Real Semigroups and Rings

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ABSTRACT. We show, among other results, that the real semigroup (RS) associated to any preordered ring, $\langle A, T \rangle$, is naturally isomorphic to the RS of reduced bounded inversion ring, canonically and functorially associated to $\langle A, T \rangle$.

Our goal here is to show that the real semigroup (RS) of any preordered ring, $\langle A, T \rangle$, is naturally isomorphic to the RS of a reduced bounded inversion ring (BIR), canonically and functorially associated to $\langle A, T \rangle$. This will be accomplished in section 3. Section 1 contains basic material concerning real semigroups, in particular, those associated to preordered rings, while section 2 describes the relations between the real semigroup associated to a p-ring and to its ring of fractions by a multiplicative set. We also take the opportunity to present interesting examples of RS-congruences in real semigroups associated to rings.

1 Preliminaries

For the theory of real semigroups, the reader is referred to [Dickmann and Petrovich, 2004] and to the more comprehensive [Dickmann and Petrovich, 2011]. For lack of a convenient reference, we give a succinct account of the natural functors from the category of preordered rings to that of real semigroups and from the latter into the category pre-special groups.

1.1. Notation and Basic Definitions. In all that follows, "ring" means commutative unitary ring. Let R be a ring.

a) If $D \subseteq R$:

(1) $R^\times = \{u \in R : a \text{ is a unit in } R\}$ is the (multiplicative) group of units in R;

(2) $D^\times = D \cap R^\times$; (3) $D^2 = \{d^2 \in R : d \in D\}$;

(3) $\Sigma D^2 = \{\sum_{i=1}^n d_i^2 \in R : n \geq 1 \text{ is an integer and } \{d_1, \ldots, d_n\} \subseteq D\}$.

b) A **preorder** on R is a subset P of R, closed under sums and products, containing R^2 and such that $-1 \notin P$. If $-1 \notin \Sigma R^2$, R is said to be **semi-real**; in this case, ΣR^2 is the least preorder on R.

c) A **preordered ring (p-ring)** is a pair $\langle A, T \rangle$, where A is a ring and T is a preorder on A.

d) A p-ring $\langle A, T \rangle$ is a **bounded inversion ring (BIR)** if $1 + T \subseteq A^\times$.

e) If $\langle A, T \rangle$, $\langle R, P \rangle$ are p-rings, a map $f : \langle A, T \rangle \longrightarrow \langle R, P \rangle$ is a **p-ring morphism** if it is a morphism of unitary rings, satisfying $f[T] \subseteq P$. ■

Forthwith, all rings will be assumed to be semi-real.

1.2. *T*-convex and *T*-radical Ideals. (cf. Chapter 4 of [Bochnak *et al.*, 1998]).

a) An ideal I in a p-ring $\langle A, T \rangle$ is

- **T-convex** if for all $s, t \in T$, $s + t \in I \Rightarrow s, t \in I$;
- **T-radical** if for all $a \in A$ and $t \in T$, $a^2 + t \in I \Rightarrow a \in I$.

A ΣA^2-radical ideal is called **real**.

By Proposition 4.2.5 in [Bochnak *et al.*, 1998] an ideal of A is T-radical iff it is T-convex and radical. In particular, a prime ideal is T-radical iff it is T-convex. ■

Note that if $T \subseteq \alpha \in \mathrm{Sper}(A)$, then the prime ideal $\mathrm{supp}(\alpha)$ is T-convex. Conversely,

PROPOSITION 1. ([Bochnak *et al.*, 1998], Prop. 4.3.8, p. 90) *If I is a proper prime ideal, T-convex for a given preorder T of A, then there is $\alpha \in \mathrm{Sper}(A, T)$ such that* $\mathrm{supp}(\alpha) = \alpha \cap -\alpha = I$. ■

PROPOSITION 2. a) ([Bochnak *et al.*, 1998], Prop. 4.2.7, p. 87) *A preorder T on a ring A is proper iff A has a proper T-convex ideal.*

b) *If $\langle A, T \rangle$ is a p-ring, any ideal of A, maximal for the property of being T-convex, is prime.* ■

PROPOSITION 3. ([Bochnak *et al.*, 1998], Prop. 4.2.6, p. 87) *Given a preorder T of A, every ideal I of A is contained in a smallest T-radical ideal (possibly improper), namely:*

$$\sqrt[T]{I} = \{a \in A : \exists\, m \in \mathbb{N} \text{ and } t \in T \text{ such that } a^{2m} + t \in I\},$$

called the ***T*-radical of *I***, *the intersection of all T-convex prime ideals containing I.* ■

REMARK 4. With notation as in 3:

a) If $a \in A$, write $\sqrt[T]{a}$ for the T-radical of the principal ideal (a). In particular, $\sqrt[T]{0}$ is the T-radical of the zero ideal. By 3, an ideal I is T-radical iff $\sqrt[T]{I} = I$.

b) If $T = \Sigma A^2$ and I is an ideal in A, we write $\sqrt[re]{I}$ for $\sqrt[T]{I}$, the **real radical** of I, equal to the intersection of all real primes of A containing I.

c) Recall that a ring A is **reduced** if it has no non-zero nilpotent elements, i.e., the intersection of all prime ideals in A is the zero ideal; the analog of this notion in the case of preordered rings appears in the next definition. ■

DEFINITION 5. A p-ring $\langle A, T \rangle$ is ***T*-reduced** if $\sqrt[T]{0} = (0)$. If $T = \Sigma A^2$, i.e., $\sqrt[re]{0} = (0)$, we say A is a **real ring**. Clearly, a T-reduced ring is reduced and semi-real [1].

Every p-ring has a **BIR hull**, as follows:

PROPOSITION 6. (Proposition 6.5.(a), p. 72ff of [Dickmann and Miraglia, 2011]) *If* $\langle A, T \rangle$ *is a p-ring, then* $S = 1 + T$ *is a proper multiplicative subset of* A. *Moreover, if* $\nu : A \longrightarrow A^* = AS^{-1}$ *is the ring of fractions of* A *by* S *and*

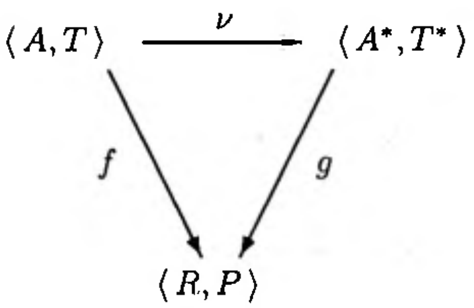
$$T^* = \{t/s^2 \in A^* : t \in T \text{ and } s \in S\},$$

then

(1) T^* *is a proper preorder of* A^* *and* $\langle A^*, T^* \rangle$ *is a BIR.*

(2) ν *is a p-ring morphism; moreover, if* A *is* T*-reduced* (cf. 5), *then* ν *is injective.*

(3) *If* $f : \langle A, T \rangle \longrightarrow \langle R, P \rangle$ *is a p-ring morphism and* $\langle R, P \rangle$ *is a BIR, there is a* unique *p-ring morphism,* $g : \langle A^*, T^* \rangle \longrightarrow \langle R, P \rangle$, *such that* $g \circ \nu = f$.



■

1.3. Ternary Semigroups ([Dickmann and Petrovich, 2004], [Dickmann and Petrovich, 2011]). A structure $\langle S, \cdot, 1, 0, -1 \rangle$ is a **ternary semigroup (TS)** if

[TS 1] $\langle S, \cdot, 1 \rangle$ is an Abelian semigroup (monoid) with identity 1;

[TS 2] $x^3 = x$, for all $x \in S$; [TS 3] $-1 \neq 1$ and $(-1)(-1) = 1$;

[TS 4] $x \cdot 0 = 0$, for all $x \in S$; [TS 5] For all $x \in S$, $x = -1 \cdot x \Rightarrow x = 0$.

If $x \in S$, write $-x$ for $-1 \cdot x$.

b) If S is a TS, $R \subseteq S^3$ is a ternary relation on S and a, b, $c \in S$, write $a \in R(b, c)$ in place of $R(a, b, c)$. Define the **transversal** of R, R^t, by

[t-rep] $a \in R^t\ (b, c) \quad \Leftrightarrow \quad a \in R(b, c) \wedge -b \in R(c, -a) \wedge -c \in R(b, -a)$.

b) If S, S' are TSs, a map, $f : S \longrightarrow S'$ is **TS-morphism** if it preserves product, 0, 1 and -1. ■

DEFINITION 7. A set-theoretic map, $f : D \longrightarrow E$, induces a map

$$f \times f : D \times D \longrightarrow E \times E, \text{ given by } \langle a, b \rangle \longmapsto \langle f(a), f(b) \rangle.$$

Define [2],

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[1] Since $\sqrt[T]{0} = (0)$, A has a proper real prime ideal, and 2.(a) guarantees that ΣA^2 is a proper preorder of A. Moreover, our definition of real ring coincides with the usual one, i.e. (0) is a real ideal (cf. 4.(a)).

