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A random walk model with a mixed memory profile: Exponential and rectangular profile

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ABSTRACT

The theory of Markovian random walks is consolidated and very well understood, however the theory of non-Markovian random walks presents many challenges due to its remarkably rich phenomenology. An important open problem in this context is to study how the diffusive properties of random walk processes change when memory-induced correlations are introduced. In this work we propose a model of a random walk that evolves in time according to past memories selected from rectangular (flat) and exponentially decaying memory profiles. In this mixed memory profile model, the walker remembers either the last B steps with equal *a priori* probability or the steps A prior to B with exponentially decaying probability, for a total number of steps equal to $A+B$. The diffusive behavior of the walk is numerically examined through the Hurst exponent (H). Even in the lack of exact solutions, we are still able to show that the model can be mapped onto a RW model with rectangular memory profile.

1. Introduction

A Random Walk (RW) is characterized by a stochastic process formed by a succession of random steps. This terminology was first introduced in 1905 by Karl Pearson, in a paper entitled "The Problem of the Random Walk" [1]. Pearson wanted to find the probability that the walker is at a distance between r and $r+\delta r$ after n repetitions of the process. The term Random Walk was also associated with the motion of particles in a fluid, observed by Robert Brown in 1828 [2] and later modeled by Einstein in 1905 [3].

Random Walks models associated with long-range memory are considered non-Markovian processes and over the years these models have been used in various fields of science ranging from physics to economics [4–19] Random walks are also able to represent real processes through a stochastic evolution equation. Schütz and Trimper developed a RW model with memory [20] which became known as the Elephant Random Walk (ERW) model, an allusion to the fact that elephants present good memory. The ERW model includes long-range memory correlations with access the full history of previous steps and incorporates a decision process that depends on a single parameter.

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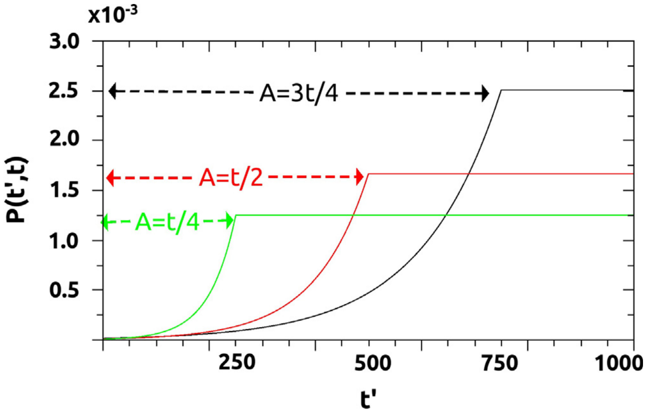

Fig. 1. Illustration of the memory profiles for several possible choices of A. This plot shows the probability distribution of $P(t', t)$ (2) for $A = t/4, t/2$ and $3t/4$ with $\lambda = 5$ and $t = 1000$. For times $A \leq t' < t$ the proximal past is well remembered, but there is exponential degradation of the memories of the distant past $t' < A$. This random walk model is one of very few (perhaps the first) to have a mixed memory profile.

An important parameter used to analyze the diffusion properties of the RW is the Hurst exponent. For the case of zero drift velocity, the Hurst exponent quantifies how the mean squared displacement scales with time t, i.e., $\langle x^2\rangle - \langle x\rangle^2 \sim t^{2H}$, where normal diffusion correspond to $H = 1/2$, superdiffusion to $H > 1/2$ and subdiffusion to $H < 1/2$.

It is a widely accepted fact that information retained in the human brain is rapidly lost. In fact, forgetfulness is a natural process that affects both short-term and long-term information [21–24]. This has been depicted by the well known Ebbinghaus forgetting curve [25,26], which describes how information is forgotten over time. Ebbinghaus hypothesis, still accepted nowadays, suggests that the forgetting mechanism of the brain can be approximated by an exponential process [27]. The aim of this paper is to present a new model of random walk with long-range memory, by introducing a mixed memory profile, exponential and rectangular in shape (Fig. 1). The exponential decay associated with ancient memory [28] represents the natural decay of long-term memory resembling the forgetting process in biological systems.

In Section 2 we describe the model and the statistical method behind the model. Section 3 presents the results and in Section 4 we discuss and conclude.

