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REGULARLY VARYING DENSITIES FOR
ST. PETERSBURG ENVELOPES

by

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Resumo

Correct conditioning is seen to explain both versions of the Two-envelope Paradox. Prior densities for the smallest amount in the envelopes that indicate as optimal decision to swap the envelopes are seen to be of regular variation. The situation is related to the St. Petersburg paradox.
Keywords: two-envelope paradox, paradoxical distributions, regularly varying distributions

1 Introduction

There is a vast material concerning the two-envelope paradox, cf. e.g. Clark and Shackel [CS00], Horgan [Hor00], Blachman [Bla03]. This problem can be described as a game where you are presented with two undistinguishable envelopes, one containing twice as much money as the other, and you select one at random; then you are offered the chance to swap and take the other instead. In a variant situation you can take a glance at the amount in the first envelope before deciding.

At the latter case, intuitively, if you know that there is an upper bound to the total amount in each envelope, say 100 units, and the first envelope contains 80 units, then, clearly, this is the highest amount, and the better decision is not to swap. If the first envelope contains 30 units, say, you have to estimate which is more probable for the second envelope: 15 units or 60 units, and then decide.

This reasoning can also be applied to the case of unbound amount in the envelope: what is your information about the amount and how a probability distribution can be elicited in terms of this information? After looking at the amount of the first envelope, wich is the better decision: to swap or not to swap, considering the full information?

More formally, let Θ and 2Θ be the amounts in the two envelopes, let X denote the value (not observed) in the selected envelope, and Y, the amount in the second envelope. Then either $Y = X/2$ or $Y = 2X$ units. The paradoxical (and fallacious) argument says that your expected utility if you swap is

$$\boldsymbol{E}(Y) = \frac{1}{2}\,\frac{X}{2} + \frac{1}{2}\,2X = \frac{5}{4}\,X > X$$

if $X > 0$. That is, swapping is recomended. However, exactly the same argument would have been available if you had picked the other envelope in the first place!

Let us define the variable $M = \mathbf{1}\{Y > X\}$. The fallacy consists in the fact that the actual expected value in the second envelope is

$$\begin{aligned}
\boldsymbol{E}(Y) &= \boldsymbol{E}[\boldsymbol{E}(Y \mid M, X)] \\
&= \boldsymbol{P}(M=0)\boldsymbol{E}(Y \mid X, M=0) + \boldsymbol{P}(M=1)\boldsymbol{E}(Y \mid X, M=1) \\
&= \frac{1}{2}\,\boldsymbol{E}\left[\frac{X}{2} \mid M=0\right] + \frac{1}{2}\,2\boldsymbol{E}[2X \mid M=1] \\
&= \frac{1}{4}\,\boldsymbol{E}\,[2\Theta] + \boldsymbol{E}[\Theta] \\
&= \frac{1}{2}\,\boldsymbol{E}\,[\Theta + 2\Theta] = \boldsymbol{E}(X),
\end{aligned}$$

so, the paradox does not exist, in that case.

Suppose now that you have $X = x$ in the selected envelope. Then the second envelope contains either $x/2$ or $2x$ units. Assuming a prior distribution to the unknown value Θ, f_Θ, we say that it is paradoxical if

$$\boldsymbol{E}_\Theta[Y \mid X = x] \geq x,$$

that is, the better decision is swapping, independent of the observed value $X = x$.

From this definition, it can be shown that a distribution with finite support cannot be paradoxical. A paradoxical distribution has to assign non-negligible probability to extreme values, in other words, has to be heavy-tailed.

In this work, a complete mathematical characterization of the paradoxical distribution is given, by means of the regularly varying distributions. In Section 2, definition and some well-known properties of the regularly varying distributions are presented without proof, cf. Feller [Fel66], Resnick [Res87]. The main result and propositions are presented in Section 3.

2 Regularly varying distributions

Regularly varying funtions were initially studied, under the mathematical point of view, by Karamata [Kar30, Kar33], and since then, after the Feller's work [Fel66], have been extensively used in probability theory and statistical models for extreme values.

A funtion, $U : \boldsymbol{R}^+ \to \boldsymbol{R}$, is said to be regularly varying (in $+\infty$) with index ρ if it holds:

$$\lim_{x\to\infty} \frac{U(tx)}{U(x)} = t^\rho, \tag{1}$$

for all $t > 0$, and denoted $U \in RV_\rho$. The number ρ is named the variation exponent. If $\rho = 0$, we say that function U is slowly varying. Denoting, generically, by $L(x)$ a slowly varying funtion, it is immediate that a regularly varying funtion with index ρ can be expressed as $x^\rho L(x)$, by defining $L(x) = U(x)/x^\rho$.

