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Classification of stable maps between 2-manifolds with given singular set image

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CLASSIFICATION OF STABLE MAPS BETWEEN 2 - MANIFOLDS WITH GIVEN SINGULAR SET IMAGE

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This paper is a continuation of [1] *. We apply to stable maps its results on extensions of immersions. We refer to it for basic notation and better explanation of the definitions. The category used is the $\mathcal{C}^\infty$ - category.

We classify the proper $\mathcal{C}^\infty$- maps $\Phi : M^2 \to N^2$ that have as singular set a given proper normal family of curves with cusps in N.

A non-compact <u>surface</u> (= $\mathcal{C}^\infty$ 2 - manifold) has <u>finite type</u> if it is equal to a compact one minus a finite number of points.

Let each of M and N be either a compact surface or a surface of finite type such that $\partial M = \partial N = \emptyset$. Let $\Phi : M \to N$ be $\mathcal{C}^\infty$. The singular set of Φ is $\Sigma_\Phi = \{ p \in M | d\Phi_p : TM_p \to TN_{(p)}$ is not injective$\}$

According to Whitney [10], a <u>generic</u> Φ is given in adequate local coordinates around $p \in \Sigma_\Phi$ by: $u=x$, $v=y^2$ (<u>fold point</u>), or $u = x$, $v = xy-y^3$ (<u>cusp point</u>). In this case, $C = \Sigma_\Phi$ is an imbedded curve in M and $f = \Phi|_C : C \to M$ has the set of cusps as singular set.

Let M and N be oriented, N connected. Consider a proper

* This work forms part of my Doctoral Thesis [2], under Prof. Mauricio Matos Peixoto. The first part is contained in [1].

generic map $\phi : M \to N$ such that, for $\Sigma_\phi = C$, $\phi|_{\Sigma_\phi} = f$, we have: (i) C has a finite number of components, (ii) f has as crossings a finite number of transverse double points in f (C-K), where $K = \Sigma_f$ is also finite.

We call a curve $f: C \to N$ satisfying (i) and (ii) a <u>proper normal curve with cusps</u>.

This work was conceived to answer a question of Haefliger [7] on the existence of a $\mathcal{E}^\infty$ map as above that extends a given normal curve with cusps $f : C \to N$, with $C \subset M$, when M is compact and $N = R^2$.

Francis-Troyer [6] solved Haefliger's problem using groupings similar to Francis' ones [5] . They also solved the case where M has finite type and $N = R^2$.

We solve the general case for either compact surfaces or surfaces of finite type. We use simple groupings as developped in [1] . This is because we can obtain the complete classification of the involved extensions and we eliminate some of the hypotheses. We also deduce a relation between invariants of the curve and invariants of the extensions, such as windings numbers and Euler characteristics.

1. Characterization of ϕ in the neighbourhood of Σ_ϕ.

Let $\phi : M \to N$ be a proper generic map satisfying the given conditions (i) and (ii).

If $\mathcal{V}_i$, $i=1,\ldots,n$, are the connected components of $M-\Sigma_\phi$, then $\phi|_{M-\Sigma_\phi}$ is an immersion of $\bigcup_{i=1}^{n} \mathcal{V}_i$ in N. We orient Σ_ϕ in such a way that ϕ preserves orientation to the left. Then, for each $p \in \Sigma_\phi$, there are local systems of coordinates $(\mathcal{V}, \phi)$ and $(\mathcal{W}, \psi)$ around p and $\phi(p)$ respectively, such that $\phi(p)=(0,0)$, $\psi\phi(p) = (0,0)$, det $(d\phi_q) > 0$ and det $(d\psi_{\phi(q)}) > 0$ for $q \in \mathcal{V}$ and where ϕ is given by one of the following forms:

(i) $u = s, \quad v = t^2$

(ii) $u = 2s^3 + st, \quad v = 3s^2 + t$

(iii) $u = 2s^3 - st, \quad v = 3s^2 - t$

We define $\theta = \theta_\phi : K \to \{-1, 1\}$ by $\theta(p) = -1$ in case (ii) and $\theta(p) = 1$ in case (iii). then $\theta(p)$ measures the <u>local degree of ϕ at p</u> as in Quine [9].

The following two lemmas characterize ϕ near Σ_ϕ. Their proofs can be found in [2, 6].