[2] Sometimes called the *fibered product of* A *over* f.

$$\ker f = (f \times f)^{-1}[\Delta_E] = \{\langle a, b\rangle \in D \times D : f(a) = f(b)\},$$

called the **kernel of** f, where Δ_E is the diagonal of $E \times E$. If $D \xrightarrow{f} E \xrightarrow{g} F$ are set-theoretic maps, we clearly have

[ker-comp] $$\ker(g \circ f) = (f \times f)^{-1}[\ker g].$$

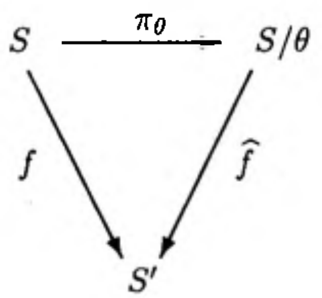
1.4. TS-Congruences. For more detailed information on this topic, the reader is referred to section 1 of Chapter I in [Dickmann and Petrovich, 2011] (cf. Definition I.1.9ff).

a) Let S be TS; an equivalence relation, θ, on S is a **TS-congruence** if it is a congruence with respect to the product in S, i.e, $a\ \theta\ a'$ and $b\ \theta\ b'$ implies (ab) θ $(a'b')$.

Let $S/\theta = \{a/\theta : a \in S\}$ be the set of equivalence classes of elements of S by θ and let $\pi_\theta : S \longrightarrow S/\theta$, $a \longmapsto a/\theta$ be the canonical quotient map. Notice that $\ker \pi_\theta = \theta$. With the operation induced by the product in S, S/θ has a natural structure of ternary semigroup, wherein 1, 0, and -1 are the classes of these constants modulo θ. Moreover, π_θ is a TS-morphism and the diagram $S \xrightarrow{\pi_\theta} S/\theta$ has the following property:

If $f : S \longrightarrow S'$ is a TS-morphism, such that $\theta \subseteq \ker f$, there is a *unique* TS-morphism, $\widehat{f} : S/\theta \longrightarrow S'$, making the diagram below left commutative:

[TS-UFP]



Indeed, it is straightforward that $\widehat{f}(a/\theta) = f(a)$, $a \in S$, has the required properties.

b) If $f : S_1 \longrightarrow S_2$ is a TS-morphism, then $\ker f$ is a TS-congruence on S_1 and there is a *unique* TS-morphism, $\widehat{f} : S_1/\theta \longrightarrow S_2$ such that $\widehat{f} \circ \pi_{\ker f} = f$. Moreover, it is straightforward to show:

(1) $\widehat{f}$ is injective $\Leftrightarrow$ $\ker f = \theta$;

(2) $\widehat{f}$ is a TS-isomorphism $\Leftrightarrow$ $\ker f = \theta$ and f is surjective. ■

1.5. Real Semigroups ([Dickmann and Petrovich, 2004], [Dickmann and Petrovich, 2011]). a) A **real semigroup (RS)** is:

• A TS, $\mathcal{G}$, together with a ternary relation, $\mathcal{D}_\mathcal{G} = \mathcal{D}$, **representation by binary forms**, satisfying, for all $a, b, c, d, e \in \mathcal{G}$, (where $\mathcal{D}^t$ is the transversal of $\mathcal{D}$):

[RS 0] : $c \in \mathcal{D}(a,b) \Leftrightarrow c \in \mathcal{D}(b,a)$; [RS 1] : $a \in \mathcal{D}(a,b)$;

[RS 2] : $a \in \mathcal{D}(b,c) \Rightarrow ad \in \mathcal{D}(bd,cd)$;

[RS 3] (Strong associativity) :

$a \in \mathscr{D}^t(b,c)$ and $c \in \mathscr{D}^t(d,e) \Rightarrow \exists\, x \in \mathscr{D}^t(b,d)$ so that $a \in \mathscr{D}^t(x,e)$.

[RS 4] : $e \in \mathscr{D}(c^2a, c^2b) \Rightarrow e \in \mathscr{D}(a,b)$;

[RS 5] : $ad = bd$, $ac = bc$ and $c \in \mathscr{D}(d,e) \Rightarrow ac = bc$;

[RS 6] : $c \in \mathscr{D}(a,b) \Rightarrow c \in \mathscr{D}^t(c^2a, c^2b)$;

[RS 7] (Reduction) : $\mathscr{D}^t(a,-b) \cap \mathscr{D}^t(-a,b) \neq \emptyset \Rightarrow a = b$;

[RS 8] : $a \in \mathscr{D}(b,c) \Rightarrow a^2 \in \mathscr{D}(b^2,c^2)$.

b) $L_{RS} = \langle \cdot, \mathscr{D}, 1, 0, -1 \rangle$ is the language of RSs.

c) If $\mathscr{G}$ is a RS, write $\mathscr{G}^\times = \{x \in \mathscr{G} : x^2 = 1\}$ for the group of **units in $\mathscr{G}$**.

d) If $\mathscr{G}_1$, $\mathscr{G}_2$ are RSs, a map, $f : \mathscr{G}_1 \longrightarrow \mathscr{G}_2$, is a **RS-morphism** if it preserves 1, 0, -1, product and representation. ∎

REMARK 8. If $\mathscr{G}$ is a RS, the representation and transversal representation are interdefinable as follows. for $a, b, c \in \mathscr{G}$:

- By [t-rep] in 1.3.(b),

$a \in \mathscr{D}^t\,(b,c) \Leftrightarrow a \in \mathscr{D}(b,c) \wedge -b \in \mathscr{D}(b,-a) \wedge -c \in \mathscr{D}(b,-a)$;

- The axioms for RSs in 1.5.(a) entail $a \in \mathscr{D}(b,c) \Leftrightarrow a \in \mathscr{D}^t(a^2b, a^2c)$.

Hence, if $f : \mathscr{G}_1 \longrightarrow \mathscr{G}_2$ is a TS-morphism and $\mathscr{G}_i$, $i = 1, 2$, are RSs, the following are equivalent:

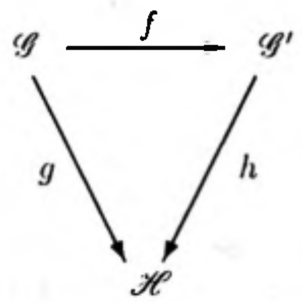
(1) f is a RS-morphism (i.e., it preserves representation);

(2) f preserves transversal representation, i.e, for all $a, b, c \in \mathscr{G}_1$,

$$a \in \mathscr{D}^t_{\mathscr{G}_1}(b,c) \Rightarrow f(a) \in \mathscr{D}^t_{\mathscr{G}_2}(f(b), f(c)).$$ ∎

1.6. RS-Congruences. For an extensive discussion of the theme, the reader is referred to Chapter II of [Dickmann and Petrovich, 2011]. We shall here mildly change the presentation, in order to emphasize the importance of the unique factorization property contained in Definition II.2.1 of [Dickmann and Petrovich, 2011]. To keep matters straight, if $\mathscr{G}$ is a RS, we write <u>$|\mathscr{G}|$ *for the ternary semigroup underlying* $\mathscr{G}$.</u>

DEFINITION 9. A RS-morphism, $f : \mathscr{G} \longrightarrow \mathscr{G}'$ has the **RS-unique factorization property (RS-UFP)** if for all RS-morphisms, $g : \mathscr{G} \longrightarrow \mathscr{H}$ such that $\ker f \subseteq \ker g$, there is a *unique* RS-morphism, $h : \mathscr{G}' \longrightarrow \mathscr{H}$ making the following diagram commute:



Notation as in 1.4, we have

DEFINITION 10. (Essentially Def. II.2.1, [Dickmann and Petrovich, 2011]) An equivalence relation θ on a RS $\mathscr{G}$ is a **RS-congruence** if

[RS-cong 1] : θ is a congruence of ternary semigroups (1.4);

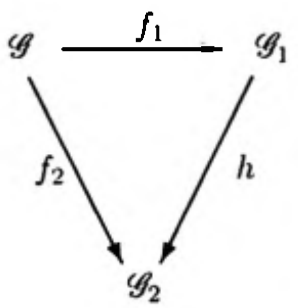
[RS-cong 2] : There is a ternary relation $\mathscr{D}_\theta$ in the quotient TS, $|\mathscr{G}|/\theta$, such that $\mathscr{G}/\theta \ := \ \langle\, |\mathscr{G}|/\theta, \cdot, \mathscr{D}_\theta, -1, 0, 1\,\rangle$ is a RS and the canonical projection, $\pi_\theta : \mathscr{G} \longrightarrow \mathscr{G}/\theta$, is a RS-morphism;

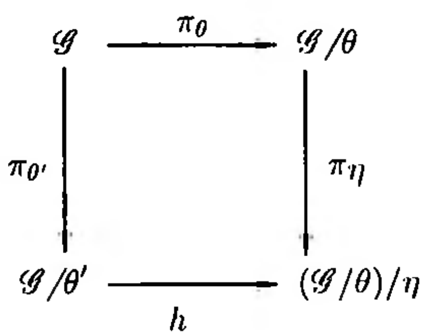
[RS-cong 3] : The map $\pi_\theta : \mathscr{G} \longrightarrow \mathscr{G}/\theta$ has the RS-UFP.

