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POISSONIAN APPROXIMATION FOR THE TAGGED PARTICLE IN ASYMMETRIC SIMPLE EXCLUSION

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Poissonian Approximation for the Tagged Particle in Asymmetric Simple Exclusion

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Abstract

We consider the position of a tagged particle in the one dimensional asymmetric nearest neighbors simple exclusion process. Each particle attemps to jump to the site to its right at rate p and to the site to its left at rate q. The jump is realized if the destination site is empty. We assume $p > q$. The initial distribution is the product measure with density ρ, conditioned to have a particle at the origin. We call $X(t)$ the position at time t of this particle. Using a result recently proved by the authors for a semi-infinite zero range process, it is shown that for all $t \geq 0$, $X(t) = N(t)+B(t)-B(0)$, where $\{N(t)\}$ is a Poisson process of parameter $(p-q)(1-\rho)$ and $\{B(t)\}$ is a stationary process satisfying $E\exp(\theta|B(t)|) < \infty$ for some $\theta > 0$. As a corollary we obtain that —conveniently centered and rescaled— the process $\{X(t)\}$ converges to Brownian motion. A previous result says that in the scale $t^{1/2}$, the position $X(t)$ is given by the initial number of empty sites in the interval $(0, \rho t)$ divided by ρ. We use this to compute the asymptotic covariance at time t of two tagged particles initially at sites 0 and rt. The results also hold for the net flux between two queues in a system of infinitely many queues in series.

Keywords and phrases. Asymmetric simple exclusion, tagged particle, Poissonian approximation, zero range process, system of queues in series

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Running head. Poissonian Approximation .

1. Introduction. Let η_t be an asymmetric simple exclusion process in $\{0,1\}^{\mathbb{Z}}$ with jump rates p and q to the right and left respectively, $p > q \geq 0$, starting with the equilibrium measure μ'_ρ, the product of Bernoulli distributions in $\mathbb{Z}$ with constant parameter ρ at all sites except at the origin, where it is 1, and let X_t denote the position of the tagged particle initially put at the origin.

When the process is totally asymmetric ($q = 0$), it is known that X_t is a Poisson process (Spitzer (1970), Liggett (1985), Kipnis (1986)). This is a consequence of a theorem of Burke (1956) for queues in series; see Kelly (1979). We study the case $p > q > 0$. Since the system is in equilibrium, the expected value of the position of the tagged particle at time t can be computed immediatly: $EX_t = (p-q)(1-\rho)t$. Arratia (1983) conjectured that the asymptotyc variance should be the same as the mean:

$$\lim_{t\to\infty} \frac{E(X_t - (p-q)(1-\rho)t)^2}{t} = (p-q)(1-\rho).$$

This was then proven by De Masi and Ferrari (1986). Since then, this identity between the asymptotic variance and the mean is puzzling us. In this paper we show that X_t can be approximated by a Poisson process so sharply that the error can be dominated by a random variable with an exponential tail uniformly in time. We have two proofs of this, both based on Ferrari and Fontes (1994) where we study a zero range process in $\{0, \infty\}$ with a sink-source of customers in -1 and with a positive average net output of customers. The zero range process is also called Jackson network, due to the early work of Jackson (1963).

We say that a random variable W has a finite exponential moment if there is a positive constant θ such that $Ee^{\theta W}$ is finite. We say that a process W_t has a bounded exponential moment if there is a positive constant θ such that $Ee^{\theta W_t}$ is bounded uniformly in t.

Our first theorem says that we can write for all $t \geq 0$,

$$X_t = N_t + B_t - B_0$$

where N_t is a Poisson process of parameter $(p-q)(1-\rho)$ and B_t has a bounded exponential moment. To show this result we have to use a coupling between the semi infinite simple exclusion process and the infinite process. In the semi infinite process there is a leftmost particle and the invariant distribution for the process as seen from this particle translated by x approaches exponentially fast the product measure ν_ρ, as x goes to infinity. The mentioned coupling was introduced by Ferrari, Kipnis and Saada (1991).

Our second theorem says that there exist Poisson processes R_t^0 and R_t^1 with the same rate $(p-q)(1-\rho)$ and processes D_t^0 and D_t^1 in $\mathbb{Z}$ with bounded exponential moments such that for all $t \geq 0$,

$$R_t^1 + D_t^1 \leq X_t \leq R_t^0 + D_t^0.$$

This result is contained in the previous one. We decided to include it anyway because we have an easier and more general proof. One of the advantages of this proof is that it can be extended to simple exclusion processes for which jump rates of different particles can be different. See Benjamini, Ferrari and Landim (1994). An application to the zero range process is discussed in the last section.

