Nonstandard Formulae and Model-Theoretic Paradoxes

Rogério Augusto dos Santos Fajardo*
University of Sao Paulo, Sao Paulo, Brazil

Keywords: inaccessible cardinals, set theory, class theory, nonstandard arithmetic, inner models, model theory.

It is well known in the literature that ZFC+I (ZFC plus the existence of a strongly inaccessible cardinal) proves the consistency of ZFC. This occurs because the κ-th level of the Von Neumann hierarchy V_κ – for a strongly inaccessible cardinal κ – is a model for ZFC. As a consequence, it follows from Gödel's second incompleteness theorem that, if ZFC is consistent, it cannot prove that its own consistency implies the consistency of ZFC+I.

However, there are several gaps – or even inaccuracies – in the proofs of the above results in the literature that can lead us to paradoxes, as incorrect proofs of ZFC's inconsistency. The problem occurs because ZFC is not finitely axiomatizable, and the proofs that V_κ is a model for ZFC is often presented as a theorems' scheme, in which for any axiom φ it is proven that φ holds when relativizated to V_κ. In this way, the quantification yields in the metalanguage, whereas the conclusion about the consistency of ZFC should be stated as a single first-order sentence.

The purpose of this talk is discussing how the literature approaches these results, analyzing the subtlety of working in different language's levels in the process and proposing new and detailed proofs of the following known (but somehow folkloric) results: $V_\kappa \models ZFC$, $V_{\kappa+1} \models KM$ (the Kelley-Morse class theory), for a strongly inaccessible cardinal κ, and $KM \vdash Con(ZFC)$. It follows from these results and Gödel's second incompleteness theorem that the consistency strength of KM is strictly between ZFC and $ZFC + I$. I.e., assuming ZFC is consistent, we cannot prove, using ZFC in metatheory, that $Con(ZFC) \to Con(KM)$ or $Con(KM) \to Con(ZFC + I)$.

Along this abstract, κ always refers to a strongly inaccessible cardinal.

Review of the literature. Some of the most renowned textbooks of Set Theory ([4], [5] and [6]) prove that V_κ is a model of ZFC by proving that each relativization φ^{V_κ} holds, for every axiom φ of ZFC. They all mention that, as a consequence of this fact, $ZFC+I \vdash Con(ZFC)$. However, only [5] points out – with no further details – that this conclusion demands a deeper metamathematical analysis. The problem is that, defining model (or *inner model*, as it frequently appears in the literature) of ZFC in this way is a metalinguistic definition, and the result is presented as a theorems' scheme, since ZFC is not finitely axiomatizable.

A finitary proof that $V_\kappa \models ZFC$ is correctly made in [3], although the author does not deepen in the logical details of the proof.

The main reference to this theorem is [1], where the author proves that $V_{\kappa+1} \models NBG$ (Neumann-Bernays-Gödel class theory). Since it is simple to verify that the elements of $V_{\kappa+1} \setminus V_\kappa$ represent the proper classes, the fact that $V_\kappa \models ZFC$ is seen as an easy consequence of Shepherdson's result. However, he uses a finite axiomatization of NBG, and, hence, considering only the relativizated formulae is correct in this context. On the other hand, the proof that NBG implies ZFC is made again in the metalanguage, as a theorems' scheme (see [7]), which make the conclusion about ZFC compromised.

*fajardo@ime.usp.br

The validity of KM in $V_{\kappa+1}$ is mentioned in a review of a Morse's paper for Mathscinet as an "easy" adaptation of Shepherdson's proof. However, again, NBG is finitely axiomatizable, whereas KM is not.

Nonstandard formulae In order to state $V_\kappa \models ZFC$ as a single first-order sentence of ZFC, we need to build a *codified language* (see [2], chapter 9, or [3], chapter 3), where the formulae are finite sequences on a predefined "set of symbols" (which can be taken as ω). However, the domain of some sequences may be *nonstandard* natural numbers (which, by Gödel's completeness and incompleteness theorems, cannot be avoided in any recursive first-order theory which extends arithmetic).

Ignoring nonstandard formulae can lead us to false proofs of the inconsistency of ZFC. In fact, let ZFC^+ be the Set Theory system (defined in [8]) consisting of the language and axioms of ZFC plus the constant M and the following axioms: "M is nonempty and transitive" and φ^M, for each axiom φ of ZFC. It follows from Reflection's Principle that ZFC^+ is relatively consistent to ZFC (see [8]). In any model for ZFC^+, φ^M holds, for every axiom φ of ZFC. Then, by completeness theorem, there is a proof from ZFC^+ that, for every axiom φ of ZFC, we have φ^M and, hence, $M \models \varphi$. Therefore, we prove in ZFC^+ that $M \models ZFC$ and using Gödel's theorem we can deduce that both ZFC^+ and ZFC are inconsistent.

Fortunately, this argument is incorrect, since we can only assume φ^M for the *standard* formulae, which are the codifications of formulae of the metalanguage. In a model where there exists nonstandard integers, M may not be a model for ZFC.

How to fix the proof. Following the steps of [2], used to formalize the definition of Gödel's constructible universe, we must define a set theory language within ZFC, taking the set of formulae $\mathcal{L}_{ST}$ as a subset of $\omega^{<\omega}$, and the relation of satisfatibility between nonempty sets and formulae. A set theory (or class theory) can be defined as a subset of $\mathcal{L}_{ST}$ consisting of all the axioms of such theory.

Going to the metalanguage, when we write $T_1 \vdash Con(T_2)$, for set (or class) theories T_1 and T_2, it means that the following statement can be proven within theory T_1:

$$\exists M((M \neq \emptyset) \wedge \forall \varphi(\varphi \in T_2 \to M \models \varphi)).$$

In our case, T_1 is ZFC+I and theory T_2 can be either ZFC or KM. Model M is V_κ in the first case and $V_{\kappa+1}$ in the second. We need to pay special attention to axioms' scheme of Replacement, in ZFC, and Compreehension, in KM.

References

[1] Shepherdson, J. C. Inner models for set theory. Part II. *Journal of Symbolic Logic* 17:225–237, 1952.

[2] Devlin, K. *Constructibility.* Springer-Verlag, Berlin, 1984.

[3] Drake, F. *Set Theory: An Introduction do Large Cardinals.* North-Holland, 1974.

[4] Jech, T. *Set Theory: The Third Milennium Edition, Revised and Expanded.* Springer, 1999.

[5] Kanamori, A. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings.* Springer-Verlag Berlin, 2003.

[6] Kunen, K. *Set Theory.* College Publications, 2011.

[7] Mendelson, E. *Introduction to Mathematical Logic.* D. Van Nostrand Company, 1964.

[8] Weaver, N. *Forcing for Mathematicians.* World Scientific, 2014.