Delta_0^\eta \cap \Delta_+ \subset \overline{y}^{-1}(\Delta_+) \cap \Delta_+$.

For $\lambda \in \mathfrak{h}^*$, we define the subset $\Delta(\lambda) = \{\alpha \in \Delta;\ \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}$ of Δ. Then we immediately get $\Delta(\lambda) = \widehat{\Delta}(\lambda + k\Lambda_0) \cap \Delta$. Further, let us assume that $\lambda \in \overline{\mathrm{Pr}}_{k,\overline{y}t_{-\eta}}$ with $\overline{y} \in W$, $\eta \in P^\vee_+$, $(\eta, \theta) \le q-1$, $\overline{y}(\theta) \in \Delta_+$, $\Delta_0^\eta \cap \Delta_+ \subset \overline{y}^{-1}(\Delta_+) \cap \Delta_+$ and $\overline{y}(\Delta_0^\eta) = \Delta_\Sigma$ for a fixed subset Σ of Π. Then we may write

$$\begin{aligned}
\Delta(\lambda) &= \widehat{\Delta}(\lambda + k\Lambda_0) \cap \Delta = \overline{y}t_{-\eta}(\widehat{\Delta}(k\Lambda_0)) \cap \Delta \\
&= \{\overline{y}(\alpha) + (mq + (\eta, \alpha))\delta;\ \alpha \in \Delta,\ m \in \mathbb{Z}\} \cap \Delta \\
&= \{\overline{y}(\alpha);\ \alpha \in \Delta,\ (\eta, \alpha) = 0\} = \overline{y}(\Delta_0^\eta) = \Delta_\Sigma.
\end{aligned}$$

Hence, by [Jan77] we have that $M^{\mathfrak{g}}_{\mathfrak{p}_\Sigma}(\lambda) \simeq L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ since λ is regular dominant, i.e. $\langle \lambda + \rho, \alpha^\vee \rangle \notin -\mathbb{N}_0$ for $\alpha \in \Delta_+$, which gives us $\overline{\mathrm{Pr}}_{k,\overline{y}t_{-\eta}} \subset \Omega_k(\mathfrak{p}_\Sigma)$.

On the other hand, if $\lambda \in \Omega_k(\mathfrak{p}_\Sigma)$, then it is easy to see that $\Delta(\lambda) = \Delta_\Sigma$. Since we also have $\lambda \in \overline{\mathrm{Pr}}_k$, there exists $y \in \widetilde{W}$ such that $\widehat{\Delta}(\lambda + k\Lambda_0) = y(\widehat{\Delta}(k\Lambda_0))$, $y(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}^{\mathrm{re}}_+$ and $\lambda \in \overline{\mathrm{Pr}}_{k,y}$. Let $y = \overline{y}t_{-\eta}$ with $\overline{y} \in W$ and $\eta \in P^\vee$. We may assume that $\overline{y}(\theta) \in \Delta_+$. Indeed, we have

$$\overline{\mathrm{Pr}}_{k,\overline{y}t_{-\eta}} = \overline{\mathrm{Pr}}_{k,\overline{y}t_{-\eta}t_{q\omega_j}w_j} = \overline{\mathrm{Pr}}_{k,\overline{y}w_jw_j^{-1}t_{q\omega_j-\eta}w_j} = \overline{\mathrm{Pr}}_{k,\overline{y}w_jt_{-\eta_j}}$$

by (6.6) for $j \in \{0, 1, \dots, n-1\}$, where $\eta_j = w_j^{-1}(\eta - q\omega_j)$. Further, if $\overline{y}(\theta) \in \Delta_-$, then there exists $j \in \{0, 1, \dots, n-1\}$ satisfying $\overline{y}w_j(\theta) \in \Delta_+$ by Lemma 6.8.

Hence, by using the assumption $\overline{y}(\theta) \in \Delta_+$, we get that $\lambda \in \overline{\mathrm{Pr}}_{k,\overline{y}t_{-\eta}}$ with $\eta \in P^\vee_+$, $(\eta, \theta) \le q-1$, $\Delta_0^\eta \cap \Delta_+ \subset \overline{y}^{-1}(\Delta_+) \cap \Delta_+$ and $\overline{y}(\Delta_0^\eta) = \Delta(\lambda) = \Delta_\Sigma$. □

Corollary 6.10. *Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$ with denominator $q \in \mathbb{N}$. Then we have*

(i) $\Omega_k(\mathfrak{b}) = \bigcup_{\substack{\overline{y} \in W,\ \eta \in P^\vee_+, \\ (\eta,\theta) \le q-1,\ \overline{y}(\theta) \in \Delta_+,\ \Delta_0^\eta = \emptyset}} \overline{\mathrm{Pr}}_{k,\overline{y}t_{-\eta}}$;

(ii) $\Omega_k(\mathfrak{g}) = \overline{\mathrm{Pr}}_{k,e}$.

Let us note that as an immediate consequence of Corollary 6.10 we have that $\Omega_k(\mathfrak{b}) \neq \emptyset$ if and only if $q \ge n$.

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