Kipnis (1986) proved a central limit theorem for the tagged particle. Ferrari (1992) and Ferrari and Fontes (1994a) established that in the scale $\sqrt{t}$, the fluctuations of the tagged particle at time t can be read in the initial configuration. More precisely, let $n_0(\eta,(p-q)\rho t)$ be the number of empty sites of the configuration η in the interval $[0,(p-q)\rho t]$. Then, under initial distribution ν'_ρ,

$$\lim_{t\to\infty} \frac{1}{t} E\left(X_t - \frac{n_0(\eta_0,(p-q)\rho t)}{\rho}\right)^2 = 0 \qquad (1)$$

This is sufficient to give weak convergence of the finite dimensional distributions to Brownian Motion, but not tightness. We show that the tightness is a consequence of the sharpness of the approximation to a Poisson process.

Finally, using (1) we compute the covariance of two particles initially put at the origin and at site (integer part of) rt. Kipnis (1986) proved that two particles initially at distance $\sqrt{t}$ have exactly the same asymptotic fluctuations, i.e. that the covariance is the same as the variance of one of them.

2. Poissonian Approximation

Theorem 1 *There exist a Poisson process R_t with rate $(p-q)(1-\rho)$ and a stationary processes B_t in $\mathbb{Z}$ with bounded exponential moment such that for all $t \geq 0$,*

$$X_t = R_t - B_t + B_0.$$

Proof. This proof is based on a coupling argument introduced by Ferrari, Kipnis and Saada (1991) which decomposes η_t into a first and second class particles system evolving in $\mathbb{Z}$ as follows. Let η_0 be the initial configuration of η_t chosen according to μ'_ρ. Let $x_0 = 0$ and denote $0 < x_1 < x_2 < \ldots$ the positions of the positive particles of η_0 and $\ldots > x_{-2} > x_{-1} > 0$ the positions of the negative particles. The particle x_i, $i \in \mathbb{Z}$, is then labeled "first class" independently of the other particles with probability $(p/q)^i/(1+(p/q)^i)$ or "second class" with the complementary probability. These particles move like ordinary particles except that when a first class particle attempts a jump over a second class one, they exchange positions. Call γ_t and ζ_t the configurations of first and second class particles respectively. Clearly we can couple the three processes in such a way that for all $t \geq 0$ we have $\gamma_t + \zeta_t = \eta_t$ coordinatewise. Under this coupling when we disregard the labels we have the doubly infinite system η_t whose particles interact by exclusion, being X_t the position of the tagged particle initially at the origin. When we add the labels, it is possible that the label of X_t change with time. At time zero there is a particle at the origin labeled first class with probability 1/2.

Proposition 3.14 in the above mentioned paper states that this initial measure for the positions of the particles plus the labels, denoted $\bar{\nu}$ is invariant for the coupled process as seen from the tagged particle $\tau_{X_t}(\gamma_t, \zeta_t)$. The key point to show this invariance is that the measure $\bar{\nu}$ is reversible with respect to the part of the generator that governs the exchanges between first and second class particles. It is clear that under $\bar{\nu}$, there is a leftmost first class particle in the system, let us denote by Y_t its (absolute) position at time t. Under the invariant measure $\bar{\nu}$, the distribution of

$$C_t = X_t - Y_t \tag{2}$$

is independent of t and a direct computation shows that C_t has a bounded exponential moment.

The process Y_t can be studied by seeing γ_t as a semi infinite sistem of queues or a zero range process (Andjel (1982), Jackson (1963)). See Ferrari (1986). The queues are indexed by $\{-1, 0, 1, \ldots\}$ and the size of the queue at $i \in \{0, 1, \ldots\}$ is given by the number of holes between the i-th and $(i+1)$-th particles, the queue at -1 having an infinite number of customers. Y_t can then be represented by the net output from 0 to -1. More formally, let ξ_t be the process defined by $\xi_t(i)$ = number of sites between the ith and $(i+1)$-th γ_t particles. It is easy to see that γ_t is Markovian, as well as the process as seen from the leftmost particle $\tau_{Y_t}\gamma_y$. The invariance property of $\bar{\nu}$ implies that the γ marginal of $\bar{\nu}$ as seen from the leftmost γ particle is invariant for $\tau_{Y_t}\gamma_t$, the γ process as seen from the leftmost γ particle. This implies that if we denote $\bar{\nu}_\rho$ the measure induced by the γ marginal on the semi-infinite zero range process, then $\bar{\nu}_\rho$ is invariant for ξ_t.

