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Resumo
Um dos grandes desafios da Modelagem Volumétrica é a definição de um arcabouço teórico que suporte a manipulação e representação de objetos volumétricos. Diferente do que é visto em modelagem de superfícies, poucas ferramentas têm sido apresentadas para assegurar robustez topológica na modelagem de volumes. Nesse artigo oferecemos um ferramental matemático para modelagem volumétrica de malhas tetraedrais. A partir de uma caracterização topológica de tetraedros derivamos um conjunto de operadores de Morse, os quais possibilitam um controle topológico global durante a inserção e remoção de tetraedros. Modelagem geométrica e reconstrução volumétrica são áreas típicas de aplicação do ferramental aqui proposto. A eficácia de nossa abordagem é demonstrada em reconstrução de volumes a partir de imagens, onde ruídos são tratados de forma topológica, evitando a etapa de pré-processamento. A etapa de pós-processamento, usualmente empregada na detecção de buracos e cavidades no modelo, também pode ser evitada com controle topológico.

Abstract
One of the challenges of Volume Modelling is the definition of theoretical frameworks to support object manipulation and representation. Opposite to what is seen in the field of surface modelling, few satisfactory tools have been presented that ensure topological robustness to volumetric models. In this paper we offer one such mathematical framework for volumetric model definition based on tetrahedral meshes. From a complete tetrahedral topological characterization, a set of Morse Operators is given, which enable global tolopogical control during tetrahedral addition or removal. A number of applications can be envisaged, from geometrical modelling to volume reconstruction. We show the effectiveness of the tetrahedral characterization framework for volume reconstruction from images, showing that the method is capable of handling certain types of noise topologically without the need for a time consuming preprocessing step, or a post-processing step to detect cavities or holes.

1 Introduction

Computational studies of topological nature of objects have attracted researchers from several fields of science. The wide interest for such subject is motivated by the large number of computational applications that need to represent topology, such as image processing, solid and volume modelling, as well as visualization and computer graphics in general.

The particular area of Volume Modelling (VM) refers to the body of tools and techniques to handle scientific data in 3D space [9]. Arising from the area of scientific visualization, volume modelling represents the next step in the search for a framework involving definition of volumetric models to support advanced visualization applications. In VM not much is available to compound such a framework, mainly because most visualizations so far have been realized as a final step in the data understanding process, whilst new applications need to have control over the structures under manipulation in various other stages of the process (such as data representation and interaction), Interaction is central to data understanding and object manipulation and changes made during this stage must be tracked.

Three main classes of VM tasks originally involved in the visualization process can benefit greatly from the support of mathematical modelling. Those are Ddata organization, volume representation, and the visualization itself. Some mathematical tools for manipulation of objects in these categories exist, but they are sparse and not specific to VM problems.

In the Data Organization category, modelling strategies must allow flexibility and access to data fields in an effective manner. These data, generated as a result of measurements (ex. Medical Imaging) or simulations (ex. Computational Fluid Dynamics), have to be structured in a way to speed up access during mapping and visualization stages as well as further simulations. It must also reveal interesting structures in the original space. This area lacks tools for robust definition of data spaces and indexing techniques for fast access to data and features.

In VM, volume representation means the collection of strategies and tools to support definition of entities characterized by their interior. Although classic modelling and graphical data structures are adapted to representation of object interior, most of them treat internal half-spaces as uniform, whilst a considerable number of applications need to specify variation of properties inside objects (e.g. finite volume analysis). Mesh generation in 3D ([12, 16]) supports volumetric definition, but consistency, robustness and efficiency are reasons for concern. In visualization applications most techniques represent volume as union of voxels (volume units). For many applications, however, other types of volume decomposition (such as using tetrahedral cells) might prove useful, particularly for those volumes that are not regular in dimension or uniform in content. Therefore in the representation field of VM, techniques are necessary to specify various forms of object interior and their features.

In the visualization stage of volume models, mapping and rendering must be realized to translate interior representations and data into visual representations. Most techniques here are adapted to regular volume units and meshes or structured grids. Tools are necessary to develop visual counterparts of these techniques for data represented in other, less uniform, volumetric structures. For those techniques, proper manipulation tools are necessary to ease access to existing objects and to generate new ones (ex. as a result of the mapping process involved in visualization or the interaction with objects).

In this context, the task of volume representation bears importance in all other areas of VM. In the current stage of development, it relies in a handful of computational techniques with little mathematical support[9]. In particular, topological aspects of volume models have been largely neglected, regardless of bearing great weight in how VM will develop. Being able to analyze topology of volumetric meshes is important to enable the use of unstructured grids throughout the visualization and volume modelling processes. For instance, strong mathematical framework is necessary to process special features of meshes during simulation, data traversal, mapping, visualization and interaction. Holes, singularities, cavities, borders and other features affect all volume modelling tasks and must be handled with flexibility and robustness, as well as efficiency.

Topological approaches to volume modelling present various advantages over more conventional, analytical or geometrical approaches. Two of the main advantages are robustness of implementation and control over object features. Geometrical procedures are typically unstable and most mathematical representations lack control of topological constraints of the model, which are important in many applications.

In order to analyze the topological properties of objects computationally it is necessary make use of a description of such object and its topological relationships. For surface models, Euler operators have been widely employed to solve problems such as consistency and robustness. They allow reliable computational treatment of object models. Objects constructed using Euler operators lend themselves to mathematical manipulation, with its advantages. With the growth of VM, the area is now in need of similar techniques that offer the same advantages for objects defined volumetrically. This makes the cell decomposition issue central to volume modelling.

An specially important kind of volumetric cell decomposition is the tetrahedral decomposition, due to the variety of applications that employ this kind of representation: three-dimensional reconstruction [12], isosurface extraction [7], mesh generation [1] and many others. Although tetrahedral representations have been widely employed, not much has been done to understand their topological properties when compared with voxel representations.

This is the main motivation for the work presented here. In this paper we present a study of tetrahedron characterization, i.e., we analyze and classify the topological changes caused by the insertion and removal of tetrahedra in a tetrahedral object model. Besides giving the mathematical framework for such characterization, we offer a set of Morse operators for tetrahedral meshes that are capable of describing tetrahedral simplicial complexes in a similar way to surface Euler operators. They provide tools for construction and manipulation of tetrahedral meshes describing objects, in a form that allows full control of object topology.

Additionally, a data structure is presented to store meshes described with these tools, and a complexity analysis of the implementation of the operators is given. As an example of application we employ the operators developed here to a volume reconstruction problem. This gives rise to an algorithm capable of modelling the volume of objects from a set of their cross sections.

This text is organized as follows: section 2 presents a brief description of previous work on topological approaches applied to the various areas related to VM. Basic concepts necessary to understand the notation and nomenclature employed in the remaining of the text is given in section 3. Section 4 presents the topological tetrahedral characterization and the Morse operators associated with it. Section 5 introduces a short discussion about the implementation and complexity of the Morse operators. Section 6 presents the application in volumetric reconstruction and the main results, followed by our conclusions and future work in Section 7.

2 Related Works

As mentioned before, the emerging area of Volume Modelling intends to provide structure and framework for modelling volumetric information, with direct impact in three related issues, originally associated with computational scientific visualization: Data organization, Visual Display and Volume Representation. Some tools for manipulation exist, but all those areas are lacking proper mathematical framework [9], a fact that impair further development of data understanding techniques.

In order to treat some of the problems and needs of areas related to volume modelling, researchers have turned to computational topology tools [3, 5], such as the one presented here. The following text identifies needs and efforts involving computational topology as a tool in VM areas.

In the issue of data organization, tools have been employed to improve efficiency of the various steps of the visualization process. Virtually all visualization processes handling data large enough not to fit in memory could benefit from proper data indexing via geometrical or topological information of the original data set. For instance, in large data sets, a re-organization of the data using topological indexing may speed up the process of finding groups of cells of interest for a particular visualization procedure, such as checking for surface intersection[2].

Isosurface extraction is another subject that has benefited from the topological approach, in order to solve various problems arising from surface fitting of data. Those include shape ambiguity, formation of holes where none existed in the original data, excess of primitives in the surface model, and bad mesh organization leading to inefficient traversal. Several topological tools have been employed to solve problems such as topological consistency [18], mesh simplification [17], and speed up of the surface fitting process [10].