<u>Lemma 1</u>. Let $f: C \to N$ be a proper normal curve with cusps such that C is diffeomorphic to S^1 or $\mathbb{R}$. Let $\phi, \psi : C \times \mathbb{R} \to N$ be proper generic maps such that $\Sigma_\phi = \Sigma_\psi = C \times \{0\} = \Sigma$, $\phi|_\Sigma = \psi|_\Sigma = f$ and $\theta_\phi = \theta_\psi$. Then there is a unique orientation preserving diffeomorphism ϕ from a neighbourhood of Σ to a neighbourhood of Σ in $C \times \mathbb{R}$, such that $\psi = \phi \circ \phi$.

Lemma 2. Let $f: C \to N$ be a proper normal curve with cusps such that C is diffeomorphic to S^1 or $\mathbb{R}$. Let $\theta : K \to \Sigma_f \to \{-1, 1\}$ be any function. Then there is a proper generic map Φ_o from a neighbourhood of $C \times \{0\}$ in $C \times \mathbb{R}$ to N such that $\Sigma_{\Phi_o} = C \times \{0\}$, $\Phi_o | \Sigma_{\Phi_o} = f$ and $\theta_{\Phi_o} = \theta$.

2. Normal curves with cusps and groupings

Let $f: C \to N$ be normal with cusps. We can decompose f into disjoint simple curves by separating and smoothing at its double points. Let $\infty \in N - \operatorname{Im} f$ be a fixed, arbitary point. A cycle (rel. ∞) c of f is any of those simple curves that bordes an open, maybe unbounded, homeomorphic disc in $N - \{\infty\}$. Denote that disc by int c. A central cycle is a cycle c such that $c' \not\subset \operatorname{int} c$ for any cycle c'.

A set of rays for f, $R = R(f, \infty)$, is a finite set $R = R_c \cup R_e \cup R_k$ of oriented simple arcs in N with end at ∞ and such that, for $r, r' \in R$, $r \neq r'$, $v \in r \cap \operatorname{Im} f$ and not origin of r, it follows that r is transverse to f, $r \cap r' = \{\infty\}$ and v is neither a double point nor a cusp point of f. Moreover there is a ray $r \in R_k$ for each cusp point, if any, and it has origin in $f(K)$ and points to the exterior of the cusp. If $r \in R_c$ it has origin in a connected component of $N - \operatorname{Im} f$. Finally, R_e is formed by arcs (i) $\alpha_1, \ldots, \alpha_{g'}, \gamma_1, \ldots, \gamma_{gN}$ that are closed and represent generators of $\pi_1(N, \infty)$, (ii) $\eta_1, \ldots \eta_{\delta N}$ that are unbounded and proper, such that their complement in N is a collection of homeomorphic open discs and that the bounded components of $N - \bigcup_{i=1}^{\delta_N} \gamma_i$ contain all double points of f and all rays of $R_c \cup R_k$.

A set of rays for f is <u>sufficient</u> if, for each central cycle c of f, R contains at least a ray with origin in int c.

A normal curve is <u>paired</u> (rel. R_e) if either $R_e = \emptyset$ or the numbers of positive and negative crossings in each closed ray $r \in R_e$ are equal.

A <u>sign function for f</u> is a function $\theta : K \{-1, 1\}$.

An <u>associated graph to f</u> is a bichromatic graph M with set of vertexes $\{v_i, i=1,\ldots,n\}$, set of edges $\{a_j, j=1,\ldots,\rho\}$, a sign for each vertex given by c: $\{v_1,\ldots,v_n\} \to \{-1,1\}$ such that adjacent vertexes have opposite signs, and a bijection $a_j \leftrightarrow f_j$, $j=1,\ldots,\rho$.

For each $i=1,\ldots,n$, take $\gamma^i = \{a_j \mid j=1,\ldots,\rho, v_i \subset a_j\}$ and let

$$f^i = \{f_j : C_j \to N \mid f_j \leftrightarrow a_j \in \gamma^i\} : C^i \to N$$

be the normal curve with cusps associated to γ^i.