We would like to use Theorem 2 of Ferrari and Fontes (1994) to write Y_t as a Poisson process plus a perturbation of order one:

$$Y_t = N_t + A_t - A_0 \tag{3}$$

whith A_t a stationary process with a bounded exponential moment. This together with the invariance of $\bar{\nu}_\rho$ and the fact that under it the distribution of the distance between Y_t and X_t is independent of time and with a finite exponential moment will conclude the proof by taking $B_t = A_t + C_t$.

To use this result, one has to verify the hypotheses about the invariant measure $\bar{\nu}_\rho$. Consider the measure ν^*_ρ on $\mathbb{N}^{\mathbb{Z}}$ defined by

$$\nu^*_\rho(\xi(x) = k(x), x \in A) = \prod_{x \in A} \rho(x)^{k(x)}(1 - \rho(x)),$$

where $\rho(x) = m(x) + (1 - m(x))(1 - \rho)$ and $m(x) = (q/p)^{x+1}$. Ferrari (1986) showed that ν^*_ρ is invariant for the semi infinite zero range process and for this measure Ferrari and Fontes proved that (3) holds. So we have two invariant measures, $\bar{\nu}_\rho$ and ν^*_ρ, the later fitting in the hypotheses where we would like to fit $\bar{\nu}_\rho$. It is then sufficient that the two measures be shown to be the same. A direct computation appears difficult, so we use an indirect approach already present in Ferrari (1986), inspired in Liggett (1976) and Andjel (1982). First consider modifications $\nu^*_{\rho_0}$ and $\nu^*_{\rho_1}$ of ν^*_ρ where

$$\rho_0 > \rho > \rho_1. \tag{4}$$

It is clear by the properties of $\bar{\nu}_\rho$ (inherited from the product nature of $\bar{\nu}$) and the product nature of $\nu^*_{\rho_0}$ and $\nu^*_{\rho_1}$ that these measures have an asymptotic average queue size, respectively, $(1-\rho)/\rho, (1-\rho_0)/\rho_0$ and $(1-\rho_1)/\rho_1$:

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \xi(i) = (1-\rho)/\rho \quad \bar{\nu}_\rho \text{ and } \nu^*_\rho \text{ almost surely} \tag{5}$$

Ferrari (1986) says that there exists a coupling λ_0 of $\bar{\nu}_\rho$ and $\nu^*_{\rho_0}$ such that if ξ denotes the first marginal and ξ^0 the second,

$$\lambda_0(\xi^0 \geq \xi \text{ or } \xi^0 \leq \xi) = 1$$

and a coupling λ_1 of $\bar{\nu}_\rho$ and $\nu^*_{\rho_1}$ such that

$$\lambda_1(\xi^1 \geq \xi \text{ or } \xi^1 \leq \xi) = 1,$$

where ξ, ξ^0 and ξ^1 are the $\bar{\nu}_\rho, \nu^*_{\rho_0}$ and $\nu^*_{\rho_1}$ coordinates, respectively. It is clear by (4) and the asymptotic averages (5) that actually $\lambda_0(\xi^0 \leq \xi) = 1$ and $\lambda_1(\xi^1 \geq \xi) = 1$. Thus, $\nu^*_{\rho_0} \leq \bar{\nu}_\rho \leq \nu^*_{\rho_1}$ stochastically. If we replace $\bar{\nu}_\rho$ by ν^*_ρ in this last expression, it clearly still holds. The equality of $\bar{\nu}_\rho$ and ν^*_ρ follows by letting ρ_0 and ρ_1 tend to ρ and the proof is complete.□

Theorem 2 *There exist Poisson processes R^0_t and R^1_t with the same rate $(p-q)(1-\rho)$ and processes D^0_t and D^1_t in $\mathbb{Z}$ with bounded exponential moments such that*

$$R^1_t + D^1_t \leq X_t \leq R^0_t + D^0_t.$$

Proof.

As in the previous proof, the process X_t can be represented as the net flux of customers between the queues 0 and -1 for a system of infinitely many queues indexed by $\mathbb{Z}$. This representation was used by Kipnis (1986), Ferrari (1986) and Ferrari and De Masi (1985). The queueing system is the zero range process on $\mathbb{N}^{\mathbb{Z}}$ for which at rate q and p respectively, one customer at queue i jumps to the right and left nearest neighbor queue respectively. We consider this system starting with the invariant measure ν_ρ, a product of geometrics indexed by $\mathbb{Z}$ with constant parameter $1-\rho$: $\nu_\rho(\zeta(i) > 0) = 1-\rho$. Let us denote this process by ζ_t. Consider the auxiliary zero range processes ζ_t^0 and ζ_t^1 in $\mathbb{N}^{\{-1,0,1,\dots\}}$ with jump rates identical to those of ζ_t except that in the first one there are no jumps from -1 to 0 (that is, the corresponding rate is 0) and the second one has infinitely many customers at -1 (that is, the corresponding rate is q independently of the configuration). Their invariant measures are product of geometrics in each site with parameters (the probability of at least one customer in site $x \geq 0$)

$$\rho_0(x) = (1 - m(x))\rho$$

$$\rho_1(x) = m(x) + (1 - m(x))\rho,$$

respectively, where $m(x) = (q/p)^{x+1}$.

