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FORECASTING LINEAR COMBINATIONS OF TIME SERIES

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(Key words)

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A brief review on the literature is presented and some of the available results are extended to more general situations. Forecast error variance is derived and forecasting efficiency is investigated for a linear combination (temporal and series aggregation) of a basic time series following a vector ARMA model. The resulting model for a linear combination (temporal aggregation) of a seasonal ARIMA model is derived.

Key words: temporal aggregation; series aggregation; vector ARMA model; seasonal ARIMA model; efficiency.

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1 - INTRODUCTION

An aggregate time series is a linear combination of the observations of one or more time series. We may aggregate over time (temporal aggregation), or over series (contemporal aggregation), or both. In this paper we refer to a general aggregate time series as a linear combination of k time series over a period of H time intervals. Temporal aggregation (univariate case) and series aggregation are treated as special cases where k=1 and H=1, respectively.

In econometric literature, there are two cases of temporal aggregation referred to as flow and stock problems: the flow case considers non-overlaping sums of observations; the stock case considers a systematic sample of the observations. In practical situations the user needs to decide on the time unit to be used for the basic observations and he often has to forecast aggregates of the original or basic time series. If, as it is often the case, monthly observations are available but yearly forecasts are needed, there are two possible approaches: a) to aggregate monthly data into yearly data and then model the latter to obtain yearly forecasts directly; b) to model and forecast monthly data and then aggregate to obtain yearly forecasts. There are at least three reasons for using the latter approach: a) when the number of observations is too small (in fact, some procedures require a reasonable number of observations in order to produce good parameter estimates); b) when both levels (e.g., monthly and yearly) are interesting to the user; c) when yearly forecasts obtained from monthly data are more precise than those obtained directly from yearly data.

Temporal aggregation has been well discussed in statistical and econometric literature. It was first investigated in econometrics by Theil(1954), Mundlak(1961), Orcutt, Watts and Edwards(1968), Moriguchi(1970), Zellner and Montmarquette (1971), Aigner and Goldfeld(1973 and 1974), Dunn, Williams and DeChaine(1976), Tiao and Wei(1976), Geweke(1978), Hsiao(1979), Palm and Nijman(1981) and others. Geweke(1979) derived procedures for optimal seasonal adjustment and aggregation.

Derivations of the resulting model for the aggregate series given the model for the original series were presented by Amemiya and Wu(1972), in the flow case for AR model, by Brewer(1973), in the flow and stock cases for ARMA and ARMAX models, by Wei(1979), in the flow case for seasonal and nonseasonal ARIMA models, by Granger and Morris(1976), for the sum of independent ARMA processes and by Rose(1977), for linear combinations of independent ARIMA processes.

The effect of aggregation on parameter estimation was considered by Tiao(1972), Tiao and Wei(1976), Wei(1978 and 1979) and Hsiao(1979). The effect of aggregation on forecasting was studied by Tiao(1972), Amemiya and Wu(1972), Tiao and Wei(1976), Granger and Morris(1976), Rose(1977), Tiao and Guttman(1980), Wei and Abraham(1981), Abraham(1982), Abraham and Ledolter(1982), and Kohn(1982).

Temporal aggregation is related to missing observation problem when time series observations may be divided into two periods: one with data in aggregate form and another with data in disaggregate form (see Harvey and Pierse, 1984).

In this paper some of the available results are extended to more general situations. We consider two approaches:

a) first model and then aggregate (I). Here the k original time series are considered to be a k-dimensional vector time series. Forecasts for the original time series are obtained from a vector ARMA model and then aggregated to obtain forecasts for the aggregate time series.

b) first aggregate and then model (II). Here the original time series are aggregated over time and over series to obtain the univariate aggregate time series. Forecasts for the aggregate time series are obtained from an univariate ARMA model.

Obtaining forecasts from individual models for each of the k series and then aggregating them to obtain forecasts for the aggregate time series should be a third approach (see Wei and Abraham, 1981), but it is not considered in this paper.

The variance of the forecast error (which should be used for confidence intervals) for the general aggregate time series, in approach I, is derived in part 2 of this paper. The resulting model for the aggregate time series, when the basic time series follows an univariate seasonal ARIMA model is derived in part 3. Efficiency of forecasting in approach I relatively to approach II for the general aggregate time series is given in part 4.