In the subject of volume representation, tetrahedral meshes are produced by cell decomposition processes, which generate a number of problems. The quality of the mesh itself and the type of connectivity amongst elements are of concern. Additionally, access to mesh elements (mesh representation) leads to difficulties during interaction, for instance achieving real time performance during rebuilding of the mesh submitted to operations such as cuts and slicing. In the cell decomposition context, computational topology has shown to be an essential mechanism to study various problems caused by mesh generation [4] and interaction [6].

The set of applications of topological representation vary widely and may carry importance to all areas of volume modelling. This fact has motivated the investigation of important questions related to the computation of homology [3, 11] and characterization of tetrahedra [14]. These issues offer immediate tools to build a VM framework. Homology and tetrahedra characterization offer construction and analysis tools to help control the structure of volumetric objects. This is very important when handling irregular objects or phenomena in an unstructured mesh setup.

In particular, the work by Saha and Majumder [14] shows a sound mathematical framework to analyze the local characterization of tetrahedra. Their characterization allows the measurement of the local topological changes caused by a deletion or insertion of a tetrahedron. In their work, the type of tetrahedron called a 'simple' tetrahedron is a type of mesh element whose elimination or insertion does not affect the mesh topology locally. For non-simple tetrahedra it measures local change in topology, without discussing what happens to the mesh globally.

Global topological changes are important for the detection (and solution) of volumetric characteristics of objects (such as elimination of noise - reflected as holes, and measurements of various quantities). Global control is also important for indexing purposes when those characteristics must be accessed frequently. In this work we propose a set of tools for global characterization of tetrahedra, that can be seen as an extension of some results obtained by Saha and Manjumdeer once it provides a framework to analyze tetrahedral meshes. Using our framework, it is possible to tell, once the mesh changes locally, what the effect will be on the whole mesh. Besides that characterization, we provide a set of operators to implement insertion and deletion of tetrahedra keeping the topological control of the meshes at all times. These are called 3D tetrahedral Morse operators, which provide a topologically consistent framework for many tasks of volume modelling. This is an initiative that offers a mathematical framework lacking in the volume modelling literature for tetrahedral meshes. It is a contribution to allow the creation and management of volume models with consistency and robustness, in much the same way as surfaces are handled today.

3 Basic Concepts

This section introduces the basic concepts and terminology used in the remaining of the text. Definitions and results presented in this and the following sections are restricted to three-dimensional Euclidean space. A reference to many of the concepts presented in this section is [11].

A *p-dimensional simplex* or p-simplex in $\mathbb{R}^3$, $0 \leq p \leq 3$, is the convex hull of $p+1$ geometrically independent points in $\mathbb{R}^3$ (a set $\{v_0, \ldots, v_p\} \subset \mathbb{R}^3$ is geometrically independent if the vectors $v_1 - v_0, \ldots, v_p - v_0$ are linearly independent). A 0-simplex is called a *vertex*, a 1-simplex is called an *edge*, a 2-simplex is called a *triangle* and a 3-simplex is called a *tetrahedron*. If $\sigma = \{v_0, \ldots, v_p\}$ is a p-simplex, $p = 0, 1, 2, 3$, then any j-simplex, $0 \leq j \leq p$, spanned by a subset S of $\{v_0, \ldots, v_p\}$ (that is, the convex hull of S) is called a *face* of σ. The faces of σ different from σ itself, are called the *proper faces* of σ; their union is called the *boundary* of σ and denoted $Bd\sigma$. The interior of a simplex σ is defined by the equation $Int\sigma = \sigma - Bd\sigma$.

A *simplicial complex* $\mathcal{K}$ is a finite collection of simplices satisfying:

1. If $\sigma \in \mathcal{K}$ then all faces of σ belong to $\mathcal{K}$.

2. If $\sigma, \tau \in \mathcal{K}$ then either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a common face of σ and τ.

The *dimension* of the simplicial complex $\mathcal{K} \neq \emptyset$, denoted $dim\mathcal{K}$, is the maximum of the dimensions of the simplices in $\mathcal{K}$. A subcollection of $\mathcal{K}$ that is itself a simplicial complex is called a (simplicial) subcomplex of $\mathcal{K}$. The subcomplex of $\mathcal{K}$ constituted by all simplices of $\mathcal{K}$ of dimension at most p is called the *p-skeleton* of $\mathcal{K}$. The subset $|K| = \cup_{\sigma \in \mathcal{K}} \sigma$ of $\mathbb{R}^3$ is called the *underlying space of* $\mathcal{K}$. The *degree* of a vertex $v \in \mathcal{K}$ is the number of edges in $\mathcal{K}$ that have v as a vertex.

A *regularized simplicial complex* $\mathcal{K}$ is a three-dimensional simplicial complex such that any p-simplex in $\mathcal{K}$ is contained in at least one 3-simplex of $\mathcal{K}$.

From now on, $\mathcal{K}$ is always a regularized simplicial complex, i.e., any p-simplex of $\mathcal{K}$ is contained in a tetrahedron of $\mathcal{K}$. A 2-simplex t (triangle) of $\mathcal{K}$ is an *interior* triangle if t is shared by two tetrahedra of $\mathcal{K}$, otherwise, t is a *boundary* triangle of $\mathcal{K}$. The vertices and edges contained in the boundary triangles are called *boundary vertices* and *boundary edges* of $\mathcal{K}$, respectively. The *boundary* of $\mathcal{K}$ is the subcomplex $\mathcal{S} \subset \mathcal{K}$ constituted by all boundary p-simplex, $p = 0, 1, 2$, of $\mathcal{K}$. Each connected component of the underling space $|\mathcal{S}|$ is called a *boundary component* of $\mathcal{K}$.

The *star* of a simplex $\sigma \in \mathcal{K}$, denoted $st\,\sigma$, is the union of all simplices in $\mathcal{K}$ having σ as a face. The *link* of σ is the union of all simplices of $\mathcal{K}$ lying in $st\,\sigma$ that are disjoint from σ. A simplex $\sigma \in \mathcal{K}$ is *singular* if its link is not homeomorphic to a sphere or to a half-sphere.

Let σ be a p-simplex. Two orderings of its vertex set are equivalent if they differ from each other by an even permutation. If $p > 0$, there are two equivalence classes (and if $p = 0$, just one). Each one of these classes is called an *orientation* of σ. An *oriented* *simplex* is a simplex σ together with an orientation of σ. The oriented simplex with vertices $v_0, \ldots, v_p$ whose orientation is the equivalence class of the particular ordering $(v_0, \ldots, v_p)$ shall be denoted by $[v_0, \ldots, v_p]$.

Let $\mathcal{K}$ be a simplicial complex. Let $C_p(\mathcal{K})$ be the free abelian group generated by the p-simplices of $\mathcal{K}$, each one of them with a fixed orientation. $C_p(\mathcal{K})$ is called the *group of (oriented) p-chains* of $\mathcal{K}$. An element of $C_p(\mathcal{K})$ is called a *p-chain*. Therefore, any p-chain can be written, in a unique way, as $\sum_{i=1}^{k} n_i\sigma_i$, where k is the number of p-simplices σ_i of $\mathcal{K}$ and, $\forall i \in \{1, \ldots, k\}$, n_i is an integer. If $p > 0, -\sigma_i$ represents σ_i with its opposite orientation. Note that the operation in $C_p(\mathcal{K})$ is given by $(\sum_{i=1}^{k} n_i\sigma_i) + (\sum_{i=1}^{k} m_i\sigma_i) = \sum_{i=1}^{k}(n_i + m_i)\sigma_i$, where $\forall i \in \{1, \ldots, k\}$, m_i and n_i are integers.

We define a homomorphism $\partial_p : C_p(\mathcal{K}) \to C_{p-1}(\mathcal{K})$, called the *boundary operator*, by $\partial_p[v_0, \ldots, v_p] = \sum_{i=0}^{p}(-1)^i[v_0, \ldots, \hat{v}_i, \ldots, v_p]$, where $[v_0, \ldots, v_p]$ is an oriented simplex of $\mathcal{K}$ and the symbol $\hat{v}_i$ means that the vertex v_i is to be deleted from the array. We complete the definition by defining ∂_p to be the trivial homomorphism if $C_p(\mathcal{K})$ or $C_{p-1}(\mathcal{K})$ is the trivial group. It can be shown that ∂_p is well defined, that $\partial_p(-\sigma) = -\partial_p(\sigma)$ for all $\sigma = [v_0, \ldots, v_p]$ and that, $\forall p \in \mathbb{Z}$, $\partial_p \circ \partial_{p+1} = 0$.

