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Analytic Integrability for a Class of Cone Potential Hamiltonian Systems*

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The main purpose of the present paper is to prove the *analytic* Liouville integrability of some classes of Hamiltonian systems, using the components of the asymptotic velocities as the constants of motion, independent and in involution. These classes include, for example, a system of n particles on the line, interacting under some potentials like the inverse r-power potential with $r > 0$ (including the Coulombian potential, $r = 1$, and the Calogero–Moser potential, $r = 2$). The method can be adapted to the Toda-like case and in particular to the non periodic Toda system. © 1991 Academic Press, Inc.

INTRODUCTION

After the work of Gutkin, presented in a series of papers (see [G1, G2, G3]) the idea of proving Liouville integrability, using the components of the asymptotic velocities as first integrals in involution, has been the purpose of some very recent papers. In the work [OC] of Oliva and Castilla is presented a proof of the C^∞-integrability of the so-called Toda-like systems, in which the forces span a cone with a width not larger than $\pi/2$, and also of other examples with wider cones but only in two or three degrees of freedom. In [G4], Gutkin raised other questions admitting somewhat more general potentials, including potentials with cones wider than $\pi/2$, but he did not address the problem of smoothness of the asymptotic velocities

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with respect to the initial data. In [GZ1], Gorni and Zampieri developed a framework in order to include the proof of C^∞-integrability for a large class of cone potentials with a width not larger than $\pi/2$, and they also obtained an interesting persistence result. In [GZ2] the same authors considered the reduction to a normal form of the scattering problem corresponding to the above class by the use of a suitable smooth canonical transformation and in [GZ3] Gorni and Zampieri solved completely the question of the C^∞-smoothness mentioned above and their results include many examples which are described in the introduction of that paper.

Since all these examples are real analytic Hamiltonian systems, we decided to discuss, in the present paper, the question of the real analyticity of the asymptotic velocities, as functions of the initial conditions. In Section 1, just for a sake of completeness, we develop some preliminaries and present Lemma 1.1, according to the results of [G4, GZ3]. In Section 2 we prove the (real) analytic Liouville integrability for potentials which are the sum of inverse r_α-powers, so that, they include the Coulombian potential ($r_\alpha \equiv r = 1$) and the Calogero–Moser potential ($r_\alpha \equiv r = 2$); our method can be easily adapted to the Toda-like case (in particular to the non periodic Toda system) but we were not able to apply it to the inverse-logarithmic long range potential also considered in [GZ3]. The idea is to use a compactification procedure already considered in [OC]. After the compactification, either the system admits a linear analysis (which is the case of the Toda-like systems) or the system is degenerate (that means it does not have linear part). In this last case, by changing coordinates and time, we were able to obtain, in a higher dimensional space, an auxiliary analytic system of ordinary differential equations depending on parameters and having a non zero linear part at the origin which is a fixed point for all the values of the parameters. Using properly, in the auxiliary system, the analytic version of the theory of strongly stable invariant manifolds (see [He]), we could finally return to the starting problem to obtain the (real) analyticity of the first integrals.

1. Preliminaries

We consider a set of vectors $\{v_\alpha\}_{\alpha=1,\dots,N}$, $v_\alpha \in \mathbf{R}^n \setminus \{0\}$, spanning a "proper cone" $\mathscr{C}$ (see [OC]), and the functions

$$f_\alpha \in \mathscr{C}^2(\mathscr{D}, \mathbf{R}), \qquad \mathscr{D} = \mathbf{R} \qquad \text{or} \qquad \mathscr{D} =]0, +\infty[, \qquad \alpha = 1, \dots, N.$$

We assume

$$\begin{aligned} &\inf f_\alpha = 0, \qquad \sup f_\alpha = +\infty, \\ &f'_\alpha(x) < 0, \qquad f''_\alpha(x) > 0, \qquad \alpha = 1, \dots, N, \; x \in \mathscr{D}. \end{aligned} \tag{H_0}$$

Defining

$$U(q) = \sum_{\alpha=1}^{N} c_\alpha f_\alpha(q \cdot v_\alpha), \qquad c_\alpha > 0, \tag{1.1}$$

for q in the set $\{q \in \mathbf{R}^n \mid q \cdot v_\alpha \in \mathscr{D},\ \alpha = 1, ..., N\}$, we consider the Hamiltonian system

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\operatorname{grad} U \end{aligned} \tag{1.2}$$

corresponding to the Hamiltonian function $H = \| p \|^2/2 + U(q)$. Then, it is easy to verify that the solutions of this Hamiltonian system are globally defined. Moreover, for any solution $(q(t), p(t))$, we have:

(a) There exists the limit: $\lim_{t \to +\infty} p(t) =: p_\infty$;

(b) Let I_0 be any non empty subset of $I = \{1, ..., N\}$. Then there exists a vector q_E, depending on the initial data (q_0, p_0) and on I_0, such that

$$q(t) \cdot v_\alpha \geqslant q_E \cdot v_\alpha, \qquad \forall t \in \mathbf{R},\ \forall \alpha \in I_0; \tag{1.3}$$

(c) $(p_\infty \cdot v_\alpha) \geqslant 0$, $\alpha = 1, ..., N$.