2 - FORECASTING LINEAR COMBINATIONS

2.1 - Univariate Time Series

Let $\{z_t,\ t = 0, \pm 1, \pm 2, \ldots\}$ be the basic univariate time series observed at equally spaced time intervals. Let $\{Y_T,\ T = 0, \pm 1, \ldots\}$ be the temporally aggregated time series defined as a non-overlaping linear combination of the basic time series observations:

$$Y_T = \sum_{h=0}^{H-1} w_h\, z_{t-h} = \left(\sum_{h=0}^{H-1} w_h\, B^h \right) z_t, \qquad (2.1)$$

where $t = TH$, B is the backward shift operator such that $B\, z_t = z_{t-1}$ and $w_0, w_1, \ldots, w_{H-1}$ are real known weights.

The following lemma is a well known result (see Box and Jenkins, 1976).

<u>Lemma 2.1</u> -

Suppose $\{z_t,\ t = 0, \pm 1, \ldots\}$ follows a stationary and invertible ARMA (autoregressive-moving average) model, written as

$$z_t = \sum_{j=0}^{\infty} \psi_j\, a_{t-j}, \qquad (2.2)$$

where $\psi_0 = 1$, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ and $\{a_t,\ t = 0, \pm 1, \ldots\}$ is a white noise process with variance σ_a^2. Then,

(i) the unbiased minimum mean square error (MMSE) forecast of z_{t+m}, at origin t, is given by

$$\hat{z}_t(m) = \sum_{j=0}^{\infty} \psi_{m+j}\, a_{t-j}; \tag{2.3}$$

(ii) the forecast error is

$$e_t(m) = z_{t+m} - \hat{z}_t(m) = \sum_{j=0}^{m-1} \psi_j\, a_{t+m-j}; \tag{2.4}$$

(iii) the variance of the forecast error is

$$V[e_t(m)] = \sigma_a^2 \sum_{j=0}^{m-1} \psi_j^2. \tag{2.5}$$

The forecasts for Y_T in approach I are given in the following theorem.

Theorem 2.1 -

Suppose z_t satisfies the conditions of Lemma 2.1 and Y_T is given by (2.1). Then,

(i) the unbiased MMSE forecast of Y_{T+M}, at origin T, is given by

$$\hat{Y}_T(M) = \sum_{h=0}^{H-1} w_h \hat{z}_{TH}(MH-h); \tag{2.6}$$

(ii) the forecast error is

$$e_T(H,M,1) = \sum_{h=0}^{H-1} w_h \sum_{j=0}^{MH-h-1} \psi_j a_{TH+MH-h-j}; \tag{2.7}$$

(iii) the variance of the forecast error is

$$V[e_T(H,M,1)] = \sigma_a^2 \sum_{h=0}^{H-1} \sum_{j=0}^{MH-h-1} \sum_{i=0}^{\min\{H-1,j+h\}} w_h w_i \psi_j \psi_{h+j-i}. \tag{2.8}$$

Proof.

(2.6) is given by Box and Jenkins(1976, p.128). Now,

$$e_T(H,M,1) = \sum_{h=0}^{H-1} w_h [z_{(T+M)H-h} - \hat{z}_{TH}(MH-h)]$$

$$= \sum_{h=0}^{H-1} w_h e_{TH}(MH-h)$$

wich produces (2.7); (2.8) follows from (2.7). □

Corolary 2.1.1 - (Special cases)

a) Aggregation. If $w_h=1$, for all h, then (2.1) is reduced to the case presented by Abraham(1982);

b) Systematic sampling. If $w_0=1$ and $w_h=0$ for $h \geq 1$, then (2.1) is reduced to the case presented by Abraham and Ledolter (1982).

