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# Indecomposables in derived categories of skewed-gentle algebras

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# Indecomposables in derived categories of skewed-gentle algebras

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**Abstract**

We give a description of the indecomposable objects in the derived category of a finite-dimensional skewed-gentle algebra.

## 1 Introduction

In [**BeMe**] we determined all the derived indecomposables of a gentle algebra and we gave an explicit description of them by using a matrix problem studied by Bondarenko. Here, we do the same for the case of algebras $A$ whose skew-group algebras $AG$ in the case of $\operatorname{char} k \neq 2$ are Morita equivalent to gentle algebras. For this we use the papers [**GePe**], [**De**] and [**BoDr**], of Geiss, de la Peña, Deng, Bondarenko and Drozd for which the general findings of Reiten and Riedtmann ( [**ReRd**]) are basically needed.

Let $A$ be a finite-dimensional algebra of the form $kQ/\langle I\rangle$ over a field $k$, where $I$ is a set of relations for a quiver $Q$ and $A-\text{mod}$ is the category of finitely generated left $A$-modules, and let $\mathbf{D}^b(A)$ be the bounded derived category of the category $A-\text{mod}$.

The category $\mathbf{D}^b(A)$ is known only for a few algebras $A$. For example, the description of indecomposable objects of $\mathbf{D}^b(A)$ is well-known for hereditary algebras of finite and tame type [**Ha**], for tubular algebras [**HaRi**], for gentle algebras [**BeMe**], for algebras with radical square zero, and for local and two-point algebras [**BeDr**].

In this paper we give a description of the indecomposable objects in the derived category when the algebra is skewed-gentle. This class of algebras was introduced in [**GePe**]. We have found that there is a connection between the derived category of a skewed-gentle algebra and a matrix problem presented by V. M. Bondarenko and Yu. A. Drozd (*see* [**BoDr**]). We show that the problem of finding the indecomposable objects of the derived category may be reduced to finding the indecomposable objects in that matrix problem.

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The structure of the paper is as follows. In Section 2 we fix notations and show that the problem of finding the indecomposables in $\mathbf{D}^b(A)$ may be reduced to the problem of finding the indecomposables in a category $\mathfrak{p}(A)$, a certain subcategory of the category of bounded projective complexes $\mathbf{C}^b(A - \text{pro})$. In Section 3 we introduce the category of S-representations of a linearly ordered poset, Bondarenko-Drozd's matrix problem. In Section 4, a functor is defined which will solve our problem. In the final section, the description of the indecomposables of $\mathbf{D}^b(A)$ is given.

# 2 Preliminaries

## 2.1 Derived representation type

Let $A$ be a finite-dimensional algebra over a field $k$ and $A - \text{mod}$ be the category of left finite-dimensional $A$-modules. We will follow in general the notations and terminology of [**Ri**] and [**Ha**].

Given $A$, we denote by $\mathbf{D}(A)$ (resp., $\mathbf{D}^-(A)$ or $\mathbf{D}^b(A)$) the derived category of $A-\text{mod}$ (resp., the derived category of right bounded complexes of $A-\text{mod}$ or the derived category of bounded complexes of $A-\text{mod}$); by $\mathbf{C}^b(A-\text{pro})$ (resp., $\mathbf{C}^-(A-\text{pro})$ or $\mathbf{C}^{-,b}(A-\text{pro})$) the category of bounded projective complexes (resp., of right bounded projective complexes or of right bounded projective complexes with bounded cohomology (that is, complexes of projective modules with the property that the cohomology groups are non zero only at a finite number of places)); and by $\mathbf{K}^b(A - \text{pro})$ (resp., $\mathbf{K}^-(A - \text{pro})$ or $\mathbf{K}^{-,b}(A - \text{pro})$) the corresponding homotopy categories.

We identify the homotopy category $\mathbf{K}^b(A - \text{pro})$ with the full subcategory of perfect complexes in $\mathbf{D}^b(A)$. Let us recall that a complex is perfect if it is isomorphic to a bounded complex of finitely generated projective $A$-modules.