2. Model

In this model we describe a discrete-time on-lattice one-dimensional random walk as a variation of the ERW Model [20], beginning at position $x_0 = 0$ at time $t = 0$. At time t the walker is at position x_t and moves a unitary step to the left or right according to a stochastic evolution equation [20] namely $x_{t+1} = x_t + v_{t+1}$ where $v_{t+1} = \pm 1$ is a random variable. Without loss of generality, it is assumed that first step always goes to the right, i.e. $\sigma_1 = +1$. At time $t+1$ the walker chooses a random variable t' $(1 \leq t' \leq t)$, whose probability distribution is given by Eq. (1), and sets the velocity as

$$v_{t+1} = \begin{cases} +v_{t'}, & \text{with probability} \quad p \\ -v_{t'}, & \text{with probability} \quad 1-p. \end{cases}$$

This equation can also be written concisely as $P(v_{t+1}) = \{p, v_{t+1} = v_{t'}, 1-p, v_{t+1} = -v_{t'}\}$. The parameter p is such that for $p < 0.5$, the walker behaves predominantly in an opposite way of what he did in the past, whereas for $p > 0.5$, the general trend is to reproduce past behavior. When $p = 0.5$, the past does not influence the walker's decision, making his decision completely aleatory.

In this variation of the ERW model [20], the walker is able to remember with uniform probability a certain amount B of recent steps, or A past steps before B with exponentially decaying probability. The total number of steps is given by $A + B = t$. The probability of t' is then derived from

$$P(t', t) = \begin{cases} \kappa \exp\left[-\lambda(A - t')/A\right] & \text{for} \quad t' < A \\ \kappa & \text{for} \quad t' \geq A \end{cases} \tag{1}$$

where the constant λ is a dimensionless parameter that controls the shape of the exponential distribution and

$$\kappa = \frac{1}{(t-A) + (A/\lambda)[1 - \exp(-\lambda)]} \tag{2}$$

is a normalization constant. Notice that, although not explicitly shown, A and B are actually time dependent. Otherwise the corresponding memory length would disappear for very large times.

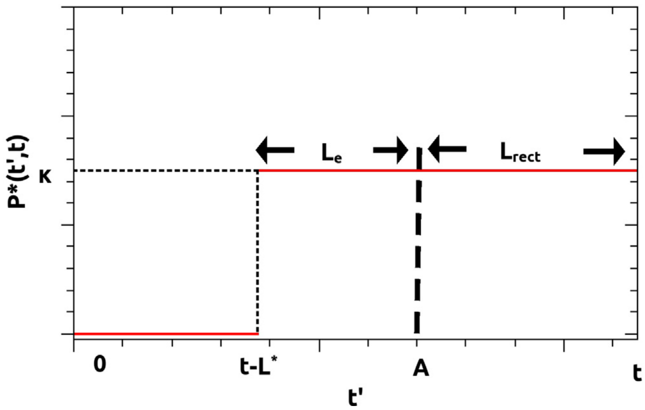

Fig. 2. Diagram illustrating the mapping of a mixed memory profile onto a recent rectangular memory profile. The memory length L_e represents the mapping corresponding to the exponential memory and L_{rect} the rectangular memory mapping.

Since the solution for the non-Markovian RW model with exponential memory profile is unknown, we derive an approximate solution to this process. We assume that this problem can be mapped onto an equivalent rectangular memory profile [28,29]. This approach requires an effective rectangular memory length to the exponential memory profile. Accordingly we define a fixed memory size $L_e = f^\star A$ where the parameter $0 < f^\star < 1$ fixes the size of the effective rectangular memory (see Fig. 2). Given that t' is within the interval $[A - L_e, A]$, the probability of finding t' is then given simply by $1/L_e$ for $(1 - f^\star)A < t' < A$. This approach must be validated by numerical analysis, which is done below.

3. Results

We can define the memory's effective length [28,29] by

$$L^\star \equiv \int_0^t \left[\frac{P(t', t)}{P_{max}(t', t)} \right] dt' \quad ,$$

where $L^\star = L_e + L_{rect} = f^* A + B$ and $P_{max}(t', t)$ is the maximum value of $P(t', t)$. Here $L_{rect} = B$ is the length of the rectangular piece of the mixed memory profile (see Fig. 2). This mapping is an "ansatz" necessary to write the equation of motion for this non-Markovian problem. Notice that it would reproduce a rectangular memory profile with the same length, if $P(t', t)$ were rectangular. Its validity has been supported by previous studies, as shown in Refs. [28,29].