The kernel of the boundary operator $\partial_p : C_p(\mathcal{K}) \to C_{p-1}(\mathcal{K})$, denoted $Z_p(\mathcal{K})$, is called the group of *p-cycles* of $\mathcal{K}$. The image of $\partial_{p+1} : C_{p+1}(\mathcal{K}) \to C_p(\mathcal{K})$ is called the group of *p-boundaries* of $\mathcal{K}$ and is denoted $B_p(\mathcal{K})$. Since $\partial_p \circ \partial_{p+1} = 0$, $B_p(\mathcal{K}) \subset Z_p(\mathcal{K})$, therefore we have the quotient group $H_p(\mathcal{K}) = Z_p(\mathcal{K})/B_p(\mathcal{K})$ called the p^{th} *homology group* of $\mathcal{K}$.

The rank of $H_p(\mathcal{K})$, denoted by $\beta_p(\mathcal{K})$, represents the *number of connected components*, when p equals 0; the *number of holes*, when p equals 1; and the *number of cavities*, when p equals 2 in $\mathcal{K}$. The $\beta_p(\mathcal{K})$ are called Betti numbers of the homology group. If n_v, n_e, n_f, and n_t are the number of vertices, edges, triangles, and tetrahedra in $\mathcal{K}$, the *Euler characteristic* of $\mathcal{K}$ is defined by $\chi(\mathcal{K}) = n_v - n_e + n_f - n_t$. It can be shown that the Euler characteristic of $\mathcal{K}$ can also be computed as $\chi(\mathcal{K}) = \beta_0(\mathcal{K}) - \beta_1(\mathcal{K}) + \beta_2(\mathcal{K})$.

A sequence of abelian groups A and homomorphisms ϕ

$$\cdots \to A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \to \cdots$$

is said to be exact at A_i if $image\phi_{i-1} = kernel\phi_i$. It is said to be an exact sequence if it is exact in all of its groups.

We finish this section with an important result known as the Mayer-Vietoris Sequence [11].

`Mayer-Vietoris Sequence`: Let $\mathcal{K}$ be a simplicial complex; let $\mathcal{K}_0$ and $\mathcal{K}_1$ be subcomplexes such that $\mathcal{K} = \mathcal{K}_0 \cup \mathcal{K}_1$. Then there is an exact sequence

$$\cdots \to H_p(\mathcal{K}_0 \cap \mathcal{K}_1) \to H_p(\mathcal{K}_0) \oplus H_p(\mathcal{K}_1) \to H_p(\mathcal{K}) \to H_{p-1}(\mathcal{K}_0 \cap \mathcal{K}_1) \to \cdots$$

of abelian groups and homomorphisms.

In the next section the concepts presented here are employed to support the characterization of tetrahedra and definition of Morse Operators for tetrahedral meshes.

4 Topological Operators in the construction and characterization of tetrahedral meshes

In this section we investigate how the insertion or removal of a tetrahedron can change the homology of a simplicial complex $\mathcal{K}$. Based on this discussion we introduce the concept of tetrahedral Morse operators, which provide a robust mechanism to control the homology during the construction of a three-dimensional simplicial complex.

4.1 Characterization of Tetrahedra

Let τ be a new tetrahedron to be added to $\mathcal{K}$. The homological change caused by adding τ in $\mathcal{K}$ is related with the intersection $\tau \cap \mathcal{K}$. A tetrahedron τ is called *adjacent* to $\mathcal{K}$ if $\tau \notin \mathcal{K}$ and $\tau \cap \mathcal{K}$ is either empty or a subcomplex of $\mathcal{K}$ and of τ. We shall say that two simplices τ_1 and τ_2 adjacent to $\mathcal{K}$ are equivalent if $\tau_1 \cap \mathcal{K}$ and $\tau_2 \cap \mathcal{K}$ are homeomorphic, that is, if there is a bijection f between the set of vertices of $\tau_1 \cap \mathcal{K}$ and the set of vertices of $\tau_2 \cap \mathcal{K}$ such that the vertices $v_0, \ldots, v_n$ of $\tau_1 \cap \mathcal{K}$ span a simplex of $\tau_1 \cap \mathcal{K}$ if and only if $f(v_0), \ldots, f(v_n)$ span a simplex of $\tau_2 \cap \mathcal{K}$. Of course this is an equivalence relation.

Lemma 1: There are twenty eight equivalence classes for the set of adjacent tetrahedra to a simplex $\mathcal{K}$.

Proof. The proof of this lemma can be done by exhaustive enumeration of the cases. In fact, let $\mathcal{L}^{(p)}$ be the p-skeletons, $p = 0, 1, 2$, of the complex $\mathcal{L} = \mathcal{K} \cap \tau$. If $\mathcal{L} = \emptyset$ we have a equivalence class 1a). Suppose that $\mathcal{L}^{(0)} \neq \emptyset$ and $\mathcal{L}^{(p)} = \emptyset$ for $p = 1, 2$. In this case, the cardinality of $\mathcal{L}^{(0)}$ can be $1, 2, 3$ or 4 and we have four equivalence classes in this case (figure 1b)). If $\mathcal{L}^{(1)} \neq \emptyset$ and $\mathcal{L}^{(2)} = \emptyset$ we can have fourteen equivalence classes as indicated in figure 1c). Note that the tetrahedra in each dashed box are in classes where the p-skeletons of the intersections have the same cardinality but the intersections are not homeomorphic as simplicial complexes. In the last case, when $\mathcal{L}^{(2)} \neq \emptyset$, we have nine equivalence classes, as represented in figure 1d).□

The equivalence classes described in lemma 1 represent the different manners of inserting a tetrahedron in $\mathcal{K}$.

Regarding homological changes, adding tetrahedra from different classes can produce either the same homological change (if any) or homologically distinct complexes. It is also worth noting that the intersection between a tetrahedron τ and $\mathcal{K}$ is not sufficient to decide about the homology of the resulting simplicial complex $\mathcal{K} \cup \tau$. An example of this fact is shown in figure 2 where the addition of a tetrahedron with three common edges with $\mathcal{K}$ can either generate a cavity (figure 2a)) or close a hole (figure 2b)) of $\mathcal{K}$.

Although the intersection $\tau \cap \mathcal{K}$ does not characterize the homological changes in $\tau \cup \mathcal{K}$, it does determine all of its possibilities. Before showing that, we analize the intersection $\tau \cap \mathcal{K}$.

Lemma 2: Let τ be a tetrahedron adjacent to a simplicial complex $\mathcal{K}$. Let $\beta_0, \beta_1, \beta_2$ be

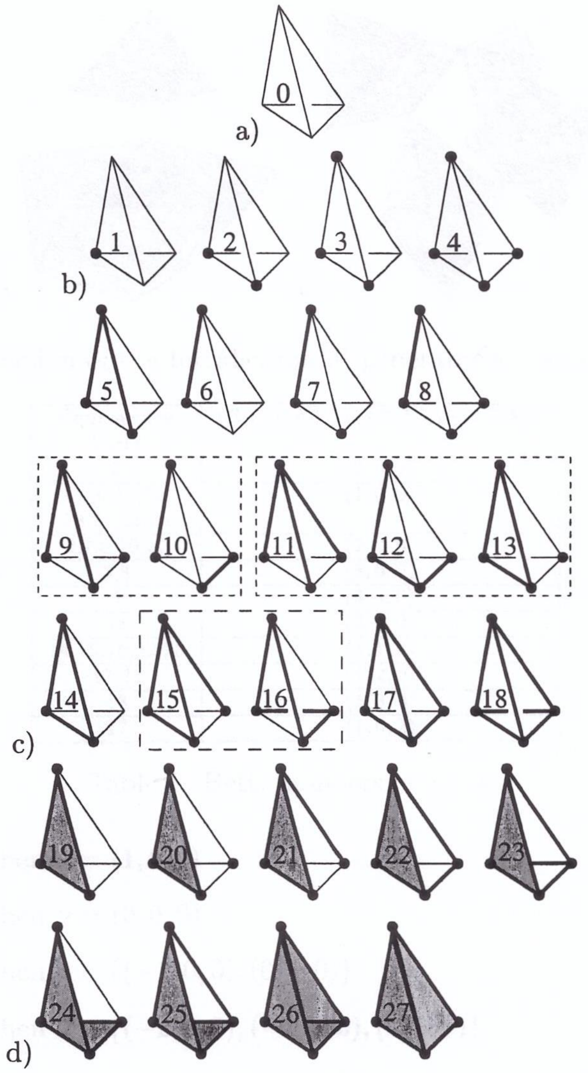


Figure 1: Equivalence classes of the tetrahedra adjacent to $\mathcal{K}$.

the ranks of the homology groups of $\tau \cap \mathcal{K}$ as defined in section 3. If $\beta = (\beta_0(\tau \cap \mathcal{K}), \beta_1(\tau \cap \mathcal{K}), \beta_2(\tau \cap \mathcal{K}))$ is the triple representing the ranks of the homology groups of $\tau \cap \mathcal{K}$ then $\beta \in \{(0,0,0),(1,0,0),(2,0,0),(3,0,0),(4,0,0),(2,1,0),(1,1,0),(1,2,0),(1,3,0),(1,0,2)\}$.