2.2 - Vector Time Series

Let $\{\underset{\sim}{z}_t = (z_{1t}, \ldots, z_{kt})', t = 0, \pm 1, \ldots\}$ be the basic time series observed at equally spaced time intervals. Let $\{Y_T, T = 0, \pm 1, \ldots\}$ be the aggregate time series defined as a non-overlaping linear combination of the basic time series

$$Y_T = \sum_{i=1}^{k} \left(\sum_{h=0}^{H-1} w_{hi} B^h \right) z_{it} = \left(\sum_{h=0}^{H-1} \underset{\sim}{w}_h' B^h \right) \underset{\sim}{z}_t, \qquad (2.9)$$

where $t = TH$ and $\underset{\sim}{w}_h$ is a $k \times 1$ vector of real known weights.

The following lemma is a well known result (see Tiao and Box, 1981).

Lemma 2.2 -

Suppose $\{\underset{\sim}{z}_t, t = 0, \pm 1, \ldots\}$ follows a vector ARMA model, stationary and invertible, written as

$$\underset{\sim}{z}_t = \sum_{j=0}^{\infty} \underset{\sim}{\Psi}_j \underset{\sim}{a}_{t-j}, \tag{2.10}$$

where $\underset{\sim}{\Psi}_j$ are k x k matrices with $\underset{\sim}{\Psi}_0 = \underset{\sim}{I}$ and $\underset{\sim}{a}_t$ is a k x 1 vector of random shocks with covariance matrix $\underset{\sim}{\Sigma}$. Then,

(i) the unbiased MMSE forecast vector of $\underset{\sim}{z}_{t+m}$, at origin t , is given by

$$\hat{\underset{\sim}{z}}_t(m) = \sum_{j=0}^{\infty} \underset{\sim}{\Psi}_{m+j} \underset{\sim}{a}_{t-j}; \tag{2.11}$$

(ii) the forecast error vector is

$$\underset{\sim}{e}_t(m) = \underset{\sim}{z}_{t+m} - \hat{\underset{\sim}{z}}_t(m) = \sum_{j=0}^{m-1} \underset{\sim}{\Psi}_j \underset{\sim}{a}_{t+m-j}; \tag{2.12}$$

(iii) the covariance matrix of the forecast error is

$$V[\underset{\sim}{e}_t(m)] = \sum_{j=0}^{m-1} \underset{\sim}{\Psi}_j \underset{\sim}{\Sigma} \underset{\sim}{\Psi}_j'. \tag{2.13}$$

The forecasts of Y_T in approach I are given in the following theorem.

Theorem 2.2 -

Suppose $\underset{\sim}{z}_t$ satisfies the conditions of Lemma 2.2 and Y_T is given by (2.9). Then,

(i) the unbiased MMSE forecast of Y_{T+M}, at origin T, is given by

$$\hat{Y}_T(M) = \sum_{h=0}^{H-1} \underset{\sim}{w}_h' \, \hat{\underset{\sim}{z}}_{TH}(MH-h); \tag{2.14}$$

(ii) the forecast error is

$$e_T(H,M,k) = \sum_{h=0}^{H-1} \sum_{j=0}^{MH-h-1} \underset{\sim}{w}_h' \, \underset{\sim}{\Psi}_j \, \underset{\sim}{a}_{TH+MH-h-j}; \tag{2.15}$$

(iii) the variance of the forecast error is

$$V[e_T(H,M,k)] = \sum_{h=0}^{H-1} \sum_{j=0}^{MH-h-1} \sum_{i=0}^{\min\{H-1,j+h\}} \underset{\sim}{w}_h' \, \underset{\sim}{\Psi}_j \, \underset{\sim}{\Sigma} \, \underset{\sim}{\Psi}_{j+h-i}' \, \underset{\sim}{w}_i. \tag{2.16}$$

Proof.

Analogous to theorem 2.1. □

3 - MODEL FOR THE AGGREGATE TIME SERIES

Let z_t and Y_T be as in (2.1) and let x_t be an overlaping linear combination of the basic time series observations:

$$x_t = \left(\sum_{h=0}^{H-1} w_h B^h \right) z_t \qquad (3.1)$$

Therefore $Y_T = x_{TH}$ (systematic sample).

Assuming $w_0 \neq 0$, let r be an integer, $1 \leq r \leq H$, defined by

$$r = \max \{h \in \{0, 1, \ldots, H-1\} : w_h \neq 0\} + 1 \qquad (3.2)$$

The model for Y_T, when z_t follows a nonseasonal ARIMA model, is given in the following theorem.