Write $Con_{RS}(\mathscr{G})$ for the set of RS-congruences in $\mathscr{G}$.

LEMMA 11. *a) Let $f_i : \mathscr{G} \longrightarrow \mathscr{G}_i$, $i = 1, 2$, be RS-morphisms with the RS-UFP. If* $\ker f_1 = \ker f_2$, *there is a* unique RS-isomorphism, $h : \mathscr{G}_1 \longrightarrow \mathscr{G}_2$, *making the triangle below left commutative.*

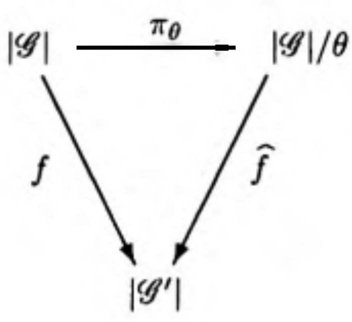
b) Let $f : \mathscr{G} \longrightarrow \mathscr{G}'$ be a surjective *RS-morphism. If f has the RS-UFP, then* $\ker f$ *is a RS-congruence in $\mathscr{G}$.*





c) (Double quotient) *Let $\mathscr{G}$ be a RS and let θ be a RS-congruence in $\mathscr{G}$. If η is a RS-congruence on $\mathscr{G}/\theta$, then, with notation as in* 10, $\theta' \ := \ (\pi_\theta \times \pi_\theta)^{-1}[\eta]$ *is a RS-congruence on $\mathscr{G}$. Moreover, there is a* unique RS-isomorphism, $h : \mathscr{G}/\theta' \longrightarrow (\mathscr{G}/\theta)/\eta$ *making the square above right commutative.*

PROOF. Item (a) is clear. For (b), by 1.4.(b), $\theta := \ker f$ is a TS-congruence on $\mathscr{G}$. Since f is surjective, the [TS-UFP] in 1.4.(a), together with (2) in 1.4.(b), yield a unique *TS-isomorphism*, $\widehat{f} \ : \ |\mathscr{G}|/\theta \longrightarrow |\mathscr{G}'|$, making the following diagram commute:



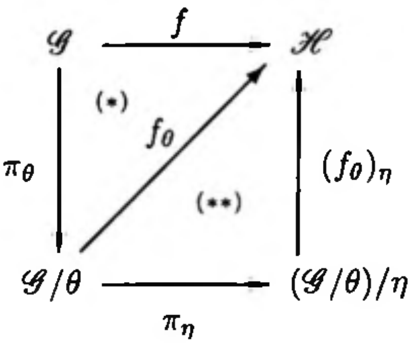
Since $\mathscr{G}'$ is a RS, $\widehat{f}$ may be made into a *RS-isomorphism*, by which $|\mathscr{G}|/\theta$ becomes a RS, $\mathscr{G}/\theta$, and $\pi_\theta : \mathscr{G} \longrightarrow \mathscr{G}/\theta$ becomes a RS-morphism. Since, θ $= \ker f = \ker \pi_\theta$ and f has the RS-UFP, the same will be true of $\pi_\theta : \mathscr{G} \longrightarrow$

$\mathscr{G}/\theta$, and θ is a RS-congruence, as claimed.

c) Let $g = \pi_\eta \circ \pi_\theta$; g is clearly surjective and $\ker g = \theta' = (\pi_\theta \times \pi_\theta)^{-1}[\eta]$. Note that

(I) $\quad (i)\ \theta \subseteq \theta'$; $\quad (ii)\ (\pi_\theta \times \pi_\theta)[\theta'] = \eta = \ker \pi_\eta$.

Indeed, (i) follows from $\Delta_{\mathscr{G}/\theta} \subseteq \eta$ (inverse image is increasing), while (ii) from the surjectivity of π_θ. We claim that g has the RS-UFP. To see this, let $f : \mathscr{G} \longrightarrow \mathscr{H}$ be a RS-morphism, such that $\theta' \subseteq \ker f$. Because θ is a RS-congruence and $\theta \subseteq \ker f$ (by (I.(i) above), there is a unique RS-morphism, $f_\theta : \mathscr{G}/\theta \longrightarrow \mathscr{H}$, making the upper triangle $(*)$ in the diagram below commutative.



Since $f = \pi_\theta \circ f_\theta$, we have $\theta' \subseteq \ker f = (\pi_\theta \times \pi_\theta)^{-1}[\ker f_\theta]$, and the surjectivity of π_θ together with (I.(ii)) above, entail $\eta \subseteq \ker f_\theta$. Now, the fact that η is a RS-congruence yields a unique RS-morphism, $(f_\theta)_\eta : (\mathscr{G}/\theta)/\eta \longrightarrow \mathscr{H}$, making the lower triangle $(**)$ in the above square commutative, establishing the RS-UFP for $g = \pi_\eta \circ \pi_\theta$. Now it follows immediately from (b) that θ' is a RS-congruence on $\mathscr{G}$. Moreover, since both $\pi_{\theta'} : \mathscr{G} \longrightarrow \mathscr{G}/\theta'$ and $g : \mathscr{G} \longrightarrow (\mathscr{G}/\theta)/\eta$ have the the same kernel and the RS-UFP, item (a) yields the unique RS-isomorphism making the displayed square in the statement commute, ending the proof. ■

In what follows, we shall see applications of the above results to RSs arising from p-rings.

1.7. The Real Semigroup of a p-Ring . Let $\langle A, T \rangle$ be a p-ring. For details on the constructions about to be presented, the reader is referred to [Dickmann and Petrovich, 2004], [Dickmann and Petrovich, 2011] and [Marshall, 1996]

a) Let Sper(A) be the real spectrum of A (cf. Chapter 7 and Chapter 4 in [Bochnak *et al.*, 1998]) and set

$$\mathrm{Sper}(A,T) = \{\alpha \in \mathrm{Sper}(A) : T \subseteq \alpha\},$$

called the **real spectrum of $\langle A, T \rangle$**.

Each $a \in A$ gives rise to map, $\overline{a}_T : \mathrm{Sper}(A,T) \longrightarrow 3 = \{-1, 0, 1\}$, given by

$$\overline{a}_T(\alpha) = \begin{cases} 1 & \text{if } a \in \alpha \setminus -\alpha; \\ 0 & \text{if } a \in \mathrm{supp}(\alpha) = \alpha \cap -\alpha; \\ -1 & \text{if } a \in -\alpha \setminus \alpha. \end{cases}$$

If T is clear from context, we write $\overline{a}$ for $\overline{a}_T$.

b) Write $\mathscr{G}_{A,T} = \{\overline{a} : a \in A\}$. With the product induced by A, $\mathscr{G}_{A,T}$ is a ternary semigroup with identity 1 (the constant function 1) and distinguished elements 0 and -1 (the corresponding constant valued maps).

Define a **representation relation** on $\mathscr{G}_{A,T}$, as follows: for $a, b, c \in A$,

$(\mathscr{D})$ $\quad \overline{a} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{b}, \overline{c}) \;\Leftrightarrow\; \exists\, t, t_1, t_2 \in T$, s.t. $\overline{at} = \overline{a}$ and $ta = t_1 b + t_2 c$.