Let $\beta_1,\ldots,\beta_n > 0$ be integers, $\beta = \sum_{i=1}^{n} \beta_i$ and $\beta = (\beta_1,\ldots,\beta_n)$. Take disjoint circles $w_1,\ldots,w_\beta$ in a small open disk $\mathcal{U}$ in N - Im f such that $= \theta$ int $w_i \subset \mathcal{U}$ and $w_i = -\theta D_i$ where

$D_i = \text{int}\ \overline{w_i}$, $i=1,\ldots,\beta$. Take an <u>augmented curve g^{β_i} for f^i</u>, that is $g^{\beta_i} = f^i$ if $\beta_i = 0$ and otherwise, g^{β_i} has for component curves those of f and also β_i from the β circles, chosen without repetition when i runs over $\{1,\ldots,n\}$. For a given $\theta : K \to \{-1, 1\}$ consider for each $i=1,\ldots,n$:

$$K_i^+ = \{ p \in K \cap C^i \mid \theta(p) = 1\}, \qquad K_i^- = p \in K \quad C^{-1} | \theta(p) = -1\}$$

$$Q_i = (\text{Im}\ g^{\beta_i}) \cap (\bigcup_{r \in R} r) - f^i(\bar{K}_i), \quad \bar{K}_i = K_i^- \text{ if } c(v_i)=1,\ \bar{K}_i = K_i^+ \text{ if } c(v_i) = -1$$

For $v \in Q_i$ define <u>sign</u> $v = 1$, if v is a positive crossing of $\text{Im}\ g^{\beta_i}$ with $r \in R$ and <u>sign</u> $v = -1$ otherwise, including $v \in f(K)$. Define <u>index</u> $v = j$ where v is the j^{-th} crossing on r.

Let $\mathcal{S}_{Q_i}$ be the symmetric group on Q_i [11] and let S_i be the <u>sucessor permutation</u> (over the components of g^{β_i}). A <u>grouping permutation</u> on Q_i is a permutation P_i given, as a product of disjoint cycles, by trivial cycles and transpositions (uv) such that $u, v \in r$ and sign $u \neq$ sign v. A <u>β_i grouping for g^{β_i}</u> is a pair $A_i = (\beta_i, P_i)$ where $\beta_i \geq 0$ is an integer and P_i is a grouping permutation on Q_i.

A <u>$\mathcal{B}$ - grouping for f</u>, $A = A(R,f) = (\tau, \theta, B, P)$ is formed by a connected graph τ with n vertexes associated to f, a sign function θ for f, and n-tuples $\mathcal{B} = (\beta_1, \ldots, \beta_n)$ of non-negative integers and $P = (P_1, \ldots, P_n)$ of grouping permutations on $Q_1, \ldots, Q_n$, respectively.

A is <u>effective</u> if P_i moves every $u \in Q_i$ with sign $u = -1$ and if index $u <$ index v whenever (uv) is a cycle of P_i and sign $u=-1$. A is <u>transitive</u> if S_i and P_i generate a transitive subgroup of Q_i for $i=1,\dots,n$.

We should use <u>reduced groupings</u> [1, 2] . For simplicity, <u>we admit from now on</u>, that whenever $u, v \in r$ and sign $u \neq$ sign v we have $u \neq vS$ and $u \neq uS$, so that the <u>groupings are already reduced</u> (*).

Two groupings are <u>equivalent</u> if they differ by permutations on the β_i-circles of g^{β_i}. We call the first and last crossings on each unbounded component of g^{β_i} <u>special crossings</u>. An <u>orbit of A</u> is a cycle of S_iP_i, $i=1,\dots,n$. A <u>negative orbit of A</u> is any cycle of $S_i P_i$ in which occurs either some negative special crossing or a crossing of some η_j, $j=1,\dots,N$, fixed by P_i. A <u>simple grouping for f</u> is an effective β-grouping A for f such that: (a) sign $u \neq$ sign v, $u \neq u P_i$, $v \neq v P_i$ whenever $u, v \in r$ occur in the same non-negative orbit of A_i ; (b) in the negative orbits of A_i $(i=1,\dots,n)$ there are only negative crossing of γ_j and crossings of η_j fixed by P_i.

Let ξ , σ , ν and k be the numbers of orbits of A, negative orbits of A, negative crossings in Q and negative cusps of f, respectively. Denote by A_s (R, f) the set of transitive simple groupings of f and by $\mathcal{A}_s$(R, f) the set of its equivalent classes.

(*) The theorems of extension of section 3 are true for reduced groupings in general, as defined in [2].

We also consider an equivalence relation between the negative special crossings of f when $\delta_N > o$ and $\rho > \rho_o$. Let u~u' if u, u' $\in \gamma_k$, for some $k = 1,\ldots, \delta_N$ and if, for some i, j= 1,...,n $u' (P_j S_j^{-1})^t P_j = u(P_i S_i^{-1})^s P_i$ is a positive special crossing such that $u (P_i S_i^{-1})^{s-h}$ and $u' (P_j S_j^{-1})^{t-\ell}$ are not special crossings, for $0 \leq h < s$ (if s > o) and $0 \leq \ell < t$ (if t > o).