We will also use the following notations. By $\mathfrak{p}(A)$ we denote the full subcategory of $\mathbf{C}^b(A - \text{pro})$ defined by the projective complexes such that the image of every differential map is contained in the radical of the corresponding projective module. Since any projective complex is the sum of one complex with this property and two complex where, alternativelly, all differential maps are 0's or isomorphisms (which is, hence, isomorphic to the zero object in the derived category) we can always assume that we reduce ourselves to consider projective complexes of this form.

It is well known that $\mathbf{D}^b(A)$ is equivalent to $\mathbf{K}^{-,b}(A - \text{pro})$ (see, for example, [**KoZi**], Proposition 6.3.1 and [**Har**]).

**Proposition 1.** *[Har]* $\mathbf{D}^-(A)$ *is equivalent to* $\mathbf{K}^-(A - \text{pro})$. *The image of* $\mathbf{D}^b(A)$ *under this equivalence is* $\mathbf{K}^{-,b}(A - \text{pro})$.

Given $M^\bullet \in \mathbf{D}^b(A)$, we denote by $P^\bullet_{M^\bullet}$ the projective cover of $M^\bullet$ (see [**KoZi**]) and by $H^i(M^\bullet)$ the $i$-th cohomology module.

We call a category $C$ *basic* if it satisfies the following conditions:

- all its objects are pairwise non-isomorphic;
- for each object $x$ there are no non-trivial idempotents in $\mathcal{C}(x,x)$.

A full subcategory $\mathcal{S} \subset \mathcal{C}$ is called a *skeleton* of $\mathcal{C}$ if it is basic and each object $x \in \mathcal{C}$ is isomorphic to a direct summand of a (finite) direct sum of some objects of $\mathcal{S}$. It is evident that if $\mathcal{C}$ is a category with unique direct decomposition property, then it has a skeleton and the last one is unique up to isomorphism. We will denote it by $\mathbf{Sk}\,\mathcal{C}$ and the set of its objects by $\mathbf{Ver}\,\mathcal{C}$.

In order to simplify our exposition, let us introduce two easy constructions, as follows.

For $P^\bullet \in \mathbf{C}^{-,b}(A-\text{pro}) \setminus \mathbf{C}^b(A-\text{pro})$, let $s$ be the maximal number such that $P^s \neq 0$ and $H^i(P^\bullet) = 0$ for $i \leq s$. Then, $\alpha(P^\bullet)^\bullet$ denotes the *brutal truncation* of $P^\bullet$ below $s$ (see [**Wb**]), i.e. the complex given by

$$\alpha(P^\bullet)^i = \begin{cases} P^i & \text{, if } i \geq s; \\ 0 & \text{, otherwise,} \end{cases}$$

$$\partial^i_{\alpha(P^\bullet)^\bullet} = \begin{cases} \partial^i_{P^\bullet} & \text{, if } i \geq s; \\ 0 & \text{, otherwise.} \end{cases}$$

For $P^\bullet \neq 0^\bullet \in \mathbf{C}^b(A-\text{pro})$, let $t$ be the maximal number such that $P^i = 0$ for $i < t$. Then, $\beta(P^\bullet)^\bullet$ denotes the *(good) truncation* of $P^\bullet$ below $t$ (see [**Wb**]), i.e. the complex given by

$$\beta(P^\bullet)^i = \begin{cases} P^i & \text{, if } i \geq t; \\ \operatorname{Ker} \partial^t_{P^\bullet} & \text{, if } i = t-1; \\ 0 & \text{, otherwise,} \end{cases}$$

$$\partial^i_{\beta(P^\bullet)^\bullet} = \begin{cases} \partial^i_{P^\bullet} & \text{, if } i \geq t; \\ i_{\operatorname{Ker} \partial^t_{P^\bullet}} & \text{, if } i = t-1; \\ 0 & \text{, otherwise,} \end{cases}$$

where $i_{\operatorname{Ker} \partial^t_{P^\bullet}}$ is the obvious inclusion.