Theorem 3.1 -

Suppose $\{z_t, t = 0, \pm 1, \ldots\}$ follows an ARIMA(p,d,q) model and Y_T is as in (2.1). Then, Y_T follows an ARIMA(p,d,q*) model with

$$q^* = \left[\frac{(H-1)(p+d) + q + r - 1}{H} \right], \qquad (3.3)$$

where [m] denotes the largest integer contained in m.

Proof.

Let $Z_t = (1 - B)^d z_t$ and $X_t = (1 - B)^d x_t$. Then, we may write

$$Z_t = \sum_{i=1}^{p} \phi_i Z_{t-i} + \sum_{j=0}^{q} \theta_j a_{t-j} \quad \text{and}$$

$$X_t = \left(\sum_{h=0}^{H-1} w_h B^h\right) Z_t$$

$$= \sum_{i=1}^{p} \phi_i \left(\sum_{h=0}^{H-1} w_h B^h\right) Z_{t-i} + \sum_{h=0}^{H-1} w_h \sum_{j=0}^{q} \theta_j a_{t-j-h}$$

$$= \sum_{i=1}^{p} \phi_i X_{t-i} + \sum_{j=0}^{q+r-1} \theta_j^* a_{t-j}^*$$

where $\theta_j^* = \sum_{i=0}^{j} \theta_i w_{j-i} / w_0$, $a_t^* = w_0 a_t$ and $a_t^* \sim (0, w_0^2 \sigma_a^2)$.

Therefore X_t follows an ARMA(p,q+r-1) model and x_t follows an ARIMA(p,d,q+r-1) model.

Now, whereas Y_T is a systematic sample of x_t, it follows (see Brewer, 1973) that Y_T follows an ARIMA(p,d,q*) model with

$$q^* = \left[p + d + \frac{q + r - 1 - p - d}{H} \right]$$

and (3.3) holds □

Remark - Under the conditions of theorem 3.1 it is easily seen that

$$\lim_{H\to\infty} q^* = p + d + \lim_{H\to\infty} r/H, \quad \text{with} \quad \lim_{H\to\infty} r/H = 0 \text{ or } 1.$$

The following theorem extends the result to the seasonal case.

Theorem 3.2 -

Suppose $\{z_t, t = 0, \pm 1, \ldots\}$ follows an ARIMA $(p,d,q) \times (P,D,Q)_s$ model, S is an integer such that SH = s, and Y_T is as in (2.1). Then, Y_T follows an ARIMA $(p,d,q^*) \times (P,D,Q)_S$ model, with q^* given by (3.3).

Proof.

By hypothesis,

$$\Phi_P(B^s)\ \phi_p(B)\ (1 - B^s)^D\ (1 - B)^d\ z_t = \Theta_Q(B^s)\ \theta_q(B)\ a_t,$$

where $\Phi_P(B^s)$ and $\phi_p(B)$ are autoregressive operators, $\Theta_Q(B^s)$ and $\theta_q(B)$ are moving average operators and a_t is a white noise process. Then,

$$\Phi_P(B^s)\ (1 - B^s)^D\ z_t = \Theta_Q(B^s)\ b_t, \qquad (3.4)$$

where $\phi_p(B)\ (1 - B)^d\ b_t = \theta_q(B)\ a_t,$

that is, b_t follows an ARIMA(p,d,q) model. Defining

$$W_T = \left(\sum_{h=0}^{H-1} w_h B^h \right) b_{TH},$$

by theorem 3.1, W_T follows an ARIMA(p,d,q*) model, with q* given by (3.3), that is,

$$\phi_p^*(\beta)\,(1 - \beta)^d\, W_T = \theta_{q^*}^*(\beta)\, a_T^*, \qquad (3.5)$$

where $\beta = B^H$, $\beta^S = B^{SH} = B^s$, $\phi_p^*(\beta)$ is an autoregressive operator, $\theta_{q^*}^*(\beta)$ is a moving average operator and a_T^* is a white noise process. Now,

$$\Theta_Q(B^s)\, W_T = \left(\sum_{h=0}^{H-1} w_h B^h \right) \Theta_Q(B^s)\, a_{TH}$$