Proof. From figure 1 we derive table 1 that proofs the lema.□

Proposition 1: Let τ be a tetrahedron adjacent to a simplicial complex $\mathcal{K}$. If $\alpha = (\beta_0(\tau \cap \mathcal{K}), \beta_1(\tau \cap \mathcal{K}), \beta_2(\tau \cap \mathcal{K}))$ and $\nu = (\beta_0(\tau \cup \mathcal{K}) - \beta_0(\mathcal{K}), \beta_1(\tau \cup \mathcal{K}) - \beta_1(\mathcal{K}), \beta_2(\tau \cup \mathcal{K}) - \beta_2(\mathcal{K}))$ then α is related with ν as follows:

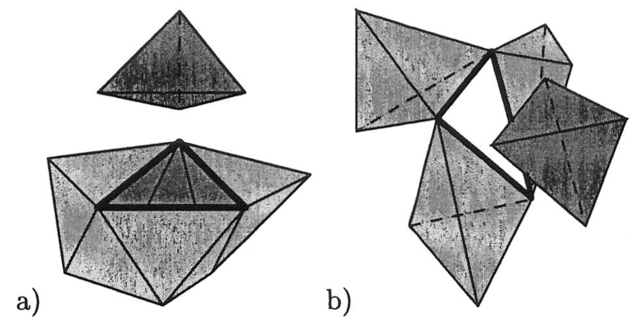


Figure 2: The insertion of the tetrahedron a) generates a cavity; b) closes a hole.

Equivalence Class	$(\beta_0(\tau \cap \mathcal{K}), \beta_1(\tau \cap \mathcal{K}), \beta_2(\tau \cap \mathcal{K}))$
0	$(0,0,0)$
1,5,6,11,12, 19,21,24,26	$(1,0,0)$
2,7,9,10,20	$(2,0,0)$
3,8	$(3,0,0)$
4	$(4,0,0)$
13	$(2,1,0)$
14,15,16,22,25	$(1,1,0)$
17,23	$(1,2,0)$
18	$(1,3,0)$
27	$(1,0,2)$

Table 1: Betti numbers of $\tau \cap \mathcal{K}$.

1. if $\alpha = (0,0,0)$ then $\nu = (1,0,0)$
2. if $\alpha = (1,0,0)$ then $\nu = (0,0,0)$
3. if $\alpha = (2,0,0)$ then $\nu \in \{(-1,0,0),(0,1,0)\}$
4. if $\alpha = (3,0,0)$ then $\nu \in \{(-2,0,0),(-1,1,0),(0,2,0)\}$
5. if $\alpha = (4,0,0)$ then $\nu \in \{(-3,0,0),(-2,1,0),(-1,2,0),(0,3,0)\}$
6. if $\alpha = (2,1,0)$ then $\nu \in \{(-1,-1,0),(-1,0,1),(0,0,0),(0,1,1)\}$
7. if $\alpha = (1,1,0)$ then $\nu \in \{(0,-1,0),(0,0,1)\}$
8. if $\alpha = (1,2,0)$ then $\nu \in \{(0,-2,0),(0,-1,1),(0,0,2)\}$
9. if $\alpha = (1,3,0)$ then $\nu \in \{(0,-3,0),(0,-2,1),(0,-1,2),(0,0,3)\}$
10. if $\alpha = (1,0,2)$ then $\nu = (0,0,-1)$

Proof. Let us consider the Mayer-Vietoris sequence:

$$0 \to H_2(\mathcal{K} \cap \tau) \stackrel{\phi_2}{\to} H_2(\mathcal{K}) \oplus H_2(\tau) \stackrel{\psi_2}{\to} H_2(\mathcal{K} \cup \tau) \stackrel{\Delta_2}{\to}$$

$$\xrightarrow{\Delta_2} H_1(\mathcal{K}\cap\tau) \xrightarrow{\phi_1} H_1(\mathcal{K})\oplus H_1(\tau) \xrightarrow{\psi_1} H_1(\mathcal{K}\cup\tau) \xrightarrow{\Delta_1}$$

$$\xrightarrow{\Delta_1} H_0(\mathcal{K}\cap\tau) \xrightarrow{\phi_0} H_0(\mathcal{K})\oplus H_0(\tau) \xrightarrow{\psi_0} H_0(\mathcal{K}\cup\tau) \xrightarrow{\Delta_0} 0$$

Let us prove item 6, that is, suppose that $(\beta_0(\tau\cap\mathcal{K}),\beta_1(\tau\cap\mathcal{K}),\beta_2(\tau\cap\mathcal{K}))=(2,1,0)$. In this particular case, the sequence above can be rewritten as follows:

$$0\to 0 \xrightarrow{\phi_2} H_2(\mathcal{K}) \xrightarrow{\psi_2} H_2(\mathcal{K}\cup\tau) \xrightarrow{\Delta_2}$$

$$\xrightarrow{\Delta_2} \mathbb{Z} \xrightarrow{\phi_1} H_1(\mathcal{K}) \xrightarrow{\psi_1} H_1(\mathcal{K}\cup\tau) \xrightarrow{\Delta_1} \tag{1}$$

$$\xrightarrow{\Delta_1} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\phi_0} H_0(\mathcal{K})\oplus\mathbb{Z} \xrightarrow{\psi_0} H_0(\mathcal{K}\cup\tau) \xrightarrow{\Delta_0} 0$$

We have two cases to analyze, either: (a) τ intersects two distinct connected components of $\mathcal{K}$ or (b) τ intersects only one connected component of $\mathcal{K}$.
Case (a): In this case $\beta_0(\mathcal{K}\cup\tau)=\beta_0(\mathcal{K})-1$ and we have exact sequences

$$0 \xrightarrow{\phi_2} H_2(\mathcal{K}) \xrightarrow{\psi_2} H_2(\mathcal{K}\cup\tau) \xrightarrow{\Delta_2} im\,\Delta_2 \xrightarrow{\phi_1} 0$$

and

$$H_2(\mathcal{K}\cup\tau) \xrightarrow{\Delta_2} \mathbb{Z} \xrightarrow{\phi_1} H_1(K) \xrightarrow{\psi_1} H_1(\mathcal{K}\cup\tau) \xrightarrow{\Delta_1} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\phi_0} H_0(\mathcal{K})\oplus\mathbb{Z}.$$

Either $\Delta_2=0$ or $im\,\Delta_2=n\mathbb{Z}$ with $n\neq 0$. If $\Delta_2=0$ we have

$$0 \xrightarrow{\phi_2} H_2(\mathcal{K}) \xrightarrow{\psi_2} H_2(\mathcal{K}\cup\tau) \xrightarrow{\Delta_2} 0 \tag{2}$$

and

$$0\to\mathbb{Z} \xrightarrow{\phi_1} H_1(K) \xrightarrow{\psi_1} H_1(\mathcal{K}\cup\tau) \xrightarrow{\Delta_1} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\phi_0} H_0(\mathcal{K})\oplus\mathbb{Z}. \tag{3}$$