**Lemma 1.** *Let $M^\bullet \in \mathbf{K}^{-,b}(A-\text{pro}) \setminus \mathbf{K}^b(A-\text{pro})$ be an indecomposable. Then $\beta(\alpha(M^\bullet)^\bullet)^\bullet$ is also indecomposable in $\mathbf{D}^b(A)$ and*

$$M^\bullet \cong P^\bullet_{\beta(\alpha(M^\bullet)^\bullet)^\bullet}.$$

*Proof.* Obvious. □

**Lemma 2.** *There exist skeletons $\mathbf{Sk}\,\mathfrak{p}(A)$ and $\mathbf{Sk}\,\mathbf{K}^b(A-\text{pro})$ of $\mathfrak{p}(A)$ and $\mathbf{K}^b(A-\text{pro})$, respectively, such that $\mathbf{Ver}\,\mathfrak{p}(A) = \mathbf{Ver}\,\mathbf{K}^b(A-\text{pro})$.*

*Proof.* Obvious. □

Let $\overline{\mathcal{X}(A)} = \{ M^\bullet \in \mathbf{Ver}\,\mathfrak{p}(A) \mid P^\bullet_{\beta(M^\bullet)^\bullet} \notin \mathbf{K}^b(A-\mathrm{pro}) \}$. Let $\cong_{\mathcal{X}}$ be the equivalence relation on the set $\overline{\mathcal{X}(A)}$ defined by $M^\bullet \cong_{\mathcal{X}} N^\bullet$ iff $P^\bullet_{\beta(M^\bullet)^\bullet} \cong P^\bullet_{\beta(N^\bullet)^\bullet}$ in $\mathbf{K}^{-,b}(A-\mathrm{pro})$. We use the notation $\mathcal{X}(A)$ for a fixed set of representatives of the quotient set $\overline{\mathcal{X}(A)}$ over the equivalence relation $\cong_{\mathcal{X}}$.

¿From lemmas 1 and 2 we obtain the following

**Corollary 1.** *There exist skeletons* $\mathbf{Sk}\,\mathbf{D}^b(A)$ *and* $\mathbf{Sk}\,\mathfrak{p}(A)$ *of* $\mathbf{D}^b(A)$ *and* $\mathfrak{p}(A)$, *respectively, such that* $\mathbf{Ver}\,\mathbf{D}^b(A) = \mathbf{Ver}\,\mathfrak{p}(A) \cup \{\beta(M^\bullet)^\bullet \mid M^\bullet \in \mathcal{X}(A)\}$.

**Remark 1.** *If $A$ has finite global dimension, we have* $\mathcal{X}(A) = \emptyset$ *and* $\mathbf{Ver}\,\mathbf{D}^b(A) = \mathbf{Ver}\,\mathfrak{p}(A)$.

Let $T$ be the translation functor $\mathbf{D}(A) \to \mathbf{D}(A)$. By analogy with [Dr] we will use the following definitions.

**Definition 1.** *Let $k$ be an algebraically closed field and $A$ be a finite-dimensional $k$-algebra. Then*

- *$A$ is called* derived wild *if there exists a complex of $A - k\langle x,y\rangle$- bimodules $M^\bullet$ such that each $M^i$ is free and of finite rank as right $k\langle x,y\rangle$-module and such that the functor $M \otimes_{k\langle x,y\rangle} -$ preserves indecomposability and isomorphism classes.*
- *$A$ is called* derived tame (see [GeKr]) *if, for each cohomology dimension vector $(d_i)_{i\in\mathbf{Z}}$, there exist a localization $R = k[x]_f$ with respect to some $f \in k[x]$ and a finite number of bounded complexes of $A - R$-bimodules $C^\bullet_1, \cdots, C^\bullet_n$ such that each $C^i_j$ is free and of finite rank as right $R$-module and such that every indecomposable $X^\bullet \in \mathbf{D}^b(A)$ with $\dim H^i(X^\bullet) = d_i$ is isomorphic to $C^\bullet_j \otimes_R S$ for some $j$ and some simple $R$-module $S$.*
- *$A$ is called* derived discrete (see [Vo]) *if for every cohomology dimension vector $(d_i)_{i\in\mathbf{Z}}$, we have up to isomorphism a finite number of indecomposables $X^\bullet \in \mathbf{D}^b(A)$ with $\dim H^i(X^\bullet) = d_i$.*
- *$A$ is called* derived finite *if we have a finite number of indecomposables*

$$X^\bullet_1, \cdots, X^\bullet_n \in \mathbf{D}^b(A)$$

*such that every indecomposable object $X^\bullet \in \mathbf{D}^b(A)$ is isomorphic to $T^i(X^\bullet_j)$ for some $i \in \mathbf{Z}$ and some $j$.*