$$= \left(\sum_{h=0}^{H-1} w_h B^h \right) \Phi_P(B^s)\,(1 - B^s)^D\, z_{TH}, \qquad (3.6)$$

by (3.4). From (2.1) and (3.6),

$$\Phi_P(\beta^S)\,(1 - \beta^S)^D\, Y_T = \Theta_Q(\beta^S)\, W_T.$$

Multiplying both sides of the latter equality by $\phi_p^*(\beta)\ (1-\beta)^d$ and using (3.5) we obtain

$$\Phi_P(\beta^S)\ \phi_p^*(\beta)\ (1-\beta^S)^D\ (1-\beta)^d\ Y_T = \Theta_Q(\beta^S)\ \phi_p^*(\beta)\ (1-\beta)^d\ W_T$$

$$= \Theta_Q(\beta^S)\ \theta_{q^*}^*(\beta)\ a_T^*$$

and the theorem is proved. □

<u>Corolary 3.2.1</u> - (Special cases)

Consider the special cases given in corolary 2.1.1, namely, aggregation (r=H) and systematic sampling (r=1). The models for Y_T, given several models for z_t, are presented in Table 1.

Table 1 - Models for z_t and Y_T in the cases of aggregation (flow) and systematic sampling (stock).

MODEL FOR z_t	MODEL FOR Y_T	
	AGGREGATION	SYSTEMATIC SAMPLING
AR(p)	ARMA(p,q*) (1) q*=[(H-1)(p-1)/H]	ARMA(p,q*) q*=[(H-1)p/H]
MA(q)	MA(q*) q*=[1+(q-1)/H]	MA(q*) q*=[q/H]
ARMA(p,q)	ARMA(p,q*) (2) $q^*=[\frac{(H-1)(p+1)+q}{H}]$	ARMA(p,q*) (2) $q^*=[\frac{(H-1)p+q}{H}]$
ARIMA(p,d,q)	ARIMA(p,d,q*) $q^*=[\frac{(H-1)(p+d+1)+q}{H}]$	ARIMA(p,d,q*) (3) $q^*=[\frac{(H-1)(p+d)+q}{H}]$
$ARIMA(p,d,q)x(P,D,Q)_S$ s = SH	$ARIMA(p,d,q^*)x(P,D,Q)_S$ (4) $q^*=[\frac{(H-1)(p+d+1)+q}{H}]$	$ARIMA(p,d,q^*)x(P,D,Q)_S$ $q^*=[\frac{(H-1)(p+d)+q}{H}]$

(1) This result was obtained by Brewer(1973). Amemiya and Wu (1972) obtained $q^*=[\{(H-1)(p+1)+1\}/H]$, if $H<p+1$, and $q^*=p$, if $H\geq p+1$.

(2) These results were obtained by Brewer(1973).

(3) This result was obtained by Abraham and Ledolter(1982).

(4) This result was obtained by Wei(1979).

4 - EFFICIENCY OF FORECASTING LINEAR COMBINATIONS

Now we shall compare the forecasts obtained in the two approaches we referred to in the introduction of this paper: a) first model and then aggregate(I); b) first aggregate and then model (II). Such comparison is made in theorems 4.1 and 4.2.

Theorem 4.1 -

Suppose $\underset{\sim}{z}_t$ and Y_T defined as in (2.9). Let $L_z = L(\underset{\sim}{z}_u : u \leq t)$ and $L_Y = L(Y_U : U \leq T)$ be the Hilbert spaces spanned by the processes $\underset{\sim}{z}_t$ and Y_T up to times t=TH and T, respectively. Let $\hat{Y}_T(M)$ and $\tilde{Y}_T(M)$ be the orthogonal projections of Y_{T+M} onto L_z and L_Y, respectively. Then,

$$E[Y_{T+M} - \hat{Y}_T(M)]^2 \leq E[Y_{T+M} - \tilde{Y}_T(M)]^2. \tag{4.1}$$

Proof.

Immediate, since $L_Y \subset L_z$. □

Wei and Abraham (1982) presented a similar result when $w_h=1$ for all h = 0, ..., H-1. This result implies that forecasts obtained in approach I are equally or more precise that those obtained in approach II.Theil(1954) also discussed some advantages of approach I. However, Aigner and Goldfeld (1974) pointed out that disaggregate data are scarce and usually have larger observation error than aggregate data.