We denote by μ the set of such equivalence classes and by m the number of negative orbits of A with no negative special crossings.

3. <u>The extension theorems</u>

Let each of M and N be oriented, connected and either a compact surface or a surface of finite type. Let $f: C \to N$ be a proper normal curve with cusps and let $\phi : M \to N$ be a proper generic map. Then ϕ is a <u>stable extension of finite type</u> of f if $\phi|_{\Sigma_\phi} \sim f$ (that is, $f = \phi|_{\Sigma_\phi} \circ \alpha$ for some orientation preserving diffeomorphism $\alpha : C \to \Sigma_\phi$). If $\Psi : M' \to N$ is another stable extension of finite type of f, then ϕ and Ψ are <u>equivalent</u> if there is an orientation preserving diffeomorphism $\varphi : M \to M'$ such that $\phi = \Psi \circ \varphi$. We denote by $E_s(f)$ the set of stable extensions of finite type of f and by $\mathcal{E}_s(f)$ the set of its equivalence classes.

We recall notations fixed on previous sections for

<u>Theorem 1</u>. Let $f: C \to N$ be proper normal with cusps and let $R=R(f, \bullet)$ be a sufficient set of rays for f. We have:

(i) if $E_s(f) \neq \emptyset$ then f is paired,

(ii) if f is paired, then $E_s(f) \neq \emptyset$ if, and only if, $A_s(R,f) \neq \emptyset$

(iii) if $E_s(f) \neq \emptyset$, then there is a natural map

$\mathcal{A}_s: E_s(f) \to \mathcal{A}_s(R, f)$ and a bijection $\mathcal{E}_s(R,f) \to \mathcal{A}_s(R, f)$

such that the diagram

$$\begin{array}{ccc} E_s(f) & \xrightarrow{\mathcal{A}_s} & \mathcal{A}_s(R, f) \\ \downarrow & \nearrow_{\sim} & \\ \mathcal{E}_s(f) & & \end{array}$$

commutes, and

(iv) if $\phi \in E_s(f)$, $\phi: M \to N$, and $A \in \mathcal{A}_s(\phi)$ then M is compact if, and only if $\delta = 0$, and

$$2 - 2g_M - \delta_M = \xi - 2\nu + k + \beta - \delta + \rho - \rho_0,$$

$$\delta_M = m + \mu .$$

<u>Remarks</u>: Setting ν^i = number of negative crossings in Q_i, $i=1,\dots,n$ and $\nu' = \sum_{i=1}^{n} \nu^i$ the formula above becomes

$$2 - 2g_M - \delta_M = \xi - \nu' + \beta - \delta + \rho - \rho_0$$

If $\rho \neq \rho_0$ then $\delta > 0$. if $\rho = \rho_0$ and $\delta = 0$ so that M is compact, then the formula becomes:

(Euler characteristic of M =)

$$\chi(M) = \xi - 2\nu + k + \beta = \xi - \nu' + \beta .$$

This is still the case when $\rho = \rho_o$ and $\delta > o$, for then $\mu = 0$ and $\delta_M = m = \delta$. For extensions $\Phi : M \to N$ with M compact and N non-compact we can ommit the rays γ_j and η_j from the start.

The proof of theorem 1 is based on the two lemmas in section 2 and the extension theorem (for immersions) in [1]. Before proving the theorem, we state a formally simpler equivalent version of it, which is based on the following corollary to the mentioned extension theorem for immersions.

Corollary [1]. Let f: C → N be proper and normal. Let R be sufficient for f. Then f has an immersed extension $\hat{F}$:V → N, with V non necessarily connected, if, and only if, f is paired and has a simple grouping that satisfies the following property: if Q' ⊂ Q is an orbit of the subgroup of $\mathcal{G}_Q$ generated by S and P then $Q' \not\subset Q^\beta = Q \; (\bigcup_{i=1}^{\beta} w_i)$. Furthermore, these groupings classify the immersed extensions of f, via the given equivalence relations.

We need some notations.

Let f: C → N be proper normal with cusps, with N connected. Let R be sufficient for f and let $\theta : K \to \{-1, 1\}$.