From (2) it follows that $\beta_2(\mathcal{K}\cup\tau)=\beta_2(\mathcal{K})$. Additionally, since the generators of each copy of $\mathbb{Z}$ in the domain of ϕ_0 are going to different components in $\mathcal{K}$, ϕ_0 is injective, so $im\,\Delta_1=ker\,\phi_0=\{0\}$ and (3) becomes

$$0\to\mathbb{Z} \xrightarrow{\phi_1} H_1(K) \xrightarrow{\psi_1} H_1(\mathcal{K}\cup\tau)\to 0,$$

therefore $\beta_1(\mathcal{K}\cup\tau)=\beta_1(\mathcal{K})-1$. On the other hand, if $im\,\Delta_2=n\mathbb{Z}$ with $n\neq 0$, then we have

$$0 \xrightarrow{\phi_2} H_2(\mathcal{K}) \xrightarrow{\psi_2} H_2(\mathcal{K}\cup\tau) \xrightarrow{\Delta_2} n\mathbb{Z}\to 0 \tag{4}$$

and

$$H_2(\mathcal{K}\cup\tau) \xrightarrow{\Delta_2} \mathbb{Z} \xrightarrow{\phi_1} H_1(K) \xrightarrow{\psi_1} H_1(\mathcal{K}\cup\tau) \xrightarrow{\Delta_1} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\phi_0} H_0(\mathcal{K})\oplus\mathbb{Z}. \tag{5}$$

From (4) it follows that $\beta_2(\mathcal{K}\cup\tau)=\beta_2(\mathcal{K})+1$. Additionally, since ϕ_0 is injective, (5) becomes

$$H_2(\mathcal{K}\cup\tau) \xrightarrow{\Delta_2} \mathbb{Z} \xrightarrow{\phi_1} H_1(K) \xrightarrow{\psi_1} H_1(\mathcal{K}\cup\tau)\to 0$$

therefore $\beta_1(\mathcal{K}\cup\tau)=\beta_1(\mathcal{K})-rank(ker\,\psi_1)=\beta_1(\mathcal{K})-rank(im\,\phi_1)$, but $rank(ker\,\phi_1)=rank(im\,\Delta_2)=1$, so $rank(im\,\phi_1)=0$, therefore $\beta_1(\mathcal{K}\cup\tau)=\beta_1(\mathcal{K})$.
Case (b): In this case $\beta_0(\mathcal{K}\cup\tau)=\beta_0(\mathcal{K})$. The computation of $H_2(\mathcal{K}\cup\tau)$ is exactly the same as in case (1), therefore if $\Delta_2=0$ then $\beta_2(\mathcal{K}\cup\tau)=\beta_2(\mathcal{K})$ and if $\Delta_2\neq 0$ then $\beta_2(\mathcal{K}\cup\tau)=\beta_2(\mathcal{K})+1$. Regarding $H_1(\mathcal{K}\cup\tau)$ we have that ϕ_0 is no longer injective. Instead, $ker\,\phi_0=\mathbb{Z}$ now. If $\Delta_2=0$, we have the exact sequence

$$0\to\mathbb{Z}\overset{\phi_1}{\to}H_1(K)\overset{\psi_1}{\to}H_1(\mathcal{K}\cup\tau)\overset{\Delta_1}{\to}\mathbb{Z}\to 0$$

therefore $\beta_1(\mathcal{K}\cup\tau)-rank(im\,\psi_1)=1$, but $rank(im\,\psi_1)=\beta_1(\mathcal{K})-rank(ker\,\psi_1)$ and $rank(ker\,\psi_1)=rank(im\,\phi_1)=1$, hence $\beta_1(\mathcal{K}\cup\tau)=\beta_1(\mathcal{K})$. Now, if $im\,\Delta_2=n\mathbb{Z}$ with $n\neq 0$, we have

$$H_2(\mathcal{K}\cup\tau)\overset{\Delta_2}{\to}\mathbb{Z}\overset{\phi_1}{\to}H_1(K)\overset{\psi_1}{\to}H_1(\mathcal{K}\cup\tau)\overset{\Delta_1}{\to}\mathbb{Z}\oplus\mathbb{Z}\overset{\phi_0}{\to}H_0(\mathcal{K})\oplus\mathbb{Z}$$

therefore $\beta_1(\mathcal{K}\cup\tau)-rank(im\,\psi_1)=1=rank(im\,\Delta_1)$ but $rank(im\,\psi_1)=\beta_1(\mathcal{K})-rank(ker\,\psi_1)$ and $rank(ker\,\psi_1)=rank(im\,\phi_1)=0$, hence $\beta_1(\mathcal{K}\cup\tau)=\beta_1(\mathcal{K})+1$.

The same kind of reasoning that we employed to prove item 6, was used to prove the remaining items.□

An important result that can be extracted from proposition 1 is:

Corollary 1: There are twenty five different forms of changing the homology of a simplicial complex $\mathcal{K}$ by attaching tetrahedra. This is straightforward from the cases shown in Proposition 1.

An immediate consequence of proposition 1 is that we can group the equivalence classes in figure 1 in accordance with the possible homological changes that their tetrahedra can produce in the complex . We arrange the equivalence classes in six categories named: 0-handle, H_0-handle, H_2-handle, H_0H_1-handle, H_1H_2-handle, and $H_0H_1H_2$-handle. Following the numbering of figure 1, each category is constituted as follows:

1. 0-handle= $\{1,5,6,11,12,19,21,24,26\}$
2. H_0-handle= $\{0\}$
3. H_2-handle= $\{27\}$
4. H_0H_1-handle= $\{2,3,4,7,8,9,10,20\}$
5. H_1H_2-handle= $\{14,15,16,17,18,22,23,25\}$
6. $H_0H_1H_2$-handle= $\{13\}$

If τ is a tetrahedron contained in a class of 0-handles then τ is called a 0-handle tetrahedron, if τ is contained in a class of H_0-handles it is called H_0-handle tetrahedron and so on.

Proposition 1 above offers a way of quantifying and describing the nature of the homological change caused by adding a handle into $\mathcal{K}$. A consequence of this fact is the characterization of *simple tetrahedra* according to this theory, i.e., tetrahedra whose insertion (removal) does not change the homologies of a simplicial complex. They are our 0-handles. Therefore, in order to decide if a tetrahedron is simple it is sufficient to analyze its intersection with the simplicial complex to which it will be added. It is worth noting that the characterization of simple tetrahedra given by proposition 1 is more straightforward than that presented in the work by Saha and Majumder [14], once it depicts all possible cases of this kind of tetrahedron.

Proposition 1 also shows another important fact: a $H_0H_1H_2$-handle can be added into a simplicial complex without altering the ranks of the homology groups, i.e., a $H_0H_1H_2$-handle can replace a hole for another one, not modifying the rank of H_1. In that respect, a $H_0H_1H_2$-handle could be mistaken for a simple tetrahedron. However this handles does change the homology, although to one with the same ranks.

Proposition 1is also the basis for table 2 below, where the first column indicates the type of handle (in accordance with figure 1), the second column displays the group in which the handle is contained and the third column shows the possible changes that such handle can produce in the number of connected components (ν_0), holes (ν_1) and cavities (ν_2) of a simplicial complex $\mathcal{K}$.

Table 2 presents all possible changes that the addition of a new tetrahedron can produce in a simplicial complex $\mathcal{K}$. As far as we know, this is so far the most complete characterization of tetrahedra described in the literature. Under this theory, a set of construction operators (and their inverses) (seen in Table 2), which are discussed in the next section.

4.2 Tetrahedral Morse Operators

From table 2 above we can define a set of topological operators we shall call *tetrahedral Morse operators.* As we are going to show, these Morse operators make it possible to add new tetrahedra into a simplicial complex while keeping control of the number of connected components, of holes, and of cavities. Morse operators enable a more robust handling of the incidence and adjacency relationship in a tetrahedral mesh, considering that all the elements affected by the addition of a new tetrahedron are completely specified for each operator.

Tetrahedral Morse operators are defined in a straightforward manner based on table 2. The fifty two operators are grouped into seven different sets according to the homological change they introduce in the simplicial complex.