We couple ζ_t^0, ζ_t and ζ_t^1 in such a way that at time zero $\zeta_0^0(x) \leq \zeta_0(x) \leq \zeta_0^1(x)$ for each $x \geq -1$, using the same "jump arrows" for the three of them (except that those from -1 to 0 are deleted for the first process). This is the basic coupling for the zero range process, see Andjel (1982).

Let X_t^0 and X_t^1 denote the net outputs of ζ_t^0 and ζ_t^1, respectively. That is the net flux of customers between queue 0 and "queue −1". By Theorem 2 in Ferrari and Fontes (1994), we have the following representation.

$$X_t^0 = R_t^0 - B_t^0 + B_0^0,$$

$$X_t^1 = R_t^1 - B_t^1 + B_0^1,$$

where R_t^0 and R_t^1 are Poisson processes with the same rate $(p-q)(1-\rho)$ and B_t^0 and B_t^1 are stationary processes on $\mathbb{N}$ with bounded exponential moments.

The following inequalities hold.

$$X_t \leq X_t^0 + C^0, \tag{6}$$

$$X_t^1 \leq X_t + C^1, \tag{7}$$

where $C^0 = \sum_{x\geq 0}(\zeta_0(x) - \zeta_0^0(x))$ and $C^1 = \sum_{x\geq 0}(\zeta_0^1(x) - \zeta_0(x))$ are finite random variables with finite exponential moments. To show the second inequality notice that since $\zeta_0(x) \leq \zeta_0^1(x)$, each time that a ζ customer jumps from 0 to −1, a ζ^1 customer accompanies it. Now assume that a ζ^1 customer exits without a ζ companion. This may be for two reasons: either it was a ζ^1 customer present at time zero, and this is taken account with the term C^1 or it is a customer that entered the system without a ζ companion after time zero, but in this case the net flux produced by this customer is null. On the other hand, each time that a ζ customer jumps from −1 to 0, a ζ^1 customer accompanies it. The first identity is shown in the same manner.

We get the result by identifying D_t^0 and D_t^1 respectively with $C^0 - B_t^0 + B_0^0$ and $-C^1 - B_t^1 + B_0^1$.□

Corollary 1 *Let* $\tilde{X}_t^\epsilon = (X_{\epsilon^{-1}t} - (p-q)(1-\rho)\epsilon^{-1}t)/\sqrt{(p-q)(1-\rho)\epsilon^{-1}t}$. *Then* $\tilde{X}_t^\epsilon$ *converges weakly to standard Brownian Motion as* $\epsilon \to 0$.

Proof.

The result follows from from the weak convergence of the standardized Poisson process $\epsilon^{1/2}R_{\epsilon^{-1}t}$ to Brownian motion and the following. Let B_t be the stationary process of Theorem 1. For any $T > 0$,

$$\sup_{0\le t\le T} \epsilon^{1/2}B_{\epsilon^{-1}t} \to 0$$

as $\epsilon \to 0$ in probablity.

To establish the latter, we argue as follows. Partition $[0,T]$ in intervals of equal lengths ϵ (except possibly the last one which will have length at most ϵ) and enumerate them I_i and its extremes t_i for $i = 1, 2, \ldots, N$. Notice that N is of the order of ϵ^{-1}. Now notice that $B_{\epsilon^{-1}t}$ will change more than once in I_i (denote this event by A_i) only if there are at least two arrows between the sites "0" and "-1" in an interval of length ϵ of the zero range representation of X_t. This has probability of the order of ϵ^2. Thus, given $\delta > 0$,

$$\begin{aligned} P(\sup_{0\le t\le T} B_{\epsilon^{-1}t} > \epsilon^{-1/2}\delta) &= P(\sup_{1\le i\le N}\sup_{t\in I_i} B_{\epsilon^{-1}t} > \epsilon^{-1/2}\delta) \\ &\le \sum_{i=1}^{N} P(\sup_{t\in I_i} B_{\epsilon^{-1}t} > \epsilon^{-1/2}\delta) \\ &\le \sum_{i=1}^{N} P(B_{\epsilon^{-1}t_i} > \epsilon^{-1/2}\delta - 1) + \sum_{i=1}^{N} P(A_i). \end{aligned}$$

The first sum is bounded above by $N \times \exp(-\theta\epsilon^{-1/2}\delta)$ with a positive constant θ given by the exponential boundedness of $B_{\epsilon^{-1}t}$. The second sum is bounded above by $N \times O(\epsilon^2)$. Letting $\epsilon \to 0$, we get the result.□

3. Two Point Covariance

In this section we get an asymptotic expression for the covariance of the positions of two tagged particles placed initially order t apart.