We consider:

$K^+ = \{p \in K \mid \Theta(p) = 1\}$, $K^- = \{p \in K \mid \Theta(p) = -1\}$,

β^+ , β^- non-negative integers,

g^+, g^- augmented curves for f by β^+ and β^-, respectively,

$Q^+ = \mathrm{Im}\, g^+ \cap (\bigcup_{r \in R} r) - f(K^-)$, $Q^- = \mathrm{Im}\, g^- \cap (\bigcup_{r \in R} r) - f(K^+)$,

$Q = Q^+ \cup Q^-$.

Let the groupings $A^+ = (\beta^+, p^+)$ and $A^- = (\beta^-, p^-)$ be given on Q^+ and Q^-, respectively. The pair $A = (A^+, A^-)$ is <u>effective</u> when A^+ and A^- are effective. An effective pair A is <u>simple</u> when A^+ and A^- are simple. Let S be the sucessor permutation on Q and consider the natural extensions of $P^+$and P^- to Q given by $vP^+= v$ if $v \notin Q^+$, $vP^-= v$ if $v \notin Q^-$. A is a <u>transitive pair</u> if the subgroup of $\mathcal{S}_Q$ generated by S, P^+ and P^- is transitive. Finally, we say that $A_1 =(A_1^+, A_1^-)$ and $A_2 = (A_2^+, A_2^-)$ are <u>equivalent</u> if $A_1^+ \sim A_2^-$.

<u>Theorem 2</u>. f has a stable extension of finite type $\Phi : M \to N$ if and only if, f is paired and has a simple transitive pair $A = (A^+, A^-) = A(R,f)$ of groupings.
Furthermore, these groupings classify the stable extensions of f , via the given equivalence relations.

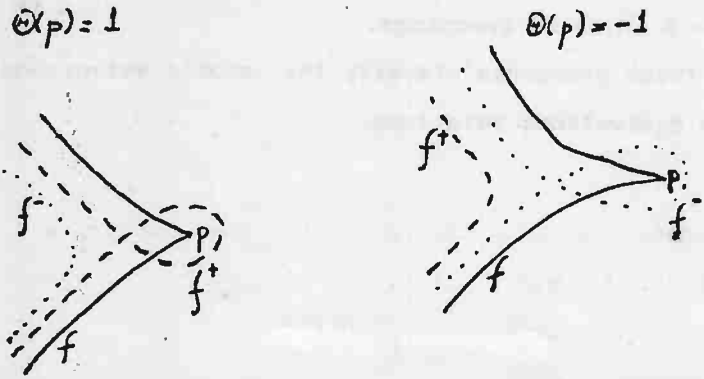
<u>Comment on the proof</u>. It can be proved by reduction to theorem 1 via the stated corollary, for instance. The graph used appears naturally as each component curve f_i of f occurs once for each of A^+ and A^-. Transitivity then implies that the graph and thus M is connected.

<u>Proof of theorem 1</u>. Let $\phi : M \to N$, $\phi \in E_s(f)$. Then

$$f = \phi|_{\Sigma_\phi} \circ \alpha : C \to \bigcup_{j=1}^{\rho} C_j \to N$$

Set $\alpha_j = \alpha|_{C_j}$. Let $\mathcal{V}_i$ be the components of $M-\Sigma_\phi$.

ϕ determines a connected graph Γ_ϕ such that $v_i \leftrightarrow \mathcal{V}_i$, $v_i \subset a_j$ if $\alpha_j(C_j) \subset \overline{\mathcal{V}_i}$, and $c(v_i) = 1$ or -1 according to ϕ preserve orientation or not. By Lemmas 1 and 2, we have for $j=1,\dots,\rho$, extensions ϕ_o^j of $f|_{C_j}$ and orientation preserving embeddings ϕ_j such that $\phi_j|_{C_j \times \{0\}} = \alpha_j$ and $\phi \circ \phi_j = \phi_o^j$. We consider suitable curves $f_j^+ : C_j^+ \to N$, $f_j^- : C_j^- \to N$ as illustrated, $j=1,\dots,\rho$.



For each $i=1,\ldots,n$, let $V'_i \subset \mathcal{V}_i$ be the submanifold of M with boundary $\partial V'_i = \bigcup_{a_j \in \gamma^i} \phi_j(C_j^+)$ if $c(\nu_i)=1$ and $\partial V'_i = \bigcup_{a_j \subset \gamma^i} \phi_j(C_j^-)$ if $c(v_i)= -1$. Let $V_i=V'_i$ if $c(v_i) = 1$, $V_i=-V'_i$ if $c(v_i)=-1$, and $F_i: V_i \to N$ be the resulting orientation preserving immersion given by $F_i(x) = \phi(x)$.