From Eqs. (1) and (2) we get

$$L^\star = \frac{A}{\lambda} [1 - \exp(-\lambda)] + (t - A) \tag{3}$$

where $A < t$ and $\lambda \neq 0$. Comparing (3) and (2) we note that $L^\star = 1/\kappa$.

For $A = t$ we recover the pure exponential model [28]. In this case, the rectangular memory length is $L^* = (1/\lambda)(1 - \exp(-\lambda))t$. The normal to superdiffusive transition starts at $p = 3/4$ for $\lambda \to 0$ (ERW model) and gets larger with growing λ. For $\lambda \to \infty$ we get the Brownian motion (regular Markovian Random Walk).

We verified that, for some values of λ, p and A, the mapping of an exponential memory profile [28] onto a rectangular memory profile (Eq. (3)), presents a deviation error larger than the error bar of the simulation. For example, for the pure exponential memory model with $\lambda = 5$ and $p = 0.95$, the Hurst exponent is $H \approx 0.75$ (see Ref. [28], Figure 1). However, for this specific model of exponential decay ($A = t$), the effective memory length is given by $L^\star = f_{eff}\, t \approx (1/\lambda)t$, i.e., $f_{eff} \approx 0.2$. For this value of f_{eff}, the recent memory model presents normal diffusion, considerably different than that obtained from computing simulations for the exact profile. Exact approaches are currently being developed to deal with these issues. While this discussion is beyond the scope of this work, we shall address this issue in future work.

Based on these comments, we can now achieve an analytical solution for the mixed memory problem, by considering the number of steps forward and backward up to time t, i.e., $n^+(t)$ and $n^-(t)$, respectively. We can then write the effective probabilities of taking a step forward and backward, i.e., $P^+_{eff}(t, x)$ and $P^-_{eff}(t, x)$, respectively, as

$$P^+_{eff} = \left(\frac{n^+}{L^\star} \right) p + \left(\frac{n^-}{L^\star} \right) (1 - p) \tag{4}$$

and

$$P^-_{eff} = \left(\frac{n^-}{L^\star} \right) p + \left(\frac{n^+}{L^\star} \right) (1 - p) \tag{5}$$

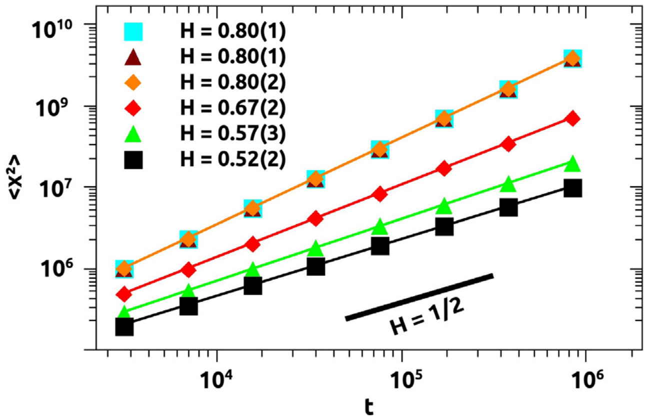

Fig. 3. Log–Log plot showing $\langle x^2 \rangle$ versus time for a sub-set of the data, namely for $p = 0.9$. The colors and symbols are the same as those used in Fig. 4. These results for H (shown with the corresponding errors) are consistent with the plots drawn in Fig. 4.

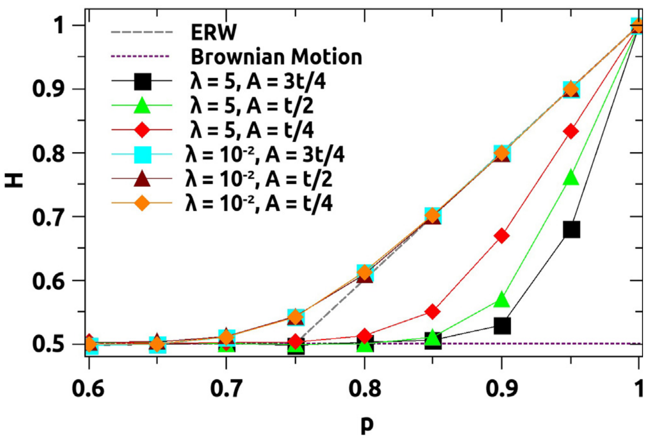

Fig. 4. The Hurst exponent as a function of the parameter p for the RW with fixed memory model and ERW model. The symbols represent numerically evaluated values of H versus p for the fixed memory model with $\lambda = 5$ and $\lambda = 10^{-2}$. Several values were used for the width of the exponential profile (A), namely: ($t/4$, $t/2$ and $3t/4$). For comparison the corresponding H values for the ERW and Brownian Walk models are also shown by the purple and gray dashed lines, respectively. Notice that the mixed memory model tends to the full memory ERW model for $\lambda \to 0$ as it should. Simulations used $10^3 < t < 10^6$ and were averaged with 10^5 runs. Computation data for $p < 0.6$ were not included because the behavior for $H = 0.5$ (normal diffusion) is similar (for the λ values showed in the figure).

