Gelfand-Kirillov Conjecture as a First-Order Formula

Hugo Luiz Mariano*
Instituto de Matematica e Estatística, Universidade de São Paulo, São Paulo, Brasil
João Fernando Schwarz †
International Center for Mathematics, Shenzhen, China

Keywords: first-order axiomatization, algebraic geometry, non-commutative algebra.

Connections between Algebra and Logic are well known. In the specific topic of Algebraic Geometry, this line of inquiry began with the work of Alfred Tarski on the decidability via quantifier elimination in the theory of algebraically closed fields, and have achieved a remarkable development through the years using more sophisticated methods of Model Theory in Algebraic Geometry, such as the work of Ax, Kochen and Ershov on Artin's Conjecture, or the celebrated proof of Mordell-Lang Conjecture by Hrushovski.

Let's fix some conventions. All our rings and fields will be algebras over a base field k.

One of the main problems of algebraic geometry is the birational classification of varieties ([4]). In case k is algebraically closed and we work in the category of affine irreducible varieties the situation is rather simple: given two finitely generated domains A and B and corresponding varieties $X = \text{Spec } A$, $Y = \text{Spec } B$, they are birationally equivalent if and only if $\text{Frac } A = \text{Frac } B$. In general, two varieties are birationally equivalent if their function fields are isomorphic fields [4]. For a state-of-the-art introduction to the subject, see [6].

In the 1966 the study of birational geometry of noncommutative objects began. In his adress at the 1966 ICM in Moscow, A. A. Kirillov proposed to classify, up to birational equivalence, the enveloping algebras $U(\mathfrak{g})$ of finite dimensional algebraic Lie algebras $\mathfrak{g}$ when k is algebraically closed of zero characteristic. This means to find canonical division rings such that every skew field $\text{Frac } U(\mathfrak{g})$ of the enveloping algebras, which are an Ore domain, is isomorphic to one of them.

The idea became mature in the groundbreaking paper [2], where A. A. Kirillov and I. M. Gelfand formulated the celebrated Gelfand-Kirillov Conjecture. Before we formulate it, lets recall some definitions:

Definition. *The rank n Weyl algebra $A_n(\mathsf{k})$ is the algebra given by generators $x_1, \dots, x_n$ and $y_1, \dots, y_n$ and relations $[x_i, x_j] = [y_i, y_j] = 0; [y_i, x_j] = \delta_{ij},\ i, j = 1, \dots, n$. We denote by $A_{n,s}(\mathsf{k})$ the algebra $A_n(\mathsf{k}(t_1, \dots, t_s))$, for $n \geq 1, s \geq 0$. For the sake of notational simplicity, call $A_{0,s}(\mathsf{k}) = \mathsf{k}(t_1, \dots, t_s)$. In characteristic 0, as is well known, the Weyl algebras are finitely generated simple Noetherian domains [7]. We denote by $\mathbb{D}_{n,s}(\mathsf{k}), \mathbb{D}_n(\mathsf{k})$ the skew field of fractions of $A_{n,s}(\mathsf{k}), A_n(\mathsf{k})$, respectively. These skew fields are called the Weyl fields.*

Conjecture. (Gelfand-Kirillov Conjecture)*: Consider the enveloping algebra $U(\mathfrak{g})$, $\mathfrak{g}$ a finite dimensional algebraic Lie algebra over k algebraically closed of zero characteristic. Its skew field of fractions,* $\text{Frac } U(\mathfrak{g})$*, is of the the form $\mathbb{D}_{n,s}(\mathsf{k})$, for some $n, s \geq 0$.*

The purpose of this work is to show that, surprisingly, given a (reduced) root system Σ (cf. [1, 11.1]) and *any* algebraically closed field k with zero characteristic, the validity of the Gelfand-Kirillov Conjecture for the finite dimensional Lie algebra $\mathfrak{g}_{\mathsf{k},\Sigma}$ — that is, the only semisimple Lie algebra

*hugomar@ime.usp.br
†jfschwarz.0791@gmail.com

over k whose associated root system is Σ — is equivalent to the provability of a certain first-order sentence in the language of rings $\mathcal{L}(0, 1, +, *, -)$ in the theory of algebraically closed fields of zero characteristic — ACF_0 (cf. [5]).