Theorem 3 *Let $r > 0$. Assume that the initial configuration is determined by first picking a configuration from ν_ρ and then adding two particles: one at the origin and the other at site $\lfloor rt \rfloor$ (regardless the configuration at those sites). Let X_t and Y_t be the positions of these two tagged particles at time t. Then*

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Cov}(X_t, Y_t) = \begin{cases} 0 & , \text{ if } r \geq (p-q)\rho, \\ ((p-q)\rho - r)(1-\rho)/\rho & , \text{ otherwise.} \end{cases} \tag{8}$$

Proof.

Equation (1.5) in Ferrari (1992) and remarks thereof establish that under μ''_ρ it holds (1) and

$$\lim_{t\to\infty} \frac{1}{t} E\left(Y_t - \frac{n_{\lfloor rt \rfloor}(\eta, (p-q)\rho t)}{\rho}\right)^2 = 0, \tag{9}$$

where $n_z(\eta, y) = \sum_{x=z}^{z+y}(1-\eta(x))$ is the number of empty sites of the initial configuration η between z and $z+y$.

These equations imply that

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Cov}(X_t, Y_t) = \lim_{t\to\infty} \frac{1}{t} \frac{\operatorname{Cov}\left(n_0(\eta, (p-q)\rho t), n_{\lfloor rt \rfloor}(\eta, (p-q)\rho t)\right)}{\rho^2} \tag{10}$$

When $r \geq (p-q)\rho$, the intervals of the summations in $n_0(\eta,(p-q)\rho t)$ and $n_{\lfloor rt \rfloor}(\eta,(p-q)\rho t)$ are disjoint, so the product nature of ν_ρ implies that (10) vanishes. When $r < (p-q)\rho$, then it is easy to see that (10) will reduce to

$$\lim_{t\to\infty} \frac{1}{t} \frac{\mathrm{Var}\left(\sum_{x=rt}^{(p-q)\rho t}(1-\rho(x))\right)}{\rho^2},$$

which is equal to $((p-q)\rho - r)(1-\rho)/\rho$.□

4. Remarks on applications to networks of queues.

Our results can be applied to a system with infinitely many queues in series with service times with rate 1 and with routing matrix $q(x,x+1) = q$, $q(x,x-1) = p$, that is, a customer after service chooses a new queue between the nearest neighbors with probabilities q to the right and p to the left. Theorems 1 and 2 say that for this system starting with the product measure ν_ρ, the net flux of customers to the right between two neighboring queues can be expressed as a Poisson process plus an order 1 stationary process. This is because, as discussed in the proofs of the theorems, the net flux in the zero range process can be represented as X_t when one transforms this process into an asymmetric simple exclusion process. The rate of the Poisson process is of course $(p-q)(1-\rho)$.

The proof of Theorem 2 can also be applied to a general network of queues labeled with the integers with arbitrary jumps between queues. It shows a Poissonian approximation for the net flux of customers between the queue 0 and the queues with negative labels and provided the three conditions below:

(a) The customers can not jump between a positive queue and a negative queue in neither direction.

(b) There exists an invariant product distribution under which there is a positive average net flux between the queue 0 and the queues with negative labels.

(c) Consider the semi-infinite system obtained by assuming that the negative labeled queues have infinitely many customers. Assume that for this system there exists an invariant measure for the non-negative queues concentrating in configurations with a finite total number of customers; furthermore under this invariant measure the number of customers has a finite exponential moment. It is not hard to see that this hypothesis puts the semi-infinite system under the conditions of Theorem 2 of Ferrari and Fontes (1994b).

Finally, the proof of Theorem 2 can be applied to systems with finitely many queues indexed $\{-N, \dots, N\}$ for the net flux of customers between queue 0 and the "negative" queues provided (a) and (b) above. If in the limit when $N \to \infty$, (c) above holds, then the approach to the Poisson process is uniform in N. See Ferrari and Fontes (1994b) for a discussion of the finite system with a sink/source of customers at -1.

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