Clearly, a regularly varying function not need to be a probability density, unless $\rho \leq -1$. In this work, we are interested in probability distributions whose survival function is regularly varying. An example is the family

$$1 - F(x) = x^{-\alpha} \qquad , \quad x > 1, \quad \alpha > 0.$$

The α-stable distributions, $0 < \alpha < 2$, are regularly varying, since that

$$1 - F(x) \sim c\, x^{-\alpha} \qquad , \quad x \to \infty, \quad c > 0,$$

and, in particular, the Cauchy distribution, with density $f(x) = (\pi(1 + x^2))^{-1}$, satisfies

$$1 - F(x) \sim (\pi x)^{-1},$$

where the relation $\sim$ between two functions indicates that the ratio between them tends to 1, when $x \to \infty$.

The normal distribution, e.g., is not regularly varying, since the survival function decays exponentially, and then the limit (1) equals zero, for all $t \in \boldsymbol{R}$, $t > 1$.

For simplicity, we will use the following formal notation:

$$U_k(x) = \int_0^x t^k\, U(t)dt \quad , \qquad U_k^*(x) = \int_x^\infty t^k\, U(t)dt.$$

The following result says that if U is regularly varying then the latter funtions are assymptotically related to U as U were a power function of x, that is, for integration purposes, a regularly varying funtion in RV_ρ behaves assymptotically like x^ρ.

Theorem 1 (Karamata's theorem) *(a) If $\rho < -1$, then $U \in RV_\rho$ implies that $U_0^*(x)$ exists, $U_0^* \in RV_{\rho+1}$, and*

$$\lim_{x\to\infty} \frac{x\,U(x)}{U_0^*(x)} = -(\rho+1).$$

Conversely, if $U_0^*(x)$ exists and

$$\lim_{x\to\infty} \frac{x\,U(x)}{U_0^*(x)} = \lambda \in (0,\infty)$$

then $U \in RV_{-\lambda-1}$.

(b) If $\rho \geq -1$, then $U \in RV_\rho$ implies that $U_0 \in RV_{\rho+1}$ and

$$\lim_{x\to\infty} \frac{x\,U(x)}{U_0(x)} = \rho+1.$$

Conversely if U satisfies

$$\lim_{x\to\infty} \frac{x\,U(x)}{U_0(x)} = \lambda \in (0,\infty)$$

then $U \in RV_{\lambda-1}$.

The proof of this result can be found in Resnick [Res87]. In Feller [Fel66] an extension of Karamata's theorem to higher moments is presented.

Part (a) of the theorem can be applied to the truncated first moment of an absolutely continuous probability distribution $F(x)$, replacing $U(x)$ by the density $f(x)$.

The same result is obtained to a general distribution $F(x)$ defining

$$Z_k(x) = \int_0^x t^k\, dF(t) \quad , \qquad Z_k^*(x) = \int_x^\infty t^k\, dF(t).$$

With this notation, observe that $Z_0^* = 1 - F$ is the survival funtion of the distribution. In the present work, we consider F concentrated in $(0,\infty)$.

3 Characterization of paradoxical distributions

From Section 1, we have the following definition.

Definition 2 *A distribution f_Θ is said paradoxical if*

$$\boldsymbol{E}_\Theta[Y \mid X = x] \geq x,$$

where Θ, Y, X are as in Section 1.

Lemma 3 *For a given density, f_Θ, be paradoxical it is a sufficient and necessary condition*

$$4f_\Theta(x) \geq f_\Theta(x/2) \qquad \text{for all } x > 0. \tag{2}$$

Proof. With the notation of Section 1, we have

$$\begin{aligned}
\boldsymbol{E}(Y \mid X = x) &= \boldsymbol{P}(M = 0 \mid X = x)\, \boldsymbol{E}(Y \mid X = x, M = 0) \\
&\quad + \boldsymbol{P}(M = 1 \mid X = x)\, \boldsymbol{E}(Y \mid X = x, M = 1) \\
&= \frac{\boldsymbol{P}(M = 0) f(x \mid M = 0)}{f(x)} \frac{x}{2} + \frac{\boldsymbol{P}(M = 1) f(x \mid M = 1)}{f(x)} 2x \\
&= \frac{1}{2} \frac{1}{f(x)} f_{2\Theta}(x) \frac{x}{2} + \frac{1}{2} \frac{1}{f(x)} f_\Theta(x)\, 2x \\
&= \frac{1}{\frac{1}{2}\frac{1}{2} f_\Theta(x/2) + \frac{1}{2} f_\Theta(x)} \left(\frac{1}{2}\frac{1}{2} f_\Theta(x/2) \frac{x}{2} + \frac{1}{2} f_\Theta(x)\, 2x \right)
\end{aligned}$$

then

$$\boldsymbol{E}(Y \mid X = x) \geq x \iff \frac{f_\Theta(x/2)}{f_\Theta(x)} \leq 4,$$

independently of the observed value x.

q.e.d.