Being more precise, we have:

Theorem. *Given a root system Σ, there is a first-order sentence ϕ_Σ in the language of rings $\mathcal{L}(1, 0, +, *, -)$ such that the below are equivalent:*

1. *For some algebraically closed field k of zero characteristic, the Gelfand-Kirillov Conjecture holds for $\mathfrak{g}_{\mathsf{k},\Sigma}$.*

2. *For all algebraically closed of zero characteristic k , the Gelfand-Kirillov Conjecture holds for $\mathfrak{g}_{\mathsf{k},\Sigma}$.*

3. *$ACF_0 \vdash \phi_\Sigma$.*

Moreover, ϕ_Σ is **naturally** *constructed as an existential closure of boolean combinations of atomic formulas in the language.*

This theorem is surprising because there is no a priori reason for the Gelfand-Kirillov Conjecture to be expressed in a first-order formula. In what follows, $\mathbb{A}$ will denote the field of algebraic numbers.

Let Σ be a root system and k an algebraically closed field of zero characteristic. We want to define the predicate $\mathcal{GK}(\mathsf{k}, \Sigma)$, that means that the Gelfand-Kirillov Conjecture is true for $\mathfrak{g}_{\mathsf{k},\Sigma}$. The initial definition is in ZFC.

If, for each k, we had a formula $\theta_{\mathsf{k},\Sigma}$ in the language $\mathcal{R}$ such that $\mathcal{GK}(\mathsf{k}, \Sigma)$ if and only if $\mathsf{k} \models \theta_{\mathsf{k},\Sigma}$, we would already have a remarkable fact.

However, there is a first-order formula θ_Σ in the language $\mathcal{R}$ such that $\mathcal{GK}(\mathsf{k}, \Sigma)$, for arbitrary k if and only if $\mathbb{A} \models \theta_\Sigma$. As AFC_0 is a complete theory, this holds if and only if $AFC_0 \vdash \theta_\Sigma$.

We remark also that the expression of a statement as a first-order sentence in ACF_0 is a very important question. One of the main applications of this idea is Lefchetz's Principle from algebraic geometry, since ACF_0 is a complete theory ([5]), in order to prove a statement for a variety over an algebraically closed field of zero characteristic, it suffices to show it for $\mathsf{k} = \mathbb{C}$, where transcendental methods are applicable. The Gelfand-Kirillov Conjecture is obviously in the realm of noncommutative algebraic geometry.

References

[1] J. Dixmier. *Algèbres Enveloppantes.* Gauthier-Villardd, Paris, 1974.

[2] I. M. Gelfand, K. K. Kirillov. *Sur les corps liés aux algèbres enveloppantes des algèbres de Lie.* Inst. Hautes Études Sci. Publ. Math. 31: 5–19, 1966.

[3] *Lectures on algebraic model theory.* Edited by Bradd Hart and Matthew Valeriote. Fields Institute Monographs, 15. American Mathematical Society, Providence, RI, 2002.

[4] R. Hartshorne. *Algebraic Geometry.* Springer-Verlag, 1977.

[5] W. Hodges. *A shorter model theory.* Cambridge University Press, Cambridge, 1997.

[6] Kollár J, Mori S. *Birational Geometry of Algebraic Varieties.* Cambridge University Press, 1998.

[7] J. C. McConnell, J. C. Robinson. *Noncommutative Noetherian rings.* With the cooperation of L. W. Small. Revised edition. Graduate Studies in Mathematics, 30. American Mathematical Society, Providence, RI, 2001. xx+636 pp.