The seven sets with their respective operators are:

TMO_0 ={MV, ME, M2E, M3E, MF, M2F, M3F, MEF}

TMO_{H_0} ={MC, M2VKC, M3VK2C, M4VK3C, MVEKC, M2VEK2C, MV2EKC, M2EKC, MVFKC}

TMO_{H_1} ={M2VH, M3V2H, M4V3H, MVEH, M2VE2H, MV2EH, M2EH, M3EKH, M4EKH, M5EK2H, M6EK3H, MVFH, M2EFKH, M3EFK2H, ME2FKH, MV3EHKH}

Handle	Equivalence Class	(ν_0, ν_1, ν_2)	Operator
0	H_0-handle	$(1,0,0)$	MC
1	0-handle	$(0,0,0)$	MV
2	H_0H_1-handle	$(-1,0,0)$	M2VKC
		$(0,1,0)$	M2VH
3	H_0H_1-handle	$(-2,0,0)$	M3VK2C
		$(-1,1,0)$	M3VHKC
		$(0,2,0)$	M3V2H
4	H_0H_1-handle	$(-3,0,0)$	M4VK3C
		$(-2,1,0)$	M4VHK2C
		$(-1,2,0)$	M4V2HKC
		$(0,3,0)$	M4V3H
5	0-handle	$(0,0,0)$	M2E
6	0-handle	$(0,0,0)$	ME
7	H_0H_1-handle	$(-1,0,0)$	MVEKC
		$(0,1,0)$	MVEH
8	H_0H_1-handle	$(-2,0,0)$	M2VEK2C
		$(-1,1,0)$	M2VEHKC
		$(0,2,0)$	M2VE2H
9	H_0H_1-handle	$(-1,0,0)$	MV2EKC
		$(0,1,0)$	MV2EH
10	H_0H_1-handle	$(-1,0,0)$	M2EKC
		$(0,1,0)$	M2EH
11	0-handle	$(0,0,0)$	M3E
12	0-handle	$(0,0,0)$	M3E
13	$H_0H_1H_2$-handle	$(-1,-1,0)$	MV3EKCH
		$(-1,0,1)$	MV3EPKC
		$(0,0,0)$	MV3EHKH
		$(0,1,1)$	MV3EHP

Handle	Equivalence Class	(ν_0, ν_1, ν_2)	Operator
14	H_1H_2-handle	$(0,-1,0)$	M3EKH
		$(0,0,1)$	M3EP
15	H_1H_2-handle	$(0,-1,0)$	M4EKH
		$(0,0,1)$	M4EP
16	H_1H_2-handle	$(0,-1,0)$	M4EKH
		$(0,0,1)$	M4EP
17	H_1H_2-handle	$(0,-2,0)$	M5EK2H
		$(0,-1,1)$	M5EPKH
		$(0,0,2)$	M5E2P
18	H_1H_2-handle	$(0,-3,0)$	M6EK3H
		$(0,-2,1)$	M6EPK2H
		$(0,-1,2)$	M6E2PKH
		$(0,0,3)$	M6E3P
19	0-handle	$(0,0,0)$	MF
20	H_0H_1-handle	$(-1,0,0)$	MVFKC
		$(0,1,0)$	MVFH
21	0-handle	$(0,0,0)$	MEF
22	H_1H_2-handle	$(0,-1,0)$	M2EFKH
		$(0,0,1)$	M2EFP
23	H_1H_2-handle	$(0,-2,0)$	M3EFK2H
		$(0,-1,1)$	M3EFPKH
		$(0,0,2)$	M3EF2P
24	0-handle	$(0,0,0)$	M2F
25	H_1H_2-handle	$(0,-1,0)$	ME2FKH
		$(0,0,1)$	ME2FP
26	0-handle	$(0,0,0)$	M3F
27	H_2-handle	$(0,0,-1)$	M4FKP

Table 2: Handles (operators) and their relation with the homology of the simplicial complex.

TMO_{H_2} ={M3EP, M4EP, M5E2P, M6E3P, M2EFP, M3EF2P, ME2FP, M4FKP}

$\text{TMO}_{H_0H_1}$ ={M3VHKC, M4VHK2C, M4V2HKC, M2VEHKC, MV3EKCH}

$\text{TMO}_{H_0H_2}$ ={MV3EPKC}

$\text{TMO}_{H_1H_2}$ ={MV3EHP, M5EPKH, M6EPK2H, M6E2PKH, M3EFPKH}

The types of homological changes caused by these classes of operators are identified by the handles expressed in their names, that is :

TMO_0 : Cause no homological changes in the complex.

TMO_{H_0} : Causes change in the H_0 number, that is, operators in this class change the number of connected components.

TMO_{H_1} : Causes change in the number of holes.

TMO_{H_2} : Causes change in the number of cavities.

$\text{TMO}_{H_0H_1}$: Causes change in the number of connected components and in the number of holes.

$\text{TMO}_{H_0H_2}$: Causes change in the number of connected components and in the number of cavities.

$\text{TMO}_{H_1H_2}$: Causes change in the number of holes and cavities.

A number of symbols are used to name the operators, and the convention for naming any particular operator provides a summary description of its action over the simplicial complex.

The seven symbols used for naming the operators are: M - make; K - kill; H - hole; P - cavity (pocket); C - component; V - vertex; E - edge; F - face. Numbers are utilized to describe the number of times that an action is executed in each entity (the number 1 is omitted in order keep the notation clean). An operator is identified by a sequence of symbols representing the actions on elements necessary to complete an operation of insertion or deletion of a tetrahedron.

During insertion or deletion of a tetrahedron, sometimes the action make (M in the sequence) refers to identification of the elements following M in the sequence that already exist in the complex. This happens when the elements that follow are vertices, edges and faces (that is, when V,E or F follows the letter M). Other times M means that the elements in the sequence are added or caused in the simplex. This occurs for holes, cavities and components (that is, when the symbols H, P and C follow the letter M). For instance, lets suppose a tetrahedron is to be inserted in an existing simplicial complex, by gluing it to a single vertex of another tetrahedron already there. In this case, this addition is done through the identification of the vertex already there, to which the new tetrahedron must be glued (MV - make 1 vertex.) Figure 3 shows the action of the MV operator.

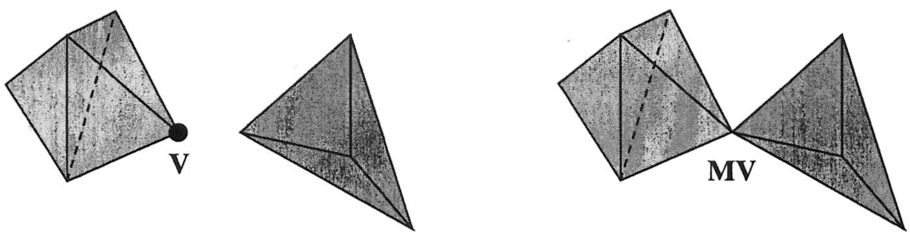


Figure 3: Tetrahedron insertion through MV operator.

As further examples of the action of an operator, lets take the symbol sequence MV2EH naming an operator in TMO_{H_1}. Its name indicates that, to add a tetrahedron, this operator makes (i.e., identifies) one vertex and two edges already in the complex (totalling three vertices), and adds a hole in the process(see Figure 4); the operator MV3EPKC in $\text{TMO}_{H_0H_2}$ indicates that, to adds a tetrahedron, it makes (i.e., identifies) one vertex and three edges (totalling four vertices), adds one cavity, and kills a component. Figure 2 given before shows the execution of tetrahedra insertion through the operators M3EP (figure 2a)) and M3EKH (figure 2b)).

Note that the presence of symbols V and E means that the operator containing such symbols introduces singular vertices and edges into the simplicial complex. From this observation follows the next proposition.

Proposition 2: It is not possible to generate a simplicial complex $\mathcal{K}$ with holes or cavities,

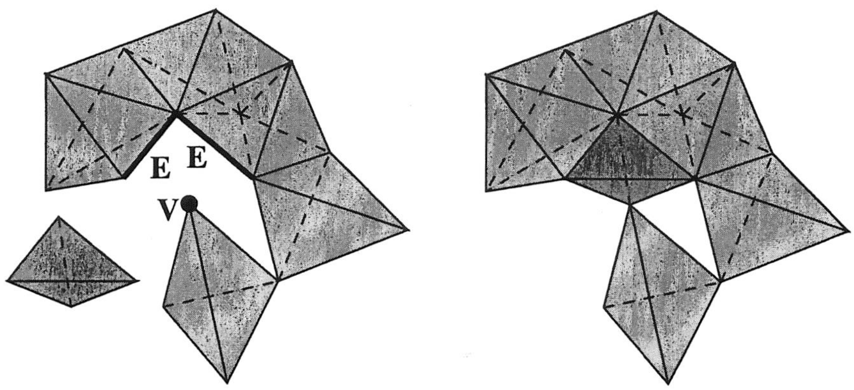


Figure 4: Tetrahedron insertion through MV2EH operator.

by gluing tetrahedra, without adding singular vertices or edges (or both) to $\mathcal{K}$ during the process.