for $t > 0$. The difference between P^+_{eff} and P^-_{eff} leads to the effective, or expected, value of the velocity ν at time $t + 1$ (for one realization up to time t), i.e.,

$$\nu^{eff}_{t+1} = P^+_{eff}(t, x) - P^-_{eff}(t, x) \quad . \tag{6}$$

Writing $n^+ + n^- = t - L^\star$ and $n^+ - n^- = x_t - x_{t-L^\star}$, and averaging over all realizations for very large t, we can write $\nu = d\langle x_t \rangle/dt$. Thus, Eq. (6) becomes

$$d\langle x_t \rangle/dt = \frac{\alpha}{L^\star}\left(\langle x_t \rangle - \langle x_{t-L^\star} \rangle\right) \tag{7}$$

where $\alpha = 2p - 1$. This is the equation of motion for this model.

The quantitative analysis of the diffusive behavior of the RW in the mixed memory model can be done by estimating the Hurst exponent as a function of the parameter p. This is done in Fig. 4, which shows numerical estimations for the Hurst exponent as a function of the parameter p for the ERW (purple dashed line), the Brownian Walk (gray dashed line) and the RW model with mixed memory (i.e., the original model with mixed square and exponentially decaying memory) For the latter we varied the profile lengths (by varying A) and the decay constant λ. Simulations were averaged with 10^5 runs and $10^3 < t < 10^6$.

Fig. 3 shows Log–Log plots of $\langle x^2 \rangle$ versus t for $p = 0.9$ (in this model, the variance and $\langle x^2 \rangle$ have the same scaling behavior). The values obtained for the Hurst exponent H, shown here with the corresponding errors, are in accordance with the results shown in Fig. 4. For λ small (e.g. $\lambda = 0.01$) the values for H agree within the statistical error in the superdiffusive region. In the normal diffusion regime the H values coincide ($H = 0.5$). Fig. 4 shows the Hurst exponent against p. According to Fig. 4, the RW model with mixed memory also shows a phase transition at $p = 3/4$ when the normal diffusion regime with $H = 1/2$ for $p < 3/4$ changes to a superdiffusion regime with $H > 1/2$ for $p < 3/4$.

4. Conclusions

In conclusion, we introduce a non-Markovian random walk with long-range memory with a rectangular profile for recent memories and exponential profile for ancient memories. This mixed memory profile is non-analytical, causing a dependence of the Hurst exponent on both A and λ. For the analytic pure exponential profile H depends on one single parameter [28]. We provide numerical solutions that present good results leading to a superdiffusive region for $p > 3/4$ and an expected behavior for models of random walk with recent memory described in the literature. The model leads to a transition from a diffusive regime to a superdiffusive regime, at $p > 3/4$, as shown in Fig. 4, which should be expected. The model approaches the full memory RW model, namely, the ERW model, for $\lambda \to 0$. For $\lambda \gg 0$, the model approaches the RW model with recent memory. The mapping proposed is very effective and can be used to obtain an approximation for the Fokker–Planck equation in cases where the exact analytical solution is still unknown [28]. The same approach used to derive Eq. (7) can be used to derive moments of higher order. This can stimulate interesting and more complex studies on this subject. The method we propose here is important to the study of the effects of long-range memory in decision making processes associated with non-Markovian models with unknown exact solution. The tools we develop to study these models will allow us make adjustments from experimental data, for representations of memory processes associated to biological systems and other applications.

CRediT authorship contribution statement

K.J.C.C. de Lacerda: Formulation of ideas, Computer simulations, Analysis of results. **L.R. da Silva:** Formulation of ideas, Computer simulations, Analysis of results. **G.M. Viswanathan:** Formulation of ideas, Computer simulations, Analysis of results. **J.C. Cressoni:** Formulation of ideas, Computer simulations, Analysis of results. **M.A.A. da Silva:** Formulation of ideas, Computer simulations, Analysis of results.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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