The corresponding **transversal representation relation** is given by

$$(\mathscr{D}^t) \qquad \overline{a} \in \mathscr{D}^t_{\mathscr{G}_{A,T}}(\overline{b}, \overline{c}) \;\Leftrightarrow\; \begin{cases} \exists\, a', b', c' \in A \text{ so that } \overline{a} = \overline{a'}, \\ \overline{b} = \overline{b'},\;\; \overline{c} = \overline{c'} \text{ and } a' = b' + c'. \end{cases}$$

With these representation relations, $\mathscr{G}_{A,T}$ **is a real semigroup** in the sense of [Dickmann and Petrovich, 2004] and [Dickmann and Petrovich, 2011]. As above (1.5.(c)), $\mathscr{G}^{\times}_{A,T} = \{\overline{a_T} \in \mathscr{G}_{A,T} : \overline{a_T}^2 = 1\}$ is the group of units in $\mathscr{G}_{A,T}$. ■

In the present setting, and with notation as in 1.7.(a), the following result is important:

THEOREM 12. (Thm. 5.4.2, Cor. 5.4.3 (p. 93ff) in [Marshall, 1996]) *Let $\langle A, T\rangle$ be a p-ring. For $a, b \in A$:*

a) $\overline{a_T} = 0$ *iff there is* $k \geq 0$ *such that* $-a^{2k} \in T$.

b) $\overline{a_T} = 1$ *iff there are* $s, t \in T$ *such that* $(1 + s)a = 1 + t$.

c) $\overline{a_T} \geq 0$ *iff there are* $s, t \in T$ *and* $k \geq 0$ *so that* $(a^{2k} + s)a = a^{2k} + t$.

d) $\overline{a_T} = \overline{b_T}$ *iff there are* $s, t \in T$ *and* $k \geq 0$ *so that* $sab = (a^2 + b^2)^k + t$. ■

1.8. The functor $\mathscr{G}$ from p-Rings to RS. Notation as above, let **p-Rings** and **RS** be the categories of p-rings and RSs, respectively. If $f : \langle A, T\rangle \longrightarrow \langle A', T'\rangle$ is a p-ring morphism, define

$$\mathscr{G}(f) : \mathscr{G}_{A,T} \longrightarrow \mathscr{G}_{A',T'}, \text{ given by } \overline{a_T} \longmapsto \overline{f(a)_{T'}}.$$

To see that $\mathscr{G}(f)$ is well-defined, let $a, c \in A$ verify $\overline{a_T} = \overline{c_T}$; by Theorem 12.(d), there are $s, t \in T$ and an integer $k \geq 0$ such that

$$\text{(I)} \qquad (a^2 + c^2)^k + t = sac.$$

Applying f to both sides of (I) and recalling the inclusion $f[T] \subseteq T'$, obtains

$$(f(a)^2 + f(c)^2)^k + f(t) = f(s)f(a)f(c),$$

with $f(t), f(s) \in T'$; whence, another application of 12.(d) yields $\overline{f(a)_{T'}} = \overline{f(c)_{T'}}$, as needed. It is straightforward that $\mathscr{G}(f)$ is a semigroup morphism, preserving 1, 0 and -1. For $\mathscr{G}(f)$ to be a RS-morphism, it suffices to prove

$$\text{(II)} \qquad \overline{a_T} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{b_T}, \overline{c_T}) \;\Rightarrow\; \overline{f(a)_{T'}} \in \mathscr{D}_{\mathscr{G}_{A',T'}}(\overline{f(b)_{T'}}, \overline{f(c)_{T'}}),$$

with $a, b, c \in A$. By $(\mathscr{D})$ in 1.7.(b), the hypothesis in (II) is equivalent to the existence of $t, t_1, t_2 \in T$ such that

$$\text{(III)} \qquad ta = t_1 b + t_2 c \text{ and } \overline{ta} = \overline{t} \cdot \overline{a} = \overline{a}.$$

Applying f to the first equation in (III) yields $f(t)f(a) = f(t_1)f(b) + f(t_2)f(c)$, with $f(t), f(t_1), f(t_2) \in T'$; moreover, since $\mathscr{G}(f)$ is a semigroup morphism, the second equation in (III) entails $\overline{f(a)_{T'}} = \mathscr{G}(f)(\overline{(at)_T}) = \mathscr{G}(f)(\overline{a_T}) = \overline{f(a)_{T'}}$, and the conclusion in (II) is immediately forthcoming from $(\mathscr{D})$ in 1.7.(b).

Clearly, the maps

$$\begin{cases} \langle A, T \rangle & \longmapsto & \mathscr{G}_{A,T} \\ \langle A, T \rangle \xrightarrow{f} \langle A', T' \rangle & \longmapsto & \mathscr{G}_{A,T} \xrightarrow{\mathscr{G}(f)} \mathscr{G}_{A',T'}, \end{cases}$$

yield we have a covariant functor from **p-Rings** to **RS**. ■

EXAMPLE 13. Let $\langle A, T \rangle$ be a p-ring. The identity map, $Id_T : \langle A, \Sigma A^2 \rangle \longrightarrow \langle A, T \rangle$ is a p-ring morphism; let $\rho_T = \mathscr{G}(Id_T) : \mathscr{G}_A \longrightarrow \mathscr{G}_{A,T}$ be the induced RS-morphism, as in 1.8. We have

$$\rho_T(\overline{a}) \;=\; \overline{a_T} \;=\; \overline{a} \restriction \mathrm{Sper}(A, T).$$

Hence, $\ker \rho_T = \{\langle \overline{a}, \overline{b} \rangle \in \mathscr{G}_A \times \mathscr{G}_A : \overline{a_T} = \overline{b_T}\}$ and ρ_T is clearly surjective. Note that the description of $\ker \rho_T$ in $\langle A, T \rangle$ is given by 12.(d).

We claim that ρ_T has the RS-UFP. Indeed, let $\mathscr{G}$ be a RS and let $f : \mathscr{G}_A \longrightarrow \mathscr{G}$ be a RS-morphism, such that $\ker \rho_T \subseteq \ker f$. Since ρ_T is onto, the uniqueness of the factor RS-morphism – if it exists at all –, is clear. For $a \in A$, define

$$\widehat{f} : \mathscr{G}_{A,T} \longrightarrow \mathscr{G} \quad \text{by} \quad \widehat{f}(\overline{a_T}) = f(a).$$

Since $\ker \rho_T \subseteq \ker f$, $\widehat{f}$ is well defined; moreover, it is straightforward that $\widehat{f}$ is a TS-morphism, verifying $\widehat{f} \circ \rho_T = f$. It remains to check that $\widehat{f}$ is a RS-morphism. taking into account the definition of $\widehat{f}$, this amount to showing for a, b, $c \in A$,

$$\text{(I)} \qquad \overline{a_T} \in \mathscr{D}^t_{\mathscr{G}_{A,T}}(\overline{b_t}, \overline{c_T}) \quad \Rightarrow \quad f(a) \in \mathscr{D}^t_{\mathscr{G}}(f(b), f(c)).$$

and we conclude by the equivalence between (1) and (2) in 8. By $(\mathscr{D}^t)$ in 1.7, the antecedent in (I) is equivalent to the existence of a', b', $c' \in A$ so that

$$\text{(II)} \quad (i) \quad \overline{a'_T} = \overline{a_T}, \quad \overline{b'_T} = \overline{a_T}, \quad \overline{c'_T} = \overline{c_T} \quad \text{and} \quad (ii) \quad a' = b' + c'.$$

From (II.(i)) and the hypothesis that $\ker \rho_T \subseteq \ker f$ we obtain $f(a) = f(a')$, $f(b) = f(b')$ and $f(c) = f(c')$, while (II.(ii)) and $(\mathscr{D}^t)$ in 1.7.(b) entail $\overline{a'} \in \mathscr{D}^t_{\mathscr{G}_A}(\overline{b'}, \overline{c'})$. Since f is a RS-morphism, the latter relation implies $f(a') \in \mathscr{D}^t_{\mathscr{G}}(f(b'), f(c'))$, which in turn yields $f(a) \in \mathscr{D}^t_{\mathscr{G}}(f(b), f(c))$, establishing (I), as desired.

By items (a) and (b) of Lemma 11, $\ker \rho_T$ **is a RS-congruence** on $\mathscr{G}_A$ and the diagram $\rho_T : \mathscr{G}_A \longrightarrow \mathscr{G}_{A,T}$ is naturally isomorphic to the projection of $\mathscr{G}_A$ onto $\mathscr{G}_A / \ker \rho_T$. ■

1.9. The functor $\mathscr{U}$ from RS to pRSG. Recall that **pRSG** is the category of reduced *pre-special* groups (pRSG) and SG-morphisms (cf. Definition 1.2, p. 2 and Definition 1.11, p. 10 in [Dickmann and Miraglia, 2000]).