Then $F_i|\partial V_i \sim \phi^i$, where ϕ^i is the normal curve given by

$$\phi^i = \{ f_j^+ : C_j^+ \to N \mid a_j \in \gamma^i \} \quad \text{if } c(v_i) = 1,$$

$$\phi^i = \{ f_j^- : C_j^- \to N \mid a_j \in \gamma^i \} \quad \text{if } c(v_i) = -1$$

As F_i is an immersed extension of ϕ^i then ϕ^i is paired, by the extension theorem for immersions [1]. Thus f is also paired and (i) is proved. Each cycle of ϕ^i contains a cycle of f [1 ,2] so R is sufficient for ϕ^i. Again by the extension theorem for immersions we get a transitive simple grouping $\bar{A}_i = (\beta_i, \bar{P}_i)$ on $\bar{Q}_i = \text{Im}\ \phi^i \cap (\bigcup_{r \in R} r)$. Exchanging crossings on Im ϕ^i by crossings on Imf^i we get transitive single groupings $A_i = (\beta_i, P_i)$ for f^i. It follows that $A = (\Gamma_\phi, \theta_\phi, \beta, P) \in A_s(R,f)$, where $B = (\beta_1,\ldots, \beta_n)$ and $P=(P_1,\ldots P_n)$. This proves the necessary condition of (iii).

Suppose that f is paired and has the transitive simple grouping $A= (\Gamma, \theta, B,P)$. Let r, r_i, $i=1,\ldots n$ be the numbers of negative crossings and s, s_i, be the numbers of positive crossings of Im f and Im f^i, respectively, with either a ray α_j or γ_ℓ. As A_i is effective then $s_i \geq r_i$, i=1,...,n. The hypothesis $s_k > r_k$ for some k implies the absurd

$$2s = \sum_{i=1}^{n} s_i > \sum_{i=1}^{n} r_i = 2r .$$

Thus the f^i are all paired. For each j=1,...,ρ we have the extension ϕ_o^j of f_j. On the other hand, from the groupings A_i on Q_i we get transitive groupings $\bar{A}_i = (\beta_i, \bar{P}_i)$ for ϕ^i, i=1,...,n. The extension theorem for immersions [1] provides immersed extension $F_i: V_i \to N$ for ϕ^i. Let $V'_i = V_i$ and $F'_i = F_i$ if $c(v_i) = 1$ and, if $c(v_i) = -1$, let $V'_i = -V_i$, $F'_i: V'_i \to N$ be the obvious orientation reversing immersions. Glueing together suitably all these extensions we get the desired stable extension $\phi : M \to N$.

To prove (iii), given $\phi \in E_s(f)$ we consider the grouping $A \in A_s(R,f)$ obtained in (ii). Define $\mathcal{A}_s(\phi)$ as the equivalence class of A in $\mathcal{A}_s(R,f)$. We remind that the extensions ϕ_o^j of the f_j are unique up to diffeomorphism. Then the extension theorem for immersions proves that the quotient map $\mathcal{E}(f) \to \mathcal{A}_s(R,f)$ is well defined and is a bijection.

To prove (iv) we chose a canonical maximal submanifold D of N. Consider $M' = \phi^{-1}(D)$, and the submanifolds W_i of $F_i^{-1}(D)$ obtained by smoothing the corners of $F_i^{-1}(D)$. We have the formulae[1]:

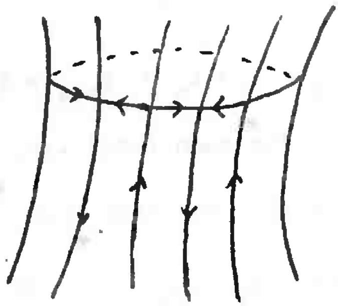
$$2 - 2g^i - \rho_o^i - \delta^i = \chi(W_i) = \xi^i - \nu^i - \beta^i - \delta^i + \rho^i - \rho_o^i$$

and $$\chi(M') = \sum_{i=1}^{n} \chi(W_i) - (\rho - \rho_o).$$

Thus $\chi(M') + (\rho - \rho_0) = \xi - (2\nu + k) + \beta - \delta + 2\rho - 2\rho_0,$

and $2 - 2g_M - \delta_M = \xi - 2\nu + k + \beta - \delta + \rho - \rho_0.$

Furthermore, each component of $\phi^{-1}(\partial D)$ is a component of $\partial M'$ that either does nor cross Σ_ϕ or cross it as illustrated. Thus $\delta_M = m + \mu$.