The properties (a), (b), (c) are well known from the literature on "cone potential" Hamiltonian systems [G1, OC, GZ1]). In fact (a) is almost obvious, due to the fact that the components of $p(t)$, in a suitable basis, are monotone functions of time as a consequence of the "cone potential" assumption.

Let us recall the way in which (b) is proved. From the proper cone assumption on $\{v_\alpha\}_{\alpha=1,...,N}$, one can show, using the Hahn–Banach theorem, that, for any I_0, a vector $\bar{v}$ can be found such that

$$\begin{aligned} \bar{v} &= \sum_{\alpha \in I_0} \rho_\alpha v_\alpha, \qquad && \rho_\alpha > 0, \\ \bar{v} \cdot v_\alpha &> 0, && \forall \alpha \in I_0. \end{aligned} \tag{1.4}$$

Denoting by E the energy (that is, the value of H) corresponding to the initial conditions (q_0, p_0), we have

$$f_\alpha(q(t) \cdot v_\alpha) \leqslant E c_\alpha^{-1}, \qquad \forall t \in \mathbf{R}, \forall \alpha \in I,$$

and, therefore, in view of the monotone decreasing property of f_α,

$$q(t) \cdot v_\alpha \geqslant f_\alpha^{-1}(E c_\alpha^{-1}), \qquad \forall t \in \mathbf{R}, \forall \alpha \in I.$$

Setting:

$$\Theta_E = \min_{\alpha \in I_0} \left\{ \frac{1}{\bar{v} \cdot v_\alpha} f_\alpha^{-1}(E c_\alpha^{-1}) \right\},$$

$$q_E = \Theta_E \bar{v},$$

property (b) is proved.

Finally (c) is an obvious consequence of (b), taking $I_0 = I$, and arguing by contradiction.

Our interest is devoted to a class of Hamiltonian functions such that the "asymptotic velocities" p_∞ belong to the interior of the dual cone of $\mathscr{C}$, i.e.,

$$p_\infty \cdot v_\alpha > 0, \qquad \alpha = 1, ..., N.$$

According to the results of [G4, GZ3] we can state the following lemma, and we give the proof for the sake of convenience of the reader.

1.1. LEMMA. *Assume that the functions f_α, $\alpha = 1, ..., N$, satisfy one of the two additional hypotheses*:

(H_1) *$f_\alpha \equiv f$, $\alpha = 1, ..., N$, and, for large x, $x|f'(x)|$ is monotone decreasing*;

(H_1') *The functions f_α are $L^1(1, +\infty)$ functions.*

Then, the asymptotic velocities of the solutions of the corresponding Hamiltonian system are in the interior of the dual cone.

Proof. Assume, by absurd, that there exist a solution $(q(t), p(t))$, with $p(t) \to p_\infty$ as $t \to +\infty$, and $I_0 \subset \{1, ..., N\}$, $I_0 \neq \varnothing$, such that

$$p_\infty \cdot v_\alpha = 0, \qquad \forall \alpha \in I_0. \tag{1.5}$$

Let us denote by I_1 the complementary set of I_0 in $\{1, ..., N\}$ and consider the vector $\bar{v}$ corresponding to I_0, as in (1.4).

We start by analyzing the case in which (H_1) holds.

Define

$$g(t) := q(t) \cdot \bar{v}.$$

We have

$$\begin{aligned} &\dot{g}(t) = p(t) \cdot \bar{v}, \\ &\ddot{g}(t) = \sum_{\alpha \in I_0} c_\alpha |f'(q(t) \cdot v_\alpha)| (v_\alpha \cdot \bar{v}) \left\{ 1 + \frac{\sum_{\sigma \in I_1} c_\sigma |f'(q(t) \cdot v_\sigma)| (v_\sigma \cdot \bar{v})}{\sum_{\alpha \in I_0} c_\alpha |f'(q(t) \cdot v_\alpha)| (v_\alpha \cdot \bar{v})} \right\}. \end{aligned} \tag{1.6}$$