If $\mathscr{G}$ is a RS, then (cf. I.2.10, p. 23 of [Dickmann and Petrovich, 2011]), $\mathscr{G}^\times = \{x \in \mathscr{G} : x^2 = 1\}$, the *group of units* of $\mathscr{G}$, with the induced representation relation [3], is a p-RSG. The canonical embedding, $u_{\mathscr{G}} : \mathscr{G}^\times \longrightarrow \mathscr{G}$, is a semigroup morphism, preserving 1 and -1. In fact, the passage from $\mathscr{G}$ to $\mathscr{G}^\times$ constitutes

[3] In $\mathscr{G}^\times$, $\mathscr{D}$ and $\mathscr{D}^t$ coincide.

a *functor*, as follows. If $f : \mathscr{G}_1 \longrightarrow \mathscr{G}_2$ is a RS-morphism, clearly we have $f[\mathscr{G}_1^{\times}] \subseteq \mathscr{G}_2^{\times}$. Hence, $f \restriction \mathscr{G}_1^{\times} := f^{\times}$ is a map from $\mathscr{G}_1^{\times}$ into $\mathscr{G}_2^{\times}$. Moreover,

- $f^{\times}$ takes 1 to 1 and -1 to -1;
- $\forall\, a, b, c \in \mathscr{G}_1^{\times},\ \ a \in \mathscr{D}_{\mathscr{G}_1}(b, c) \ \Rightarrow\ f(a) \in \mathscr{D}_{\mathscr{G}_2}(f(b), f(c))$,

i.e, $f^{\times}$ is a pRSG-morphism and the following diagram commutes:

$$\begin{array}{ccc}
\mathscr{G}_1^{\times} & \xrightarrow{\ f^{\times}\ } & \mathscr{G}_2^{\times} \\
\Big\downarrow u_{\mathscr{G}_1} & & \Big\downarrow u_{\mathscr{G}_2} \\
\mathscr{G}_1 & \xrightarrow[\ f\ ]{} & \mathscr{G}_2
\end{array}$$

It is easily seen that the maps $\left\{ \begin{array}{ccc} \mathscr{G} & \longmapsto & \mathscr{G}^{\times} \\ \mathscr{G}_1 \xrightarrow{f} \mathscr{G}_2 & \longmapsto & \mathscr{G}_1^{\times} \xrightarrow{f^{\times}} \mathscr{G}_2^{\times}, \end{array} \right.$

constitute a covariant functor, $\mathscr{U}$: **RS** $\longrightarrow$ **pRSG**. ■

2 The Real Semigroup of a Ring of Fractions

We here describe the basic relations between the real semigroups associated to a p-ring and to its ring of fractions by a multiplicative set. Firstly, we register the following (well-known) result:

LEMMA 14. *Let $\langle A, T \rangle$ be a p-ring and let S be a multiplicative subset of A and let $R := AS^{-1}$ be the ring of fractions of A by S and let $\iota_A : A \longrightarrow R$ be canonical ring morphism.*

a) The following are equivalent:

(1) $P \ = \ \left\{ \frac{t}{s^2} \in R : t \in T \text{ and } s \in S \right\}$ *is a proper preorder of R;*

(2) $S \cap (T \cap -T) \ = \ \emptyset$.

If these equivalent conditions are met, then S is a proper multiplicative subset of A and ι_A is a morphism of unitary p-rings.

b) If $S \cap \sqrt[T]{0} = \emptyset$, then the set P in (1) of item (a) is a proper preorder and ι_A is a p-ring morphism.

PROOF. Since $(T \cap -T\ ) \subseteq \sqrt[T]{0} \ = \ \bigcap \{\mathrm{supp}(\alpha) : \alpha \in \mathrm{Sper}(A,T)\}$, (b) is follows immediately from (a). For (a), it is clear that ι_A is a p-ring morphism (whether $\langle R, P \rangle$ is proper or not). To prove (1) $\Rightarrow$ (2), assume that (2) fails; hence, there is $s \in S$ so that $s, -s \in T$. Thus, $-s^2 \in S$ and $-1 = \dfrac{-s^2}{s^2} \in P$, contradicting (1). To show that (2) $\Rightarrow$ (1), if $-1 = \dfrac{t}{s^2} \in P$, there is $w \in S$ such that $-ws^2 = w\,t$; multiplying through by w obtains $-w^2s^2 = w^2t$ and so w^2s^2 is in $(T \cap -T) \cap S$, contradicting (2). ■

THEOREM 15. *Let $\langle A, T \rangle$ be a p-ring and let S be a multiplicative subset of A, such that $S \cap (T \cap -T) = \emptyset$. Let $R := AS^{-1}$ be the ring of fractions of A by S and let $\iota_A : A \longrightarrow R$ be the canonical ring morphism.*

a) Let $h := \mathscr{G}(\iota_A) : \mathscr{G}_{A,T} \longrightarrow \mathscr{G}_{R,P}$ be the induced RS-morphism. Then, for all $a \in A$ and $s \in S$, $\overline{\left(\frac{a}{s}\right)} = \overline{\left(\frac{as}{1}\right)}$. In particular, h is surjective.

b) For $a, b \in A$, the following are equivalent:

(1) $\overline{\left(\frac{a}{1}\right)} = \overline{\left(\frac{b}{1}\right)}$ *in $\mathscr{G}_{R,P}$;*

(2) *There is $s \in S$ so that $\overline{as} = \overline{bs}$ in $\mathscr{G}_{A,T}$;*

(3) *There is $s \in S$ such that $\overline{as^2} = \overline{bs^2}$ in $\mathscr{G}_{A,T}$.*

c) h is injective $\Leftrightarrow$ $\{\overline{s} \in \mathscr{G}_{A,T} : s \in S\} \subseteq \mathscr{G}_{A,T}^{\times}$.

d) For $a, b \in A$, the following are equivalent:

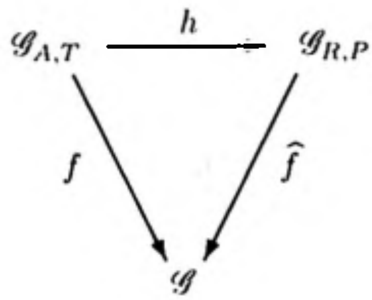
(1) $\overline{\left(\frac{a}{1}\right)} \in \mathscr{D}_{\mathscr{G}_{R,P}}\left(\overline{\left(\frac{b}{1}\right)}, \overline{\left(\frac{c}{1}\right)}\right)$;

(2) *There are $s_1, s_2, s_3 \in S$ such that, with $a' = s_1^2 a$, $b' = s_2^2 c$ and $c' = s_3^2 c$, we have $\overline{a'} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{b'}, \overline{c'})$.*

e) Suppose $\mathscr{G}$ is a RS, $f : \mathscr{G}_{A,T} \longrightarrow \mathscr{G}$ is a RS-morphism and f verifies the following condition

[ker] *For all $a, b \in A$, if there is $s \in S$ such that $\overline{as} = \overline{bs}$, then $f(\overline{a}) = f(\overline{b})$.*

Then, there is a unique RS-morphism, $\widehat{f} : \mathscr{G}_{R,P} \longrightarrow \mathscr{G}$, making the following diagram commutative:



f) $\ker h$ is a RS-congruence on $\mathscr{G}_{A,T}$ and the diagram $h : \mathscr{G}_{A,T} \longrightarrow \mathscr{G}_{R,P}$ is canonically RS-isomorphic to the quotient $\pi_{\ker h} : \mathscr{G}_{A,T} \longrightarrow \mathscr{G}_{A,T}/\ker h$.

g) With notation as in 13, the kernel of the composition $\mathscr{G}_A \xrightarrow{pr} \mathscr{G}_{A,T} \xrightarrow{h} \mathscr{G}_{R,P}$ is a RS-congruence, θ, on $\mathscr{G}_A$, canonically RS-isomorphic to the quotient $\pi_\theta : \mathscr{G}_A \longrightarrow \mathscr{G}_A/\theta$.