Proof. In order to generate a hole in $\mathcal{K}$ it is necessary to add a new tetrahedron into $\mathcal{K}$ through an operator from TMO_{H_1}, $\mathrm{TMO}_{H_0H_1}$, or $\mathrm{TMO}_{H_1H_2}$. For a cavity, it is necessary to make use of a operator from TMO_{H_2}, $\mathrm{TMO}_{H_0H_2}$, or $\mathrm{TMO}_{H_1H_2}$. Since all operators of these classes introduce singular vertices or edges the proposition is proved.□

Proposition 2 above has an important practical meaning regarding topological data structures, namely, that any data structure dedicated to represent regularized simplicial complexes generated by gluing tetrahedra must be able to handle singularities in vertices and edges in order to be effective.

The above operators are called *direct operators*, as they may be used to add tetrahedra to an existing simplical complex. Operators for removing tetrahedra are described in terms of the *inverse operators*, which may be obtained by replacing letters M with K (and vice-versa) in the above description. The inverse operators can also be grouped into the same seven sets according to the homological change they introduce. The homological changes produced are also in the same classes of the direct operatos.

We finish this section with an interesting theoretical result about a minimal number of operators to build a tetrahedral mesh.

Proposition 3: To build any simplicial complex by gluing tetrahedra it is necessary the use of at least six operators (and their inverses).

Proof. Let W be the submodule in $\mathbb{Z}^7$ whose elements (v, e, f, t, c, h, p) satisfy the equation $v - e + f - t - c + h - p = 0$. Note that any simplicial complex $\mathcal{K}$ may be represented as a vector in W and each Morse operator can also be described as a vector in W. For example, the operator MVFH may be represented by vector (0,3,3,1,0,1,0), meaning that it introduces zero new vertices, three new edges, three new faces, one new tetrahedron, zero components, one hole, and zero cavities into $\mathcal{K}$.

Let $x_1 = (1,3,3,1,0,0,0)$, $x_2 = (0,1,2,1,0,0,0)$, $x_3 = (0,0,1,1,0,0,0)$, $x_4 = (0,3,3,1,0,1,0)$,

$x_5 = (1,3,4,1,0,0,1)$, and $x_6 = (4,6,4,1,1,0,0)$ be the vector representations of operators MF, M2F, M3F, MVFH, M3EP, and MC, respectively. A straightforward computation shows that $x_i, i = 1, \ldots, 6$ are linearly independent and that they are a basis for W, thus generating any vector contained in W. As any simplicial complex $\mathcal{K}$ is represented by a vector in W, theoretically these six operators constitute a minimal set of operators to generate $\mathcal{K}$.□

Next section presents some important issues of implementation os TMOs.

5 Implementation and Computational Complexity

In this section we are concerned with the computational aspects of the Tetrahedral Morse Operators.

One issue that bears importance to all areas of Volume Modelling is that of Data Structure. In the case of our computational representation, one particular data structure, called SHF (Singular Half-Face) is under development. Some of its requirements are still under inspection and evolution so that its use during various stages of visualization processes can be accomplished. However, in the current stage of development, SHF is capable of storing and indexing meshes built via TMOs. Its description and discussion are given in the next section.

The following section discusses other computational issues such as complexity and storage requirements.

5.1 Data Structure

The Data Structure named *Singular Handle-Face* (SHF)is capable of representing regularized simplicial complexes with singular vertices and edges.

A SHF data structure is organized in terms of seven explicitly represented entities (or nodes) which are:

- **Solid** - Representing each connected component of the regularized simplicial complex.
- **Cells** - Representing the tetrahedra .
- **Vertices** - Representing the vertices.
- **Half-Faces** - Representing the face contained in a cell.
- **Half-Edges** - Representing an edge contained in a Half-Face.
- **Boundary_Components** - Representing each boundary components.
- **Star_Vertex** - Representing each edge incident on a vertex.

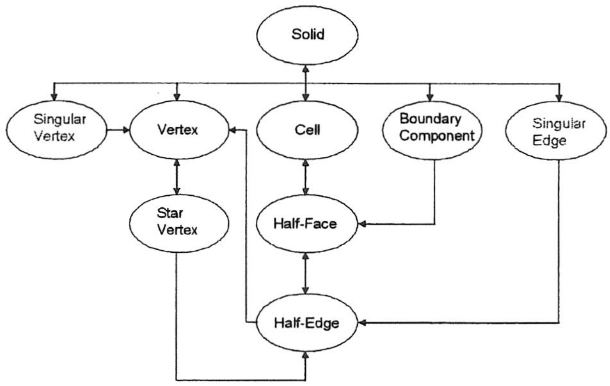


Figure 5: Hierarchical organization of SHF

Figure 5 shows the hierarchical relationship among the nodes of SHF.

Given a regularized simplicial complex $\mathcal{K}$, the solid, cells, and vertices nodes of SHF are linked lists representing each connected component, the tetrahedra, and the vertices of $\mathcal{K}$ respectively.

The nodes half-faces and half-edges are circular linked lists that store the simplices of dimension two and one contained in each tetrahedron. In other words, each tetrahedron is a cell containing its lists of half-faces and half-edges. The adjacency relationship among simplices is also stored in the nodes half-faces and half-edges. For example, each half-face "knows" the adjacent half-face in the neighbor tetrahedron, as shown in figure 6a), and each half-edge knows the adjacent half-edges in its tetrahedron and in the neighbor tetrahedron, as in figure 6b). If a half-face lies on the boundary of $\mathcal{K}$, its half-edges are employed to give access to the adjacency relationships in the boundary surface, as in a B-Rep representation. Figure 6c) presents this schema.

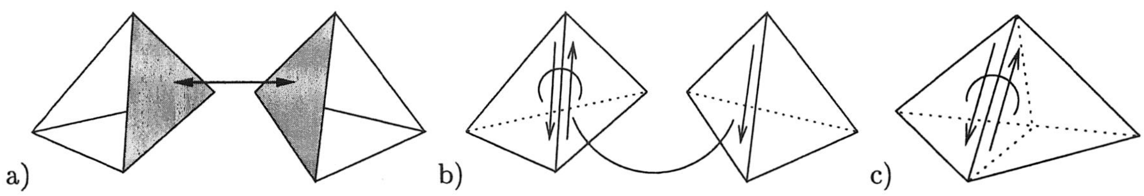


Figure 6: Adjacency relationship through Half-Face and Half-Edge nodes.

The Boundary Components node is a linked list containing a half-face of each boundary component. As the Tetrahedral Morse Operators act on the boundary of $\mathcal{K}$, it is important that SHF offers an efficient mechanism to give access to boundary components.

Each vertex in the SHF contains a Star Vertex node which is a linked list containing a half-edge of each edge incident to the vertex. The Singular Vertex node could be replaced by an algorithm to find the edges but, in order to improve the performance, we represent them explicitly. Figure 5 represents the structure, its nodes and references.

The realization of each TMO on a mesh has specific effects on the data structure that were implemented as actual operators on the data structure, one for each named operator shown before. This data structure was implemented using Object-oriented concepts and its use is open for general users. The data structure and its application shown in section 6 were also implemented in the context of an open-source visualization system, the VTK (*The Visualization Toolkit*) [15], and are available on the internet (*http://lcad.icmc.sp.br/ powervis*).

5.2 Computational Complexity

Suppose that a new tetrahedron is to be added in a regularized simplicial complex $\mathcal{K}$ represented by SHF data structure. Also suppose that the position where the new tetrahedron must be added has already been found. As the computational cost to find the correct position of the new tetrahedron is dependent on the application, we do not take it into account. For example, if the tetrahedra of $\mathcal{K}$ are given by vertex coordinates, for each new tetrahedron it is necessary perform a search in the boundary surfaces to find the position where to insert it. However, in some situations, as in the example given in the next section, the position of the tetrahedron is given.