The following are examples of such densities:

1. $f(x) = c \log(1 + x)/(1 + x)^2$
2. $f(x) = c \log(\log(e + x))/(e + x)^2$
3. $f(x) = c/(1 + x^{3/2})$
4. $f(x) = c \exp(\log^\alpha(1 + x))/(1 + x)^2$, with $\alpha \in (0, 1)$

Next results characterize paradoxical distributions as regularly varying distributions for all x, but in a finite interval. The first proposition guarantees the construction of paradoxical distributions from a regularly varying distribution, redefining it, if needed, in a finite interval, in order to satisfy inequality (2).

Proposition 4 *Given $f \in RV_\rho$, if $\rho > -2$ then there exists $x_0 > 0$ such that f satisfies (2) for all $x > x_0$.*

Proof.

If $f \in RV_\rho$, then $U(x) := xf(x) \in RV_{\rho+1}$. By definition,

$$\lim_{x\to\infty} \frac{U(2x)}{U(x)} = 2^{\rho+1} > 2^{-1}$$

since that $\rho > -2$. Then, there exists $x_0 > 0$ such that

$$\frac{U(2x)}{U(x)} > 2^{-1} \qquad \text{for all } x > x_0$$

that is,

$$2xf(2x) > 2^{-1}xf(x) \Longrightarrow 4f(2x) > f(x) \qquad \text{for all } x > x_0$$

q.e.d.

Theorem 5 *Let f be a p.d.f. on $(0,\infty)$ satisfying (2). Then there exist functions $g_1, g_2 \in RV_\rho$, with $-2 < \rho < -1$, such that $g_1 \leq f \leq g_2$.*

Proof.

Let us suppose, at first, that there exists $x_0 > 0$ such that f satisfies the equality in (2), for all $x > x_0$.

Defining $U(x) := xf(x)$, observe that given $k > 0$, for all $x > x_0$, it holds

$$\frac{U(kx)}{U(x)} = \frac{U(2x)}{U(x)}\frac{U(kx)}{U(2kx)}\frac{U(k2x)}{U(2x)} = \frac{U(k2x)}{U(2x)} = \frac{U(k4x)}{U(4x)} = \ldots = \text{constant} =: c(k)$$

In particular

$$\lim_{x\to\infty} \frac{U(kx)}{U(x)} = c(k).$$

Then, for $x, y > 0$ and $t > x_0$,

$$\frac{U(xyt)}{U(t)} = \frac{U(xyt)}{U(xt)}\frac{U(xt)}{U(t)},$$

and letting $t \to \infty$,

$$c(xy) = c(x)c(y). \tag{3}$$

So, c satisfies Hamel equation and is of the form $c(k) = k^p$. Since that

$$c(k) = \frac{1}{k}, \tag{4}$$

for k integer power of 2, $k = 2^i$, $i \in \mathbb{N}$, for the condition on f, it follows that $c(k) = 1/k$.

From

$$U(kx) = k^{-1}U(x), \qquad x > x_0 \tag{5}$$

we conclude that $U \in RV_{-1}$ and then $f \in RV_{-2}$.

In the case $4f(2x) \geq f(x)$, we can find a lower bound to f, $g_1 \in RV_\rho$, with $\rho > -2$.

By other side, the survival function $\bar{F}(x) = 1 - F(x)$ varies dominatedly, since that

$$\frac{\bar{F}(2x)}{\bar{F}(x)} < 1, \text{ for all } x > 0.$$

By a known result, cf. Feller [Fel66], there exist constants A, p and x_0 such that

$$\frac{\bar{F}(tx)}{\bar{F}(x)} < A\,t^p, \text{ for all } x > x_0, t < 1.$$

Combining those inequalities, it is straightforward to show that there is $p \leq 0$ such that

$$\frac{\bar{F}(tx)}{\bar{F}(x)} < A\,t^p, \text{ for all } x > x_0, t > 0.$$

From this follows that $\bar{F}$ is dominated by a regularly varying function with index $p \leq 0$.

So, f is dominated by a regularly varying function, g_2, with index $p - 1 \leq -1$.

q.e.d.

Theorem 6 *Let X be random variable with density function f such that $supp(f) \subset (0, \infty)$. If f is paradoxical, then $\boldsymbol{E}X = \infty$.*

Proof.

By Theorem 5, if f is paradoxical, then there is $g \in RV_\rho$, with $\rho > -2$, such that $g(x) \leq f(x)$.

By Karamata's theorem, $g_0^*(x)$ is regularly varying with index $\rho + 1 > -1$. So,

$$f_0^*(x) \geq g_0^*(x) \sim x^{\rho+1} \quad, \text{ with } \rho + 1 > -1.$$

Then g_0^* is not integrable and

$$\boldsymbol{E}X = \int_0^\infty f_0^*(x)dx \geq \int_0^\infty g_0^*(x)dx = \infty.$$

q.e.d.

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