PROOF. By Lemma 14.(a), $\langle R, P \rangle$ is a proper p-ring and ι_A is a morphism of unitary p-rings.

a) For $a \in A$ and $s \in S$, we have

$$\frac{a^2s^2}{1} + \frac{a^2}{s^2} = \frac{a^2}{1}\left(\frac{s^2}{1} + \frac{1}{s^2}\right) = \frac{a^2}{1}\frac{1+s^4}{s^2} = \frac{as}{1}\frac{a}{s}\frac{1+s^4}{s^2}.$$

with $\frac{1+s^4}{s^2} \in P$ and Theorem 12.(d) yields the desired conclusion.

b) Since for all $\xi \in \mathscr{G}_{A,T}$, $\xi^3 = \xi$, (2) and (3) are clearly equivalent. For (2) $\Rightarrow$ (1), suppose $\overline{as} = \overline{bs}$ in $\mathscr{G}_{A,T}$, with $a, b \in A$ and $s \in S$. Then,

$$\overline{\left(\frac{a}{1}\right)} = \overline{\left(\frac{as}{s}\right)} = \overline{\left(\frac{bs}{s}\right)} = \overline{\left(\frac{b}{1}\right)},$$

as needed. It remains to establish (1) $\Rightarrow$ (2). If $\overline{\left(\frac{a}{1}\right)} = \overline{\left(\frac{b}{1}\right)}$ in $\mathscr{G}_{R,P}$, by Theorem 12.(d) there are $t, t_1 \in T$, $u, v \in S$ and an integer $k \geq 0$ such that, in $R = AS^{-1}$,

$$\text{(II)} \qquad \left(\frac{a^2}{1} + \frac{b^2}{1}\right)^k + \frac{t}{u^2} = \frac{t_1}{v^2}\,\frac{a}{1}\,\frac{b}{1}.$$

Since $\frac{a^2}{1} + \frac{b^2}{1} \in P$, we may assume that $k \geq 2$ (and in fact, to be any prescribed positive integer greater than to the original k). The definition of ring of fractions yields $w \in S$ so that, after clearing denominators, we obtain [4]

$$\text{(III)} \qquad u^2v^2w^2(a^2 + b^2)^k + tv^2w^2 = t_1u^2w^2ab.$$

Multiplying (III) by $(uvw)^{2k-2}$, obtains, with $t' = tv^2w^2(uvw)^{2k-2} \in T$,

$[(auvw)^2 + (buvw)^2]^k + t' = t_1u^2w^2(uvw)^{2k-2}ab = t_1\,(uw)^{2k-2}\,v^{2k-4}\,(auvw)\,(buvw)$

$$= t''\,(auvw)(buvw),$$

with $t'' \in T$; setting $s := uvw \in S$, the immediately preceding equality and Theorem 12.(d) entail $\overline{as} = \overline{bs}$ in $\mathscr{G}_{A,T}$, as needed.

c) Suppose h is injective and $s \in S$. Then, (a) yields

$$\overline{1} = \overline{\left(\frac{1}{s}\right)}\,\overline{\left(\frac{s}{1}\right)} = \overline{\left(\frac{s}{1}\right)}\,\overline{\left(\frac{s}{1}\right)} = \overline{\left(\frac{s^2}{1}\right)},$$

and the injectivity of h entails $\overline{1} = \overline{s^2}$ in $\mathscr{G}_{A,T}$, i.e, $\overline{s} \in \mathscr{G}_{A,T}^{\times}$. The converse is an immediate consequence of the equivalence in (b).

d) (1) $\Rightarrow$ (2). By ($\mathscr{D}$) in 1.7.(b), there are $t, t_1, t_2 \in T$ and $x, y, z \in S$ such that, in $R = AS^{-1}$,

$$\text{(IV)} \qquad \begin{cases} (i) & \frac{t}{x^2}\,\frac{a}{1} = \frac{t_1}{y^2}\,\frac{b}{1} + \frac{t_2}{z^2}\,\frac{c}{1}, \text{ and} \\ (ii) & \overline{\left(\frac{t}{x^2}\right)\left(\frac{a}{1}\right)} = \overline{\left(\frac{ta}{x^2}\right)} = \overline{\left(\frac{a}{1}\right)}. \end{cases}$$

By (a) above, $\overline{\left(\frac{ta}{x^2}\right)} = \overline{\left(\frac{tax^2}{1}\right)}$ and so IV.(ii) entails $\overline{\left(\frac{tax^2}{1}\right)} = \overline{\left(\frac{a}{1}\right)}$, whence, by (c), there is $s \in S$ such that

$$\text{(V)} \qquad \overline{tax^2s^2} = \overline{as^2}.$$

[4] Recall: $a/s = a'/s'$ in R iff there is $w \in S$ so that $was' = wsa'$; multiplying by w, yields $w^2as' = w^2sa'$.

By IV.(i), there is $w \in S$, so that, after clearing denominators, we get

$$w^2y^2z^2\ ta \;=\; t_1\ (w^2x^2z^2b) \;+\; t_2\ (w^2x^2y^2c).$$

Multiplying this equality by x^2s^2 yields

(VI) $$(tx^2)(w^2y^2z^2s^2a) \;=\; t_1\ (w^2x^4z^2s^2b) \;+\; t_2\ (w^2x^4y^2s^2c).$$

Set $s_1 = wyzs$, $s_2 = wx^2zs$ and $s_3 = wx^2ys$; clearly $s_i \in S$, $i = 1, 2, 3$. Moreover, if $a' = s_1^2a$, $b' = s_2^2b$ and $c' = s_3^2c$, (VI) takes the form

(VII) $$tx^2\ a' \;=\; t_1b' + t_2c'.$$

Now note that (V) yields, multiplying by $\overline{w^2y^2z^2}$,

$$\overline{tx^2a'} \;=\; \overline{tx^2as^2(w^2y^2z^2)} \;=\; = \;\overline{s^2a(w^2y^2z^2)} \;=\; \overline{a'},$$

which, together with (VII), entails $\overline{a'} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{b'}, \overline{c'})$, as desired.

(2) $\Rightarrow$ (1). Since h is a RS-morphism and $h(\overline{a}) = h(\overline{a'})$, $h(\overline{b}) = h(\overline{b'})$ and $h(\overline{c}) = h(\overline{c'})$ (by (b)), (2) entails, because $h(\overline{c}) = \overline{\left(\frac{c}{1}\right)}$ $(c \in A)$, $h(\overline{a}) \in \mathscr{D}_{\mathscr{G}_{R,P}}(h(\overline{b}), h(\overline{c}))$, as needed.

e) The uniqueness of a map (if it exists) making the diagram commutative is clear. Define $\widehat{f} : \mathscr{G}_{R,P} \longrightarrow \mathscr{G}$ by

$$\widehat{f}\left(\overline{\left(\frac{a}{x}\right)}\right) \;=\; f(\overline{ax}).$$

To see $\widehat{f}$ is well-defined, assume $\overline{\left(\frac{a}{x}\right)} = \overline{\left(\frac{b}{y}\right)}$; by (a) we have $\overline{\left(\frac{ax}{1}\right)} = \overline{\left(\frac{by}{1}\right)}$ and so (b) yields $s \in S$ such that $\overline{axs} = \overline{bys}$. Since f verifies [ker] in the statement, we obtain $f(\overline{ax}) = f(\overline{by})$, as needed. It is straightforward that $f = \widehat{f} \circ h$ and $\widehat{f}$ preserves product, as well as the constants 1, 0 and -1. It remains to check that $\widehat{f}$ preserves representation. Since h is surjective, by the definition of $\widehat{f}$ it suffices to prove, for $a, b, c \in A$:

(VIII) $$\overline{\left(\frac{a}{1}\right)} \in \mathscr{D}_{\mathscr{G}_{R,P}}\left(\overline{\left(\frac{b}{1}\right)}, \overline{\left(\frac{c}{1}\right)}\right) \;\Rightarrow\; f(\overline{a}) \in \mathscr{D}_{\mathscr{G}}(f(\overline{b}), f(\overline{c})).$$

By item (d), the antecedent in (VIII) implies the existence of $s_i \in S$, $i = 1, 2, 3$, such that

(IX) $$\overline{as_1^2} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{bs_2^2},\ \overline{cs_3^2}).$$

Now, note that for $u \in A$ and $x \in S$, $\overline{ux^2} = \overline{u}\,\overline{x^2}$, and so condition [ker] implies $f(\overline{ux^2}) = f(\overline{u})$; this observation and (IX) entail, because f is a RS-morphism, the conclusion in (VIII), as needed.

f) By item (b),

$$\ker h = \{\langle \overline{a}, \overline{b}\rangle \in \mathscr{G}_{A,T} \times \mathscr{G}_{A,T} : \exists\ s \in S \text{ so that } \overline{as} = \overline{bs}\}.$$

Hence, condition [ker] in (e) is equivalent to $\ker h \subseteq \ker f$ and the conclusion in (e) shows that h is a surjective RS-morphism with the RS-UFP. The conclusion in (f) then an immediate consequence of Lemma 11.(b), while item (g) follows from (f) and Lemma 11.(c), ending the proof. ■

3 A Representation Theorem. Applications

Our first step in representing the RS of any p-ring by that of a BIR, is to show that the RS of a p-ring is (naturally) isomorphic to the RS of a *reduced* p-ring.

PROPOSITION 16. *Let $\langle A, T \rangle$ be a p-ring and let $I = \sqrt[T]{0}$ be the T-radical of the zero ideal. Set $R = A/I$. Then,*

a) $\langle R, T/I \rangle$ is a reduced p-ring and the canonical projection, $\pi_I : \langle A, T \rangle \longrightarrow \langle R, T/I \rangle$ is a p-ring morphism.

b) The RS-morphism $\mathscr{G}(\pi_I) : \mathscr{G}_{A,T} \longrightarrow \mathscr{G}_{R,T/I}$ is an isomorphism of real semigroups.