The position of the tetrahedron defines the type of handle that it needs to be added. From the handle we decide which operator must be employed through local and global searches (note from table 1 that a same handle can give rise to different operators). As the SHF data structure maintains the local adjacency and connected components explicitly represented, only local searches are necessary to decide the operators from the 0-handle, H_0-handle, and H_0H_1-handle.

In order to decide the operators from H_2-handle, H_1H_2-handle, and $H_0H_1H_2$-handle it is necessary make use of global searches, i.e., traverse the boundary surfaces and verify if new cavities are being created. That way, the cost of such operators is proportional to the number of boundary faces.

6 Volumetric Reconstruction

Aiming at illustrating the applicability of tetrahedra characterization, we present in this section an application of TMO in volumetric reconstruction from a sequence of images.

The reconstruction process starts from bi-dimensional images taken from measurement devices (such as MRI) or other scalar data sets representing consecutive data planes in 3D space. The sequence of images compound a regular 3D grid of cuboid cells, such as illustrated in figure 7. Each pixel value is stored at a cell vertex. Each cuboid is interpreted as a set o 6 tetrahedra, adjusted to fill out this volume unit. Tetrahedral adjustment inside a volume unit is illustrated in figure 8.

Given a range of values indicative of objects of interest inside the regular grid, the reconstruction algorithm checks for the presence of these values at each vertex. Tetrahedra inside the volume unit whose vertices checked 'yes' for the presence of values of interest

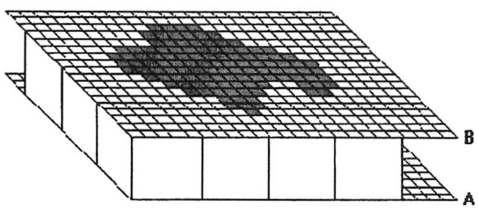


Figure 7: Volume cell organization from images.

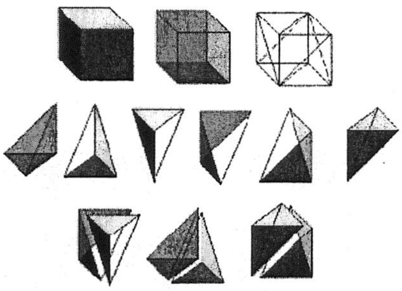


Figure 8: Tetrahedra positioning inside a cell.

are candidates to be added to the object model. Two possible approaches were taken for the selection of tetrahedra to be included. In one of them all tetrahedra with one 'yes' vertex are added. In the second approach, an average of the vertices values is taken and compared with the values of interest. The average check produces smoother models.

In all cases the final model may be smoothed out by applying a filter that changes positions of vertices in the boundary of the object by averaging its coordinates with those belonging to the neighboring vertices.

Figure 9 shows the reconstruction of a cashew nut from MRI images taken 2 mm apart after smoothing. The holes formed in this picture can be seen in the input images presented in figure 10. The larger 'hole' is actually the nut core, and is of interest for analysis of the object. The others were manually created to illustrate the reconstruction of cavities, and their elimination. Apart from these holes, there are many other, very small holes, present in the original images due to noise generated by the data collection process. Those were also reconstructed but are not visible due to their size.

Figure 11 presents a cut of small number of slices from the cashew nut reconstruction, which illustrates the tetrahedral mesh, as well as some of the small details generated during the reconstruction process.

Due to the complete control of the topological structure of the constructed objects exerted by the modelling TMOs, during reconstruction the holes formed are registered so that they can be recovered later for analysis and other purposes. Even those lost visually due to their small size are registered because of the capabilities of the method, and easily

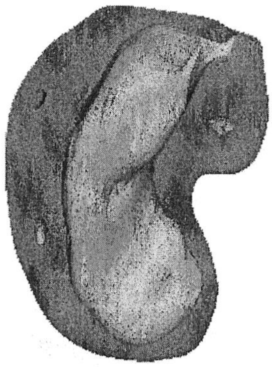
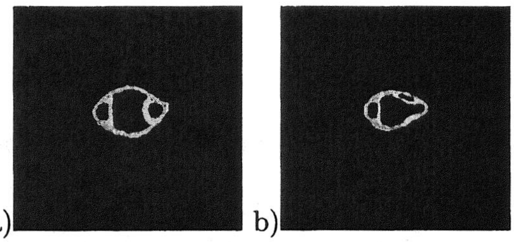
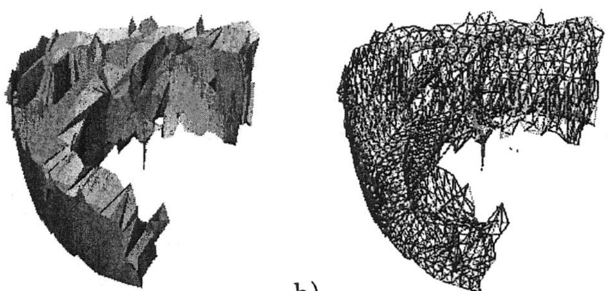


Figure 9: Cashew Nut Reconstruction with four holes.



a) b)

Figure 10: Cashew nut images before reconstruction. a) with three holes b)with two holes



a) b)

Figure 11: A slice of the tetrahedral mesh for the cashew nut reconstruction a) surface display; b) wireframe display.

recovered due to the organization of the data structure.

A particular procedure available to handle holes is to fill out any holes the user chooses not to consider part of the object. In figure 9, for instance, from the three visible holes, the larger one is actually the cavity formed by the nut core, while the others are not useful objects. After processing the filling out procedure it is possible to eliminate any number of them. Figure 12 shows the same model after elimination of the two smaller holes from figure 9 together with all the other holes formed due to noise in the original images.

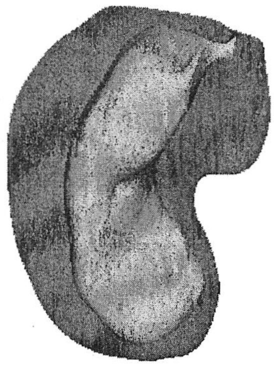


Figure 12: Cashew Nut after elimination of the three smaller holes.

7 Discussion and Conclusions

This paper presents a body of techniques to handle Volumetric Modelling via tetrahedral meshes. This is done through a body o mathematical tools from the field of topology, with which it is possible to characterize tetrahedra and tetrahedral meshes completely. This gives rise to a number of operators capable of creating and updating any non manifold simplicial complex, keeping the topology of the object formed under control. The technique offers global control, that is, any element created or deleted in the whole mesh due to the addition (or deletion) of a tetrahedron is predicted the operator used during the operation. The type of homological change is also registered by the operator class.

Thus, due to this strict topological control, at any time it is possible to tell how many holes, cavities and connected components there are in a particular object under construction. With the use of proper data structure (such as the one also presented here) it is also possible to tell where those elements are and handle them adequately (e. g. filling out their undesirable holes). The data structure presented here is also capable of representing singularities explicitly, a feature that is useful when the mesh is being used for numerical processing.

Many applications can make use of this framework. In this paper it was offered as example a case of reconstruction of a cashew nut from planar MRI images. In this particular case, the interior of the object must be modelled, once the application needs to simulate forces on the shell during post-harvest processing of the nut. In cases such as this, where the interior of the object is to be modelled in a non-uniform way (for visual or simulation reasons), the tools presented here can support many types of processing that would be difficult (or too slow) otherwise.

For the reconstruction case, we also illustrated that the TMO's lend themselves to modellling the interior of objects via a one-pass procedure, that is, in contrast with other volumetric modelling tools, here it is not necessary to reconstruct the border first, or to generate the object's contours, and then fill out the internal parts with tetrahedra, to obtain the model of the object's interior.

The main advantage of the method for modelling is its topological control. Handling topology instead of geometry improves robustness of computational procedures. Additionally, the topological control of the homology allows indexing of important features. This effort should support volume modelling of objects through tetrahedra in an integrated way, so that objects can be handled using these meshes in many stages of the visualization process (from simulation to interaction) without the need for change in representation. For instance, tracking changes and their effects during interaction should be supported by these tools.

TMO's are being extended to handle voxel meshes, taking as basis a technique developed for 2D digital surfaces by the authors [13].

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