We want to prove that $\ddot{g}(t)$ is positive for large t. Indeed, this is obvious if $I_1 = \emptyset$. Assume $I_1 \neq \emptyset$; then for t large enough, we have

$$q(t) \cdot v_\sigma \geqslant \gamma t, \qquad \forall \sigma \in I_1, \qquad \text{for} \quad \gamma = \tfrac{1}{2} \min_{\sigma \in I_1} \{p_\infty \cdot v_\sigma\}, \tag{1.7}$$

and (1.5) implies

$$\frac{q(t) \cdot v_\alpha}{t} \to 0 \qquad \text{as} \quad t \to +\infty, \qquad \forall \alpha \in I_0. \tag{1.8}$$

Then, by using (H_1), (1.7) and (1.8), we can prove that

$$\lim_{t \to +\infty} \frac{|f'(q(t) \cdot v_\alpha)|}{f'(q(t) \cdot v_\sigma)|} = +\infty, \qquad \forall \sigma \in I_1, \forall \alpha \in I_0.$$

Therefore $\ddot{g}(t) > 0$, for t large enough. Since

$$\dot{g}(t) \to 0 \qquad \text{as} \quad t \to +\infty, \tag{1.9}$$

one sees that, for large t, $g(t)$ is monotone decreasing, and therefore, by property (b)

$$K_1 \leqslant q(t) \cdot v_\alpha \leqslant K_2, \qquad \forall \alpha \in I_0,$$

where

$$K_1 = \min_{\alpha \in I_0} \{q_E \cdot v_\alpha\}.$$

Finally, setting

$$m = \min_{\substack{\alpha \in I_0 \\ K_1 \leqslant x \leqslant K_2}} \{c_\alpha \, |f'(x)|(v_\alpha \cdot \bar{v})\} > 0,$$

we have, by (1.6), $\ddot{g}(t) \geqslant m/2$ which contradicts (1.9).

Now we consider the case in which (H_1') holds. By setting

$$g_0(t) := (q(t) \cdot \bar{v}) + \sum_{\sigma \in I_1} c_\sigma (v_\sigma \cdot \bar{v}) \int_t^{+\infty} ds \int_s^{+\infty} f_\sigma'(q(\zeta) \cdot v_\sigma) \, d\zeta,$$

we have, by (1.7), that the integral above converges; also

$$\dot{g}_0(t) = (p(t) \cdot \bar{v}) - \sum_{\sigma \in I_1} c_\sigma (v_\sigma \cdot \bar{v}) \int_t^{+\infty} f_\sigma'(q(\xi) \cdot v_\sigma) \, d\xi,$$

and

$$\ddot{g}_0(t) = \sum_{\alpha \in I_0} c_\alpha \, |f_\alpha'(q(t) \cdot v_\alpha))|(v_\alpha \cdot \bar{v}).$$

Since $\dot{g}_0(t) \to 0$, as $t \to +\infty$, and $\ddot{g}_0(t)$ is a positive function, one concludes that, for t large enough, $\dot{g}_0(t)$ is negative. Therefore, for large t, $g_0(t)$ is bounded, which implies that $q(t) \cdot v_\alpha$ is bounded for all $\alpha \in I_0$. From this point, the proof follows as in the first part.

2. Main Result

Here we specialize the functions f_α, choosing

$$f_\alpha(x) = x^{-r_\alpha}, \qquad x > 0, \qquad \alpha = 1, ..., N, \tag{2.1}$$

where either $r_\alpha > 1$, or $r_\alpha = r$, $0 < r \leqslant 1$, $\forall \alpha = 1, ..., N$.

We will now prove the following result:

2.1. Theorem. *Let us consider the analytic Hamiltonian system* (1.2) *with potential* (1.1), *in which the functions f_α satisfy the assumptions* (2.1) *and the non zero vectors v_α span a proper cone. Then, the components of the asymptotic velocities of the solutions are analytic functions of the initial conditions, and, moreover, they are first integrals, independent and in involution, that is the system is analytically Liouville integrable.*

Proof. We observe, first of all, that, according to Lemma 1.1,

$$p_\infty \cdot v_\alpha > 0, \qquad \forall \alpha = 1, ..., N.$$

To obtain the compactification of the Hamiltonian system we set for $q \cdot v_\alpha > 0$, $\alpha = 1, ..., N$,

$$z_\alpha := (q \cdot v_\alpha)^{-r_\alpha/2}, \qquad \alpha = 1, ..., N. \tag{2.2}$$

Then we have

$$\begin{aligned} \dot{z}_\alpha &= -\frac{r_\alpha}{2} z_\alpha^{(r_\alpha+2)/r_\alpha}(p \cdot v_\alpha) \\ \dot{p} &= \sum_{\alpha=1}^{N} c_\alpha r_\alpha v_\alpha z_\alpha^{2(r_\alpha+1)/r_\alpha} \end{aligned} \tag{2.3}$$

for which $H = \| p \|^2/2 + \sum_{\alpha=1}^{N} c_\alpha z_\alpha^2$ is a first integral.