PROOF. a) Clearly, T/I is closed under sums, products and contains the squares in R; if $\mathfrak{p}$ is a proper T-convex prime ideal in A, then for all $t \in T$, $1 + t \notin \mathfrak{p}$ (otherwise, $1 \in \mathfrak{p}$). By Proposition 3, for all $t \in T$, $1 + t \notin \sqrt[T]{0}$; whence $-1/I \notin T/I$, and T/I is a proper preorder of R. It is clear that π_I is a p-ring morphism. The verification that R is T/I-reduced is the same as that of item (3) in the proof of part (b.3) of Proposition 6.5 (p. 73ff) in [Dickmann and Miraglia, 2011].

b) To ease notation, we shall write $\overline{a}$ for the elements of $\mathscr{G}_{A,T}$ and $\mathscr{G}_{R,T/I}$, omitting the subscripts T and T/I, respectively. By 1.8, $h := \mathscr{G}(\pi_I)$ is a RS-morphism. To show it to be an isomorphism we must verify:

(1) h is bijective;

(2)[5] For all $a, b, c \in A$, $h(\overline{a}) \in \mathscr{D}^t_{\mathscr{G}_{R,T/I}}(h(\overline{b}), h(\overline{c})) \quad \Rightarrow \quad \overline{a} \in \mathscr{D}^t_{\mathscr{G}_{A,T}}(\overline{b}, \overline{c})$.

Note that, because h is a RS-morphism, the converse of (2) also holds.

Proof of (1). It is clear that h is surjective. To show h is injective, let $x, y \in A$ and suppose

$$h(\overline{x}) = \overline{x/I} = \overline{y/I} = h(\overline{y}).$$

By Theorem 12.(d), there are $t, s \in T$ and an integer $k \geq 0$ such that

$$((x/I)^2 + (y/I)^2)^k + t/I = (s/I)(x/I)(y/I),$$

that is, setting $u := (x^2 + y^2)^k + t$, we have $u - sxy \in I = \sqrt[T]{0}$. Note that $u \in T$. By Proposition 3, there are $m \geq 0$ and $t' \in T$ such that

$$\text{(I)} \qquad (u - sxy)^{2m} + t' = 0.$$

But we have $(u - sxy)^{2m} = E - F$, where:

$$E = \textstyle\sum_{k\,even} \binom{n}{k} u^{2m-k}(sxy)^k \quad \text{and} \quad F = \textstyle\sum_{k\,odd} \binom{n}{k} u^{2m-k}(sxy)^k.$$

Then:

- $E = \sum_{k\,even} \binom{n}{k} u^{2m-k}(sxy)^k = u^{2m} + \sum_{k\,even \geq 2} \binom{n}{k} u^{2m-k}(sxy)^k$;

Since $u \in T$ and k is even, $\sum_{k\,even\geq 2} \binom{n}{k} u^{2m-k}(sxy)^k$ is in T and we may write

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[5] Because representation and transversal representation are interdefinable in a RS, $\mathscr{G}$, with $\mathscr{I}^t_{\mathscr{G}} \subseteq \mathscr{I}_{\mathscr{G}}$; see (†), [RS4] and [RS6] in Def. 2.1, p. 106 of [Dickmann and Petrovich, 2004] or [t-rep], [RS4] and [RS6] in Def. I.2.1, p. 19 of [Dickmann and Petrovich, 2011]

(II) $$E = u^{2m} + t^*, \text{ with } t^* \in T;$$

• $F = \sum_{k\, odd} \binom{n}{k} u^{2m-k}(sxy)^k = sxy \sum_{k\, odd} u^{2m-k}(sxy)^{k-1}$; just as above, $\sum_{k\, odd} u^{2m-k}(sxy)^{k-1}$ is in T and we obtain

(III) $$F = s^*xy, \text{ with } s^* \in T.$$

Substituting (II) and (III) into (I) yields

(IV) $$u^{2m} + t^* + t' = s^*xy.$$

Now observe that $u^{2m} = [(x^2 + y^2)^k + t]^{2m} = (x^2 + y^2)^{2mk} + s'$, with $s' \in T$. Hence, (IV) entails

$$(x^2 + y^2)^{2mk} + s' + t^* + t' = s^*xy,$$

whence, by Theorem 12.(d), $\overline{x} = \overline{y}$, establishing the injectivity of $h = \mathscr{G}(\pi_I)$.

Proof of (2). By ($\mathscr{D}^t$) in 1.7.(b), there are a', b', c' $\in A$, such that

(V) $$\begin{cases} (i) \ \ \overline{a/I} = \overline{a'/I}, \ \ \overline{b/I} = \overline{b'/I}, \ \ \overline{c/I} = \overline{c'/I} \\ \text{and} \\ (ii) \ \ a'/I = b'/I + c'/I. \end{cases}$$

By (V).(ii), there is $r \in I = \sqrt[T]{0}$ such that

(VI) $$a' = b' + c' + r = b' + (c' + r).$$

We now register the following

FACT 17. *Let S be a semi-real ring, let u, $v \in S$ and let $\beta \in$ Sper(S). If $v - u \in supp(\beta)$, then $\overline{u}(\beta) = \overline{v}(\beta)$.*

Proof. Clearly, $u \in$ supp(β) iff $v \in$ supp(β). Next, if $u \in \beta \setminus (-\beta)$, then $v = u + (v - u)$ and so $v \in \beta$. If $-v \in \beta$, then $-u = -v + (v - u)$ implies $-u \in \beta$, which is impossible. Hence, $v \in \beta \setminus (-\beta)$. Because supp$(\beta)$ is an ideal, the argument is symmetric in u and v and so $u \in \beta \setminus (-\beta)$ iff $v \in \beta \setminus (-\beta)$. Since $(-u - (-v)) = (v - u) \in$ supp(β), the reasoning above applies to yield $-u \in \beta \setminus (-\beta)$ iff $-v \in \beta \setminus (-\beta)$. Hence, $\overline{u}(\beta) = \overline{v}(\beta)$, as desired. □

Since r is in the intersection of all T-convex ideals in A, Fact 17 yields, with $c^* := c' + r$,

(VII) $$\overline{c' + r} = \overline{c^*} = \overline{c'}.$$

Now the injectivity of h, the equalities in (V).(i), (VI) and (VII) entail

$$\overline{a} = \overline{a'}, \ \ \overline{b} = \overline{b'}, \ \ \overline{c^*} = \overline{c'} \ \ \text{and} \ \ a' = b' + c^*,$$

which, by ($\mathscr{D}^t$) in 1.7.(b), guarantee $a \in \mathscr{D}^t_{\mathscr{G}_{A,T}}(b, c)$, establishing (2) and ending the proof. ■

The next step in our construction is the following

THEOREM 18. *Let $\langle A, T \rangle$ be a T-reduced p-ring. Let*

$$\mathcal{U} = A \setminus \bigcup_{\alpha \in Spec_R(A,T)} \text{supp}(\alpha)$$

be the complement of the union of all T-convex prime ideals in A. Let S be a multiplicative set contained in $\mathcal{U}$, let $A_S = AS^{-1}$ be the ring of fractions of A

by S and set $T_S = \left\{\frac{t}{s^2} \in A_S : t \in T \text{ and } s \in S\right\}$. *Then,*

a) $\mathcal{U}$ is a proper saturated [6] *multiplicative set in A, whose elements are all non-zero divisors.*

b) $\langle A_S, T_S \rangle$ is a proper p-ring and the canonical morphism, $\iota_S : A \longrightarrow A_S$, is a p-ring embedding.

c) A_S is T_S-reduced.

d) The map $\mathcal{G}(\iota_S) : \mathcal{G}_{A,T} \longrightarrow \mathcal{G}_{A_S,T_S}$ is a RS-isomorphism.

e) Consider the following conditions:

(1) *For all $s \in S$ and $t \in T$, $s^2 + t \in S$;* (2) *$\langle A_S, T_S \rangle$ has bounded inversion.*

Then, (1) $\Rightarrow$ (2); if S is saturated, these conditions are equivalent.