In the case of an arbitrary field $k$ we will say that a $k$-algebra $A$ is derived wild (resp., derived tame, derived discrete, derived finite) if the $\bar{k}$-algebra $A \otimes_k \bar{k}$ is derived wild (resp., derived tame, derived discrete, derived finite), where $\bar{k}$ is the algebraic closure of $k$.

## 2.2 Quivers and relations

A quiver $Q$ is a tuple $(Q_0, Q_1, s, e)$, where $Q_0$ is the set of *vertices*, $Q_1$ is the set of *arrows* and $s, e$ are functions $s, e : Q_1 \to Q_0$ which determine, resp., the starting and ending vertex of the arrows.

Given two vertices $a$ and $b$ we define $Q_1[a, b]$ as the set of all arrows from $a$ to $b$.

A *path* $p$ in $Q$ of length $l(p) = n \geq 1$ is a sequence $a_1 \ldots a_n$ of arrows such that $s(a_{i+1}) = e(a_i)$ for $1 \leq i \leq n-1$. We set $s(p) = s(a_1)$ and $e(p) = e(a_n)$. The concatenation $p_1p_2$ of paths $p_1$ and $p_2$ is defined (in the natural way) if and only if $e(p_1) = s(p_2)$. Additionally, for every $a \in Q_0$, we introduce $1_a$, a path (of length 0) with $s(1_a) = e(1_a) = a$. The set of all paths (resp., all paths of length $\geq m$) in $Q$ is denoted by $\mathcal{P}(Q)$ (resp., $\mathcal{P}_{\geq m}(Q)$).

Let $k$ be a field. A *relation* in $Q$ is a non-zero $k$-linear combination of paths of length at least 2 having the same starting vertex and the same ending vertex. A *zero relation* in $Q$ is a relation of the form $w$ where $w$ is a path. A *commutative relation* in $Q$ is a relation of the form $u - v$ where $u$ and $v$ are paths.

If $Q$ is a quiver, then we denote by $kQ$ the corresponding path algebra with basis the set of paths in $Q$. The multiplication is induced from the concatenation of paths.

As usual, if $I$ is a set of relations in $Q$, let $(Q, I)$ denote $kQ/\langle I \rangle$, the path algebra modulo the ideal generated by the elements in $I$. Note that our algebras have a unit element if and only if the set of vertices $Q_0$ is finite.

We call a path $p$ in $Q$ a *path in* $(Q, I)$ if $p \notin \langle I \rangle$. The set of all paths (resp., all paths of length $\geq m$) in $(Q, I)$ is denoted by $\mathcal{P}(Q, I)$ (resp., $\mathcal{P}_{\geq m}(Q, I)$). Note that if $u - v$ is a commutative relation, then we identify the paths $u$ and $v$. It is clear that if $I$ consists of zero and commutative relations, then the set of elements of the algebra $A = kQ/\langle I \rangle$ which correspond the elements of $\mathcal{P}(Q, I)$ is a bases of $A$. We warn the reader that we are taking only one element for each commutative relation.

A path $w = w_1 \cdots w_n$ of length $l(w) \geq 1$ in $(Q, I)$ is called *maximal in* $(Q, I)$, or simply, *maximal* if $uw$ and $wv$ are not paths in $(Q, I)$ for each $u, v \in Q_{\geq 1}$.

We will denote by $\mathbf{M} = \mathbf{M}(Q, I)$ the set of maximal paths in $(Q, I)$.