Efficiency of approach I relatively to approach II may be measured comparing the respective forecast errors or, alternatively, relating their variances:

$$E(H,M,k) = V[e_T(H,M,k)] / V[e_T^*(H,M,k)], \qquad (4.2)$$

where $e_T(H,M,k)$ states for approach I and $e_T^*(H,M,k)$ for approach II. Theorem 4.2 shows how to evaluate this efficiency measure.

For the next theorem and corolaries define

$$A_\ell = \sum_{h=0}^{H-1} \sum_{j=J}^{(\ell+1)H-h-1} \sum_{i=0}^{G} \underset{\sim}{w}_h' \underset{\sim}{\Psi}_j \underset{\sim}{\Sigma} \underset{\sim}{\Psi}_{j+h-i}' \underset{\sim}{w}_i, \quad \text{and} \qquad (4.3)$$

$$A_\ell^* = \sum_{h=0}^{H-1} \sum_{j=J}^{(\ell+1)H-h-1} \sum_{i=0}^{G} w_h \, w_i \, \psi_j \, \psi_{j+h-i}, \qquad (4.4)$$

for $\ell = 0, 1, \ldots,$ with

$G = \min\{H-1, j+h\}$, and $J = \max\{0, \ell H-h\}$.

Theorem 4.2 -

Let Y_T be as in (2.9) and suppose

$$Y_T = \sum_{m=0}^{\infty} \gamma_m b_{T-m}, \tag{4.5}$$

where $\gamma_0 = 1$, $\sum_{m=0}^{\infty} \gamma_m^2 < \infty$ and $\{b_T,\ T = 0, \pm 1, \ldots\}$ is a white noise process with variance σ_b^2. Then, the efficiency measure given in (4.2) may be evaluated by

$$E(H,M,k) = \frac{\sum_{m=0}^{\infty} \gamma_m^2}{\sum_{m=0}^{M-1} \gamma_m^2} \quad \frac{\sum_{j=0}^{M-1} A_j}{\sum_{j=0}^{\infty} A_j}, \tag{4.6}$$

where A_j is given in (4.3).

Proof.

$$Y_T = \sum_{h=0}^{H-1} \underset{\sim}{w}_h' \underset{\sim}{z}_{t-h}$$

$$= \sum_{h=0}^{H-1} \sum_{j=0}^{\infty} \underset{\sim}{w}_h' \underset{\sim}{\Psi}_j \underset{\sim}{a}_{t-h-j} \tag{4.7}$$

From (4.5) and (4.7).

$$\sigma_b^2 \sum_{m=0}^{\infty} \gamma_m^2 = E(Y_T)^2$$

$$= \sum_{h=0}^{H-1} \sum_{j=0}^{\infty} \sum_{i=0}^{G} \underset{\sim}{w}_h' \underset{\sim}{\Psi}_j \underset{\sim}{\Sigma} \underset{\sim}{\Psi}_{j+h-i}' \underset{\sim}{w}_i$$

$$= \sum_{j=0}^{\infty} A_j \qquad (4.8)$$

From (2.5) and (4.8),

$$V[e_T^*(H,M,k)] = \sigma_b^2 \sum_{m=0}^{M-1} \gamma_m^2$$

$$= \frac{\sum_{m=0}^{M-1} \gamma_m^2}{\sum_{m=0}^{\infty} \gamma_m^2} \cdot \sum_{j=0}^{\infty} A_j$$

From (2.16),

$$V[e_T(H,M,k)] = \sum_{j=0}^{M-1} A_j$$

and (4.6) holds. □

Note that $e_T(H,M,k) = e_T^*(H,M,k)$ implies $E(H,M,k)= 1$ and that $E(H,M,k) = 1$ if and only if $V[e_T(H,M,k)]=V[e_T^*(H,M,k)$. Corolaries 4.2.1 and 4.2.2 give necessary and sufficient conditions for these situations.