Due to the fact that the first integral levels for (2.3) are compact, any solution of (2.3) has a non empty ω-limit set. In particular, to any solution of system (1.2), there corresponds a solution $(z(t), p(t))$, of the compactified system (2.3), having as ω-limit set the singleton $\{(0, p_\infty)\}$ because $\{v_\alpha\}$ span a proper cone and the solutions lying on this ω-limit set satisfy $\dot{p} \equiv 0$.

For any asymptotic velocity p_∞ of system (1.2) let us define

$$\beta_\alpha = (p_\infty \cdot v_\alpha)^{-r_\alpha/2} \tag{2.4}$$

and for a fixed p_∞ we consider the following change of variables:

$$\begin{aligned} z_\alpha &= t^{-r_\alpha/2}\{\beta_\alpha + Z_\alpha\}, \qquad t \geqslant t_0, \text{ for some } t_0 > 0; \\ p &= p_\infty + Y. \end{aligned} \tag{2.5}$$

We observe that the function $z_\alpha = \beta_\alpha t^{-r_\alpha/2}$ gives a solution of Eq. $(2.3)_1$, with p replaced by p_∞. From (2.5) we have the corresponding differential system:

$$\begin{aligned} t\dot{Z}_\alpha &= \frac{r_\alpha}{2}\left\{-\frac{2}{r_\alpha} Z_\alpha - \beta_\alpha^{(r_\alpha+2)/r_\alpha}(Y \cdot v_\alpha) + N_\alpha(Z, Y, \beta)\right\} \\ t\dot{Y} &= \sum_{\alpha=1}^{N} t^{-r_\alpha} c_\alpha r_\alpha v_\alpha [\beta_\alpha + Z_\alpha]^{2(r_\alpha+1)/r_\alpha}, \end{aligned} \tag{2.6}$$

where the higher order terms $N_\alpha(Z, Y, \beta)$ are analytic functions for $\beta_\alpha > 0$, $\beta_\alpha + Z_\alpha > 0$, $\alpha = 1, ..., N$.

We introduce the auxiliary system:

$$\begin{aligned} \frac{du_\alpha}{d\varphi} &= -\frac{r_\alpha}{2} u_\alpha \\ \frac{dx_\alpha}{d\varphi} &= -x_\alpha - \frac{r_\alpha}{2}\beta_\alpha^{(r_\alpha+2)/r_\alpha}(y \cdot v_\alpha) + \frac{r_\alpha}{2} N_\alpha(x, y, \beta) \\ \frac{dy}{d\varphi} &= \sum_{\alpha=1}^{N} c_\alpha r_\alpha v_\alpha [\beta_\alpha + x_\alpha]^{2(r_\alpha+1)/r_\alpha} u_\alpha^2. \end{aligned} \tag{2.7}$$

By choosing $\bar{t} \geqslant t_0$ as initial time for the solutions of (2.6), it is easy to verify that the solutions of (2.7) satisfying the initial conditions $\bar{\varphi} = \log \bar{t}$ $u_\alpha(\bar{\varphi}) = e^{-(r_\alpha/2)\bar{\varphi}}$, correspond, in a one-to-one way, to the solutions of (2.6) by means of the change of variable $t = e^\varphi$, $\varphi \geqslant \bar{\varphi}$.

Now, by an obvious computation, we can see that $(u = 0,\ x = 0,\ y = 0)$, which is a fixed point of (2.7) for all values of the parameter p_∞ (through β), has a strongly stable manifold of dimension $2N$, tangent at the origin to the hyperplane $y = 0$. More precisely, for a fixed $p_\infty = \bar{p}_\infty$ there exist a neighborhood $\mathcal{O}_1$ of $u = 0$, $x = 0$, $p = \bar{p}_\infty$, and a function $F \in \mathscr{C}^\omega(\mathcal{O}_1, \mathbf{R}^n)$, such that the graph of the analytic function

$$y = F(u, x, p_\infty) \tag{2.8}$$

gives the strongly stable manifold of the origin of system (2.7), for p_∞ close enough to $\bar{p}_\infty$ (see [He]).