It is clear that if $kQ/\langle I \rangle$ is finite-dimensional, then any path $w \in \mathcal{P}(Q, I)$ is a subpath of a maximal path in $(Q, I)$.

## 2.3 Gentle algebras

Let $Q$ be a quiver and $I$ a set of relation for $Q$.

**Definition 2.** *The pair $(Q, I)$ is said to be* special biserial *[SkWa] if the following holds:*

- *At every vertex of $Q$ at most two arrows stop and at most two arrows start;*
- *For each arrow $b$ there is at most one arrow $a$ with $e(a) = s(b)$ and $ab \notin I$ and at most one arrow $c$ with $e(b) = s(c)$ and $bc \notin I$.*

*The pair $(Q,I)$ is said to be* gentle *[AsSk] if, it is special biserial, and moreover the following holds:*

- *$I$ is generated by zero-relations of length 2;*
- *For each arrow $b$ there is at most one arrow $a$ with $e(a)=s(b)$ and $ab\in I$ and at most one arrow $c$ with $e(b)=s(c)$ and $bc\in I$.*

*A k-algebra $A$ is* special-biserial, *or* gentle, *if it is Morita-equivalent to a factor algebra $kQ/\langle I\rangle$, where the pair $(Q,I)$ is special-biserial or gentle, respectively.*

## 2.4 Skewed-gentle algebras

We now define some basic notion and fix some notation. Let $Q$ be a quiver with a fixed distinguished set of vertices which we denote by $Sp$, and $I$ a set of relations for $Q$. We call the elements of $Sp$ *special vertices*, the remaining vertices are called *ordinary*.

For a triple $(Q,Sp,I)$ let us consider the pair $(Q^{sp},I^{sp})$, where $Q_0^{sp}:=Q_0$, $Q_1^{sp}:=Q_1\cup\{a_i\,|\,i\in Sp\}$, $s(a_i):=e(a_i):=i$ and $I^{sp}:=I\cup\{a_i^2\,|\,i\in Sp\}$.

**Definition 3.** *A triple $(Q,Sp,I)$ as above is called skewed-gentle if the corresponding pair $(Q^{sp},I^{sp})$ is gentle.*

Let $(Q,Sp,I)$ be a skewed-gentle triple. We associate to each vertex $i\in Q_0$ a set, which we will denote by $Q_0(i)$, on the following way: If $i$ is an ordinary vertex then $Q_0(i)=\{i\}$, if $i$ is special then $Q_0(i)=\{(i,-),(i,+)\}$.

$(Q^{sg},I^{sg})$ will denoted the pair defined in the following way:

$$Q_0^{sg}:=\cup_{i\in Q_0}Q_0(i),$$

$$Q_1^{sg}[\alpha,\beta]:=\{(\alpha,a,\beta)\,|\,a\in Q_1,\alpha\in Q_0(s(a)),\beta\in Q_0(e(a))\},$$

$$I^{sg}:=\{\sum_{\beta\in Q_0(s(b))}\lambda_\beta(\alpha,a,\beta)(\beta,b,\gamma)\,|\,ab\in I,\alpha\in Q_0(s(a)),\gamma\in Q_0(e(b))\},$$

where $\lambda_\beta=-1$ if $\beta=(i,-)$ for some $i\in Q_0$, and $\lambda_\beta=1$ otherwise.

Note that the relations in $I^{sg}$ are zero relations or commutation relations.

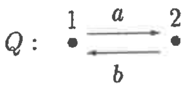
**Definition 4.** *A k-algebra $A$ is called skewed-gentle, if it is Morita-equivalent to a factor algebra $kQ^{sg}/\langle I^{sg}\rangle$, where the triple $(Q,Sp,I)$ is skewed-gentle.*

**Remark 2.** *We use signs $\lambda_\beta$ in the definition of $I^{sg}$ for technical reasons. Consider the algebra $B=kQ^{sg}/\left\langle\widetilde{I^{sg}}\right\rangle$, where*