Corolary 4.2.1 - (Conditions for $V[e_T(H,M,k)]=V[e_T^*(H,M,k)]$)

a) $V[e_T(H,M,k)] = V[e_T^*(H,M,k)]$, for all $T=0,\pm1,...,$ if and only if

$$\sigma_b^2 = \sum_{j=0}^{M-1} A_j \Big/ \sum_{m=0}^{M-1} \gamma_m^2; \tag{4.9}$$

b) $V[e_T(H,M,k)] = V[e_T^*(H,M,k)]$, for all $T=0,\pm1,...,$ and for all $M\geq1$, if and only if

$$A_\ell = \gamma_\ell^2 A_0, \text{ for all } \ell\geq1. \tag{4.10}$$

Proof.

(4.9) follows from (4.6) and (4.8).

$V[e_T(H,M,k)] = V[e_T^*(H,M,k)]$, for all $M\geq1$, if and only if (4.9) holds for all $M\geq1$. For $M=1$ and $M=2$,

$$\sigma_b^2 = A_0/\gamma_0^2 = A_0 = (A_0 + A_1) / (1 + \gamma_1^2)$$

and hence $A_1 = \gamma_1^2 A_0$. Therefore, (4.10) holds for $\ell=1$.

Suppose (4.10) holds for $\ell=M-1$. Then, it holds for $\ell=M$,

$$A_0 = \frac{A_{M-1} \sum_{j=0}^{M-2} A_j}{\gamma_{M-1}^2 \sum_{m=0}^{M-2} \gamma_m^2}, \text{ and hence } A_{M-1} = \gamma_{M-1}^2 \cdot A_0.$$

Therefore, (4.10) holds for all $\ell \geqq 1$. □

Corolary 4.2.2 - (Conditions for $e_T(H,M,k)=e_T^*(H,M,k)$)

a) $e_T(H,M,k) = e_T^*(H,M,k)$, for all $T=0,\pm1,\ldots$, if and only if

$$\sum_{h=0}^{H-1}\sum_{j=0}^{MH-h-1} \underset{\sim}{w}_h' \underset{\sim}{\Psi}_j \underset{\sim}{a}_{TH+MH-h-j} = \sum_{j=0}^{M-1} \gamma_j \, b_{T+M-j}, \qquad (4.11)$$

for all $T = 0, \pm1,\ldots$;

b) $e_T(H,M,k) = e_T^*(H,M,k)$, for all $T=0,\pm1,\ldots$, and for all $M\geq1$, if and only if

$$\sum_{h=0}^{H-1}\sum_{j=0}^{MH-h-1} \underset{\sim}{w}_h' \underset{\sim}{\Psi}_j \underset{\sim}{a}_{TH+MH-h-j} = \sum_{m=0}^{M-1}\sum_{h=0}^{H-1}\sum_{j=0}^{H-h-1} \gamma_m \underset{\sim}{w}_h' \underset{\sim}{\Psi}_j \underset{\sim}{a}_{(T+M-m)H-h-j};$$

for all $M\geq1$. (4.12)

Proof.

(4.11) follows from (2.4), (2.15) and (4.5).
Putting M=1 in (4.11) we obtain

$$b_T = \sum_{h=0}^{H-1}\sum_{j=0}^{H-h-1} \underset{\sim}{w}_h' \underset{\sim}{\Psi}_j \underset{\sim}{a}_{TH-h-j}, \quad \text{for all } T=0,\pm1,\ldots$$

and substituting in (4.11) we obtain (4.12). □

The special case where H=1 is presented in the following corolary.

Corolary 4.2.3 - (Contemporal aggregates)

Efficiency of contemporal aggregates may be measured by

$$E(1,M,k) = \frac{\sum_{m=0}^{\infty} \gamma_m^2}{\sum_{m=0}^{M-1} \gamma_m^2} \cdot \frac{\sum_{j=0}^{M-1} \underset{\sim}{w}' \underset{\sim}{\Psi}_j \underset{\sim}{\Sigma} \underset{\sim}{\Psi}_j' \underset{\sim}{w}}{\sum_{j=0}^{\infty} \underset{\sim}{w}' \underset{\sim}{\Psi}_j \underset{\sim}{\Sigma} \underset{\sim}{\Psi}_j' \underset{\sim}{w}} \qquad (4.13)$$