Obviously

$$F(0, 0, p_\infty) \equiv 0, \qquad \frac{\partial F}{\partial u_\alpha}(0, 0, p_\infty) \equiv 0,$$

$$\frac{\partial F}{\partial x_\alpha}(0, 0, p_\infty) \equiv 0, \qquad \alpha = 1, ..., N.$$

Now set

$$y = p - p_\infty = F(u, x, p_\infty).$$

Then by the implicit function theorem we can obtain p_∞ as an analytic function of (u, x, p),

$$p_\infty = G(u, x, p), \tag{2.9}$$

defined in a neighborhood $\mathcal{O}_2$ of $u = 0$, $x = 0$, $p = \bar{p}_\infty$. Moreover

$$p \equiv G(0, 0, p), \qquad \frac{\partial G_i}{\partial x_\alpha}(0, 0, p) \equiv 0, \qquad i = 1, ..., n. \tag{2.10}$$

Now we come back to the system (2.6). Fix $\bar{t}$ large enough and consider (Z, p), such that

$$((\bar{t})^{-r_1/2}, ..., (\bar{t})^{-r_N/2}, Z_1, ..., Z_N, p) \in \mathcal{O}_2.$$

Then, the solution of (2.6) starting at $\bar{t}$ from $(Z_1, ..., Z_N, p)$ admits the asymptotic velocity

$$p_\infty = G((\bar{t})^{-r_1/2}, ..., (\bar{t})^{-r_N/2}, Z_1, ..., Z_N, p) =: g(\bar{t}\, Z, p).$$

We consider the change of variables

$$\begin{aligned} z_\alpha &= \bar{t}^{-r_\alpha/2} \{(v_\alpha \cdot g(\bar{t}\, Z, p))^{-r_\alpha/2} + Z_\alpha\}, \\ p &= p. \end{aligned} \tag{2.11}$$

It is an easy computation to verify that, if $\bar{t}$ is large enough and (Z, p) is close to $(0, \bar{p}_\infty)$, then the transformation can be locally inverted. Indeed,

$$\frac{\partial z_\alpha}{\partial Z_\beta} = \bar{t}^{-r_\alpha/2} \left\{ \delta_{\alpha\beta} - \frac{r_\alpha}{2} (v_\alpha \cdot g(\bar{t}\, Z, p))^{-(r_\alpha + 2)/2} \cdot \frac{\partial g}{\partial Z_\beta}(\bar{t}\, Z, p) \right\},$$

and, recalling (2.10), we can prove that

$$\det\left(\left(\frac{\partial z_\alpha}{\partial Z_\beta}(\bar{t}\, Z, p)\right)\right) \neq 0.$$

Therefore, let $h(\bar{t}\,\cdot, \cdot) : \mathcal{O}_3(\bar{t}) \to \mathbf{R}^{N+n}$

$$(z, p) \xrightarrow{h(\bar{t}\cdot,\cdot)} (Z, p)$$

be the analytic function implicitely defined by the system (2.11).

Let us denote by ϕ_t the flow given by the solutions of (2.3); having fixed $\bar{t}$ and $(z, p) \in \mathcal{O}_3(\bar{t})$, we consider (z_0, p_0) such that $\phi_{\bar{t}}(z_0, p_0) = (z, p)$. The map $\phi_{\bar{t}}$ is analytic; therefore

$$p_\infty = l(\bar{t},\, z_0, p_0) := g(\bar{t},\, h(\bar{t},\, \phi_{\bar{t}}(z_0, p_0)) \tag{2.12}$$

is an analytic function of (z_0, p_0). Now consider an initial condition $(\bar{q}_0, \bar{p}_0)$ of a solution of the starting system (1.2) and call $\bar{p}_\infty$ its asymptotic velocity. Then, one can find a suitable neighborhood of $(\bar{q}_0, \bar{p}_0)$ such that if (q_0, p_0) is in this neighborhood, by means of (2.2) and (2.12) we have $p_\infty = l(\bar{t},\, (q_0 \cdot v_1)^{-r_1/2}, ..., (q_0 \cdot v_N)^{-r_N/2},\, p_0)$. This shows that p_∞ is an analytic function of the initial data. Finally, the independence and the involution of the first integrals $(p_\infty)_i$, $i = 1, ..., n$, follow by arguing as in [G1, OC].

Remark. The method used above can be easily applied to prove the analytic integrability of the Toda-like systems, which include the classical (non periodic) Toda system and whose C^∞-smoothness were previously considered in [OC, GZ3]. As special cases of systems considered in the framework of this paper one can also mention the Coulombian potential $(r_\alpha \equiv r = 1)$ and the Calogero–Moser potential $(r_\alpha \equiv r = 2)$ acting repulsively on a system of n particles on the line.

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