4. Related formulae

Haefliger [7], Marx-Ezell [4] and Levive [8] among others provided global formulae for extensions, that include winding numbers as well as Euler characteristics. Our goal is to present a similar formula for the stable case. We use a double tangent winding number that generalizes the tangent winding number for normal curves on surfaces [3] and the double or projective winding number for curves in the plane [8].

Let N be compact, $f: S^1 \to N$ normal with cusps. Assume for simplicity that $N \subset R^3$ and so has a natural structure of Riemannian manifold. Let $\infty \in N - \text{Im } f$ be fixed and let $\mathcal{U}' \subset \mathcal{U}$ be small neighbourhoods of ∞ in N-Im f. Let $X: N \to TR^3$ be a vector field tangent to N, with a finite number of zeros, all in $\mathcal{U}'$.

Consider $\alpha : [0,1] \to N$ given by $\alpha(t)=f(\cos 2\pi t, \text{sen } 2\pi t)$, where $\alpha(0) = \alpha(1) \notin f(K)$. If $\bar{z} = (\cos 2\pi \bar{t}, \text{sen } 2\pi\bar{t}) \in K$ then $\dfrac{d\alpha_t(1)}{\|d\alpha_t(1)\|}$ has well defined limits v^+, $v^- = -v^+ \in T_{f(\bar{z})}N$ when $t \to \bar{t}^{\pm}$. Let $0 = t_o < t_1 < \ldots < t_n < t_{n+1} = 1$ where $\{\alpha(t_1),\ldots, \alpha(t_n)\} = f(K)$. For each $i=0,\ldots,n$, let $\theta_i : [t_i, t_{i+1}] \to R$ be a $\mathcal{C}^\infty$-map such that $\theta_i(t)$ is a determination of the angle from $\dfrac{X(\alpha(t))}{\|X(\alpha(t))\|}$ to

$$\begin{cases} v^+ & \text{if } t = t_i \neq 0 \\ v^- & \text{if } t = t_{i+1} \neq 1 \\ \dfrac{d\alpha_t(1)}{\|d\alpha_t(1)\|} & \text{in all other cases.} \end{cases}$$

The numbers $\theta_i(t_{i+1}) - \theta_i(t_i)$ do not depend on the choice of the θ_i's. Furthermore $\theta_n(1) - \theta_o(0)$ is a multiple of 2π and for $i=1,\ldots,n-1$ the numbers $\theta_{i+1}(t_{i+1}) - \theta_i(t_{i+1})$ are multiples of π. Thus $\sum_{i=0}^{n} [\theta_i(t_{i+1}) - \theta_i(t_i)]$ is a multiple of π.

We define the <u>double winding number of f relative to X</u> by

$$\tau_X^{(2)}(f) = \sum_{i=0}^{n} \frac{\theta_i(t_{i+1}) - \theta_i(t_i)}{\pi} = 2 \sum_{i=0}^{n} \frac{\theta_i(t_{i+1}) - \theta_i(t_i)}{2\pi}$$

Remark. Consider the C^∞-map $\phi: S' \to S^1$ given by $\phi(z) = (\cos 2\theta_i(t), \text{sen}\, 2\theta_i(t))$, for $Z = (\cos 2\pi t, \text{sen}\, 2\pi t)$ and $t \in [t_i, t_{i+1}]$, $i=0,\dots,n$. Then $\tau_X^{(2)}(f)$ is the degree of ϕ.

Let $f:C \to N$ be normal with C diffeomorphic to S^1. If $g,h: S^1 \to N$ are normal curves with cusps equivalent to f then $\tau_X^{(2)}(g) = \tau_X^{(2)}(h)$. So we define $\tau_X^{(2)}(f) = \tau_X^{(2)}(g)$. Let $f:C \to N$ be normal with cusps, with ρ components. We define the double winding number of relative to X by

$$\tau_X^{(2)}(f) = \sum_{i=1}^{\rho} \tau_X^{(2)}(f_i).$$

Lemma. If g can be obtained from f by a homotopy fixed on a neighbourhood J of K in S^1 and regular outside J then $\tau_X^{(2)}(g) = \tau_X^{(2)}(f)$.

Proof. Suppose $C = S^1$ and let $\phi : S' \to S^1$ as in the remark. A homotopy from f to g gives rise to a homotopy starting on ϕ. As the degree of ϕ does not change, the result follows.