In this case,

a) $V[e_t(1,M,k)] = V[e_t^*(1,M,k)]$ if and only if

$$\sigma_b^2 = [\sum_{j=0}^{M-1} \underset{\sim}{w}' \underset{\sim}{\Psi}_j \underset{\sim}{\Sigma} \underset{\sim}{\Psi}_j' \underset{\sim}{w}] / \sum_{m=0}^{M-1} \gamma_m^2; \qquad (4.14)$$

b) $V[e_t(1,M,k)] = V[e_t^*(1,M,k)]$, for all $M \geq 1$, if and only if

$$\underset{\sim}{w}' \underset{\sim}{\Psi}_M \underset{\sim}{\Sigma} \underset{\sim}{\Psi}_M' \underset{\sim}{w} = \gamma_M^2 \underset{\sim}{w}' \underset{\sim}{\Sigma} \underset{\sim}{w} \text{ ,for all } M \geq 1; \qquad (4.15)$$

c) $e_t(1,M,k) = e_t^*(1,M,k)$, for all $t = 0, \pm 1, \ldots,$ if and only if

$$\sum_{j=0}^{M-1} \underset{\sim}{w}' \underset{\sim}{\Psi}_j \underset{\sim}{a}_{t+M-j} = \sum_{j=0}^{M-1} \gamma_j b_{t+M-j}, \text{ for all } t=0,\pm 1,\ldots \quad (4.16)$$

d) $e_t(1,M,k) = e_t^*(1,M,k)$, for all $t=0,\pm 1,\ldots$, and for all $M \geq 1$, if and only if

$$\underset{\sim}{w}' \underset{\sim}{\Psi}_j = \gamma_j \underset{\sim}{w}', \text{ for all } j \geq 0, \quad (4.17)$$

that is, $\underset{\sim}{w}$ is an eigenvector for each $\underset{\sim}{\Psi}_j$ and γ_j is the corresponding eigenvalue.

Remark -

Kohn (1982, theorem 1 and corolary 6) showed part (d) of corolary 4.2.3 for M=1 and showed that (4.17) is a necessary condition for $e_t(1,M,k) = e_t^*(1,M,k)$.

The special case where k=1 is presented in the following corolary.

Corolary 4.2.4 - (Univariate case)

Efficiency in the univariate case considered in theorem 2.1 may be measured by

$$E(H,M,1) = \frac{\sum_{m=0}^{\infty} \gamma_m^2}{\sum_{m=0}^{M-1} \gamma_m^2} \cdot \frac{\sum_{j=0}^{M-1} A_j^*}{\sum_{j=0}^{\infty} A_j^*} \qquad (4.18)$$

In this case,

a) $V[e_T(H,M,1)] = V[e_T^*(H,M,1)]$, for all $T=0,\pm1,\ldots$, if and only if

$$\sigma_b^2 = \sigma_a^2 \sum_{j=0}^{M-1} A_j^* \Big/ \sum_{j=0}^{M-1} \gamma_m^2; \qquad (4.19)$$

b) $V[e_T(H,M,1)] = V[e_T^*(H,M,1)]$, for all $T=0,\pm1,\ldots$, and for all $M\geq1$, if and only if

$$A_\ell^* = \gamma_\ell^2 A_0^*, \text{ for all } \ell\geq1; \qquad (4.20)$$

c) $e_T(H,M,1) = e_T^*(H,M,1)$, for all $T=0,\pm1,\ldots$, if and only if

$$\sum_{h=0}^{H-1} \sum_{j=0}^{MH-h-1} w_h \psi_j a_{(T+M)H-h-j} = \sum_{m=0}^{M-1} \gamma_m b_{T+M-m} \qquad (4.21)$$

d) $e_T(H,M,1) = e_T^*(H,M,1)$, for all $T=0,\pm1,\ldots$, and for all $M\geq1$ if and only if

$$\sum_{h=0}^{H-1} \sum_{j=J}^{MH-h-1} w_h \psi_j a_{(T+M)H-h-j} = \gamma_{M-1} \sum_{h=0}^{H-1} \sum_{j=0}^{H-h-1} w_h \psi_j a_{(T+1)H-h-j} \qquad (4.22)$$

for all $T=0,\pm1,\ldots$, and for all $M\geq1$.

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