REMARK 19. a) If $\langle A, T \rangle$ is a p-ring, the set $\mathcal{U}$ in the statement of Theorem 18 consists of the elements $a \in A$ satisfying $\overline{a}^2 = 1$, i.e., $\mathcal{U} = \{a \in A : \overline{a} \in \mathcal{G}^{\times}_{A,T}\}$, where $\mathcal{G}^{\times}_{A,T}$ is the group of units of the RS $\mathcal{G}_{A,T}$.

b) If $\langle A, T \rangle$ is a BIR, then $\mathcal{U} = A^{\times}$, the group of units in A. Indeed, by Proposition 6.3 (p. 71) of [Dickmann and Miraglia, 2011], every maximal ideal in A is T-convex and so the set of elements outside every T-convex prime ideal in A is $A^{\times}$. Hence, with notation as in 18, if $\langle A, T \rangle$ is a BIR, $\langle A, T \rangle$ and $\langle A_S, T_S \rangle$ are naturally isomorphic. ■

Proof of Theorem 18. a) It is well-known that the complement of a union of prime ideals is a proper saturated multiplicative set in A. For $x \in S$, suppose $xu = 0$, for some $u \in A$. Since x is outside all T-convex primes in A, we get $u \in \bigcap_{\alpha \in Spec_R(A,T)} \text{supp}(\alpha) = \sqrt[T]{0}$ and so $u = 0$ because A is T-reduced. It now clear that no element of S is a zero-divisor.

b) Since $\mathcal{U} \cap \sqrt[T]{0} = \emptyset$, we also have $S \cap \sqrt[T]{0} = \emptyset$ and so, by 14.(b), $\langle A_S, T_S \rangle$ is a proper p-ring and ι_S a p-ring morphism. Moreover, since no element of S is a zero-divisor, it is well-known that ι_S is an embedding.

c) For $a \in A$ and $x \in S$, suppose $\frac{a}{x}$ is in the T_S-radical of 0 in A_S. By Proposition 3, there are $t \in T$, $y \in S$ and an integer $m \geq 0$ such that $\frac{a^{2m}}{x^{2m}} + \frac{t}{y^2} = 0$. Hence, in A we obtain

(I) $$y^2 a^{2m} + x^{2m} t = 0.$$

Multiplying the equation in (I) by y^{2m-2} yields

$$(ay)^{2m} + y^{2m-2} x^{2m} t = 0,$$

and another application of 3 entails $ay \in \sqrt[T]{0}$ in A. Since $y \in S$ is outside all T-convex primes in A, we get $a \in \sqrt[T]{0}$; whence, the T-reducibility of A implies $a = 0$ and, in turn, the T_S-reducibility of A_S.

[6] $xy \in \mathcal{U} \Rightarrow x, y \in \mathcal{U}$.

d) Write $\mathscr{G}$ for $\mathscr{G}_{A,T}$ and $\mathscr{G}_S$ for $\mathscr{G}_{A_S,T_S}$. By 1.8, $h := \mathscr{G}(\iota_S)$ is a RS-morphism. To show it is an isomorphism, it suffices to prove:

($\natural$.1) h is surjective; ($\natural$.2) h is injective;

($\natural\natural$) For all a, b, $c \in A$,

$$h(a) = \overline{\left(\frac{a}{1}\right)} \in D_{\mathscr{G}_S}\left(\overline{\left(\frac{b}{1}\right)}, \overline{\left(\frac{c}{1}\right)}\right) \quad \Rightarrow \quad \overline{a} \in D_{\mathscr{G}}(\overline{b}, \overline{c}).$$

Since h is a RS-morphism, the converse of ($\natural\natural$) is also true. Property ($\natural$.1) follows from item (a) in Theorem 15, while ($\natural$.2) is a consequence of 15.(c) and 19.(a). By item (d) in Theorem 15, the hypothesis in ($\natural\natural$) yields $s_i \in S$, $i = 1, 2, 3$, such that $\overline{as_1^2} \in D_{\mathscr{G}}(\overline{s_2^2 b}, \overline{s_3^2 c})$, which is equivalent to $\overline{a} \in D_{\mathscr{G}}(\overline{b}, \overline{c})$, because $\overline{s_i} \in \mathscr{G}^{\times}$, $i = 1, 2, 3$, as needed.

e) For $s \in S$ and $t \in T$, $1 + \dfrac{t}{s^2} = \dfrac{s^2 + t}{s^2} \in A_S^{\times}$ iff there is $u \in S$ such that $u(s^2 + t) \in S$ (recall that ι_S is an embedding). It is now clear that (1) $\Rightarrow$ (2) (with $u = 1$), while, if S is saturated, the converse is immediately forthcoming. ■

3.1. Notation. Let $\langle A, T \rangle$ be p-ring and let $\mathcal{U}$ be the complement of the T-convex primes in A, as in 18. Write $\mathscr{G}^\natural$ for $\mathscr{G}_{A^\natural, T^\natural}$, where

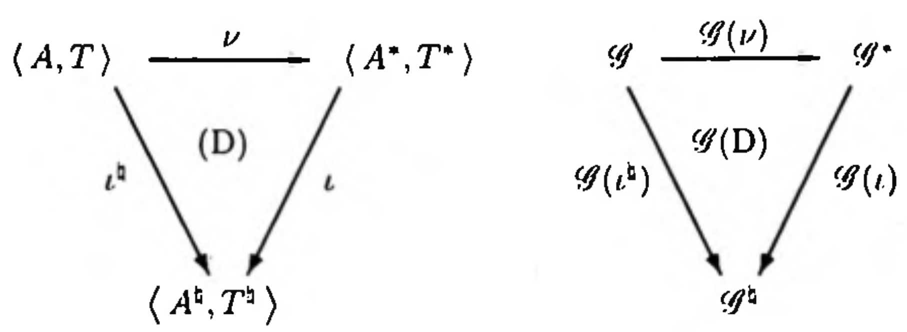
- $A^\natural = A\mathcal{U}^{-1}$ for the ring of fractions of A by $\mathcal{U}$;
- $T^\natural = \left\{ \dfrac{t}{s^2} \in A\mathcal{U}^{-1} : t \in T \text{ and } s \in \mathcal{U} \right\}$;
- $\iota^\natural : A \longrightarrow A^\natural$ for the canonical p-ring embedding. ■

We now have

COROLLARY 20. *Notation as above, let $\langle A, T \rangle$ be a p-ring. Let*
- *$\nu : \langle A, T \rangle \longrightarrow \langle A^*, T^* \rangle$ be its BIR hull* (cf. Proposition 6)*;*
- *$\iota^\natural : \langle A, T \rangle \longrightarrow \langle A^\natural, T^\natural \rangle$ be as in* 3.1.

Write $\mathscr{G}$ for $\mathscr{G}_{A,T}$ and $\mathscr{G}^*$ for $\mathscr{G}_{A^*,T^*}$. Then,

a) $\langle A^\natural, T^\natural \rangle$ is a reduced BIR and $\iota : \langle A^*, T^* \rangle \longrightarrow \langle A^\natural, T^\natural \rangle$ is the unique p-ring embedding making the diagram (D) *commutative.*



b) Diagram $\mathscr{G}$(D) *is commutative and its arrows are* RS-isomorphisms.

PROOF. a) By items (a), (c) in 18, $\langle A^\natural, T^\natural \rangle$ is a $T^\natural$-reduced proper p-ring and $\iota^\natural$ is an injective p-ring morphism. Note that $s \in \mathcal{U}$ and $t \in T$ implies $s^2 + t$

$\in \mathcal{U}$; otherwise, $s^2 + t \in \mathfrak{p}$, for some T convex prime ideal in A, and so t, $s^2 \in \mathfrak{p}$, which entails $s \in \mathfrak{p}$, an impossibility. Hence, by 18.(e), $\langle A^\natural, T^\natural \rangle$ is a BIR. By the universal property of the BIR hull in 6.(3), there is a unique p-ring morphism, $\langle A^*, T^* \rangle \xrightarrow{\iota} \langle A^\natural, T^\natural \rangle$, making diagram (D) commute. Since, in fact, $1 + T \subseteq \mathcal{U}$ [7], ι is an embedding.

b) Diagram $\mathscr{G}$(D) arises by applying the functor $\mathscr{G}$ to diagram (D), whence it is commutative. As noted above, the multiplicative set $1 + T \subseteq \mathcal{U}$ and so, by 18.(d), $\mathscr{G}(\nu)$ and $\mathscr{G}(\iota^\natural)$ are both RS-isomorphisms; consequently, the same must be true of $\mathscr{G}(\iota)$, ending the proof. ■

REMARK 21. a) If $\langle A, T \rangle$ is a proper p-ring, all constructions employed to go from $\langle A, T \rangle$ to $\langle A^\natural, T^\natural \rangle$ – quotients and rings of fractions –, are functorial. In fact, it is well-known that these constructions commute with each other.

b) The isomorphism $\mathscr{G}(\pi_I)$ of 16 yields a conclusion analogous to that of 20 for *all* proper p-rings. More precisely, given $\langle A, T \rangle$, let $I = \sqrt[T]{0}$ and let $\langle R, P \rangle$ be the BIR hull of $\langle A/I, T/I \rangle$; then, $\mathscr{G}_{A,T}$ is RS-isomorphic to $\mathscr{G}_{R,P}$. ■

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[7] $\overline{(1+t)} = \overline{1}$, for all $t \in T$.