Lemma. If $g_1, \dots g_r$ are the simple curves obtained in the decomposition of f then

$$\tau_X^{(2)}(f) = \sum_{i=1}^{r} \tau_X^{(2)}(g_i)$$

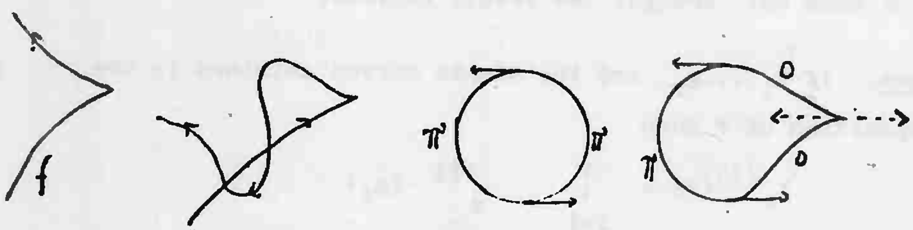
Proof. Let $f(z_1) = f(z_2)$, $z_1, z_2 \in C$. Let g be obtained from f by small homotopy H, fixed on z_1, z_2 and outside neighbourhood of z_1 and z_2, and otherwise regular. We can choose H so that $dg_{z_1} = dg_{z_2}$. Anyway, $\tau_X^{(2)}(g) = \tau_X^{(2)}(f)$. Repeat for the other double point of f and reparametrize conveniently.

Lemma. Let the positive cycle $f:S^1 \to N$ be normal with k cusps. If f is contained in a disc in N where $X(x) \neq 0$ and if no cusps points to int (Imf) then

$$\tau_X^{(2)}(f) = 2-k.$$

Proof. By a convenient homotopy we get a normal curve with cusps g having, for simple curves, one positive normal cycle, k normal negative cycles and k positive cycles with one cusp each. Using a new homotopy, if necessary, we may assume that each of these curves lie in an open coordinate neighbourhood of N conformal to a subset of the plane. Thus

$$\tau_X^{(2)}(f) = 2 + \sum_1^k (-2) + \sum_1^k 1 = 2 - k .$$



<u>Lemma</u>. If f is normal then $\tau_X^{(2)}(f) = 2\ \tau_X(f)$ where $\tau_X(f)$ is the tangent winding number of f [3] .

<u>Proof</u>. Immediate

<u>Theorem</u>. Let $\phi : M \to N$ be generic such that $\phi|_{\Sigma_\phi}$ is normal with cusps, M and N compact. Let β be the cardinality of $\phi^{-1}(\infty)$. Then

$$\chi(M) = \tau_X^{(2)}(\phi|_{\Sigma_\phi}) + \beta\ \chi(N).$$

<u>Proof</u>. For each i=1,...,n , we have

$$\chi(V_i) = \tau_X(F_i|_{\partial V_i}) + \beta_i\ \chi(N),$$

thus

$$\chi(M) = \sum_{i=1}^{n} \tau_X(F_i|_{\partial V_i}) + \beta\ \chi(N).$$

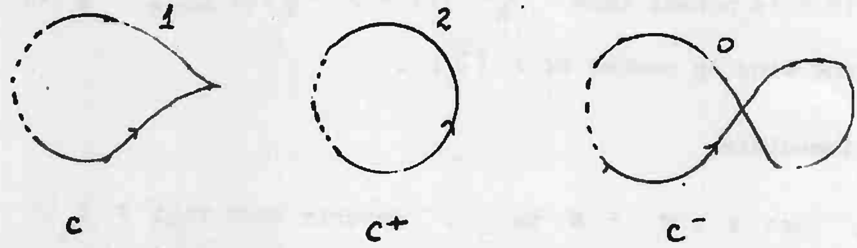
According to the sign function θ_ϕ , consider the normal maps f^+, f^-, f_j^+, f_j^- , i=1, ...,ρ. Then

$$\sum_{i=1}^{n} \tau_X(F_i|_{\partial V_i}) = \sum_{j=1}^{n} (\tau_X(f_j^+) + \tau_X(f_j^-)) =$$

$$= \frac{1}{2}\left[\sum_{j=1}^{\rho} (\tau_X^{(2)}(f_j^+) + \tau_X^{(2)}(f_j^-))\right] =$$

$$= \frac{1}{2}\left[\tau_X^{(2)}(f^+) + \tau_X^{(2)}(f^-)\right] = \tau_X^{(2)}(\phi|_{\Sigma_\phi}).$$

To justify the last equality we illustrate the case of a cusp of sign -1 .



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