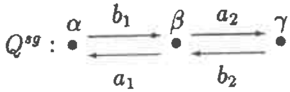
$$\widetilde{I^{sg}}:=\{\sum_{\beta\in Q_0(s(b))}(\alpha,a,\beta)(\beta,b,\gamma)\,|\,ab\in I,\alpha\in Q_0(s(a)),\gamma\in Q_0(e(b))\}.$$

*It is easy to see that the algebras $A=kQ^{sg}/\langle I^{sg}\rangle$ and $B$ are isomorphic.*

**Example 1.**



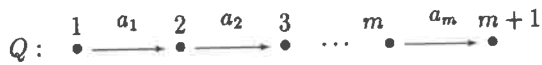

$I = \{ab, ba\}$
$Sp = \{2\}$



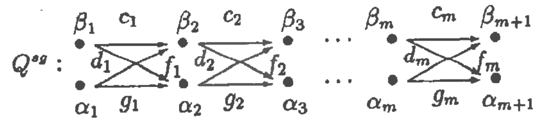

$I^{sg} = \{a_1 b_1 - a_2 b_2,$
$b_i a_j, i, j = 1, 2\}$

where $\alpha = (2,+)$, $\beta = 1$, $\gamma = (2,-)$, $a_1 = (\beta, a, \alpha)$ $a_2 = (\beta, a, \gamma)$, $b_1 = (\alpha, b, \beta)$ and $b_2 = (\gamma, b, \beta)$.

**Example 2.**




$I = \{a_i a_{i+1} \mid 1 \le i < m\}$
$Sp = Q_0$




$I^{sg} = \{c_i c_{i+1} - d_i f_{i+1},$
$c_i d_{i+1} - d_i g_{i+1}, f_i c_{i+1} - g_i f_{i+1},$
$f_i d_{i+1} - g_i g_{i+1} \mid 1 \le i < m\}$

where $\alpha_i = (i,-)$, $\beta_i = (i,+)$, $c_i = (\beta_i, a_i, \beta_{i+1})$, $d_i = (\beta_i, a_i, \alpha_{i+1})$, $f_i = (\alpha_i, a_i, \beta_{i+1})$ and $g_i = (\alpha_i, a_i, \alpha_{i+1})$.

Let $(Q, Sp, I)$ be a skewed-gentle triple. Let us consider a group of order 2, like $G = \{e, g \mid g^2 = e\}$, with the following left and right action on $A = kQ^{sg}/\langle I^{sg}\rangle$: $g(i,+) := (i,+)g := (i,-)$, $g(i,-) := (i,-)g := (i,+)$, $gj := jg := j$ for all $i \in Sp$ and all $j \in Q_0 \setminus Sp$, $g(\alpha, a, \beta) := (g\alpha, a, \beta)$, $(\alpha, a, \beta)g := (\alpha, a, \beta g)$, $g(uv) := (gu)v$ and $(uv)g := u(vg)$ for all $\alpha, \beta \in Q_0^{sg}$, $a \in Q_1$ and $u, v \in \mathcal{P}(Q^{sg}, I^{sg})$. We denote by $A_+$ the algebra generated by the elements of the form $(\alpha_1, a, \alpha_2)$ and $1_{\alpha_3}$, where $\alpha_i \ne (j,-), j \in Sp$. We remark that in general the units in $A$ and in $A_+$ are different.

**Lemma 3.** *The algebra $A_+$ is gentle.*

*Proof.* It is ease to see that $A_+ \cong kQ/\langle J\rangle$, where $J = I \setminus \{ab \mid ab \in I, e(a) \in Sp\}$. □

We set $\mathrm{M}_+(A) := \mathrm{M}_+(Q^{sg}, I^{sg}) := \mathrm{M}(A_+) = \mathrm{M}(A) \cap A_+$.

We define a $k$-linear map $\epsilon : A \to A_+$ by the following rule: $\epsilon((i,+)) := (i,+)$, $\epsilon((i,-)) := (i,+)$, $\epsilon(j) := j$ for all $i \in Sp$ and all $j \in Q_0 \setminus Sp$, $\epsilon((\alpha, a, \beta)) := (\epsilon(\alpha), a, \epsilon(\beta))$, which extends to paths by the rule: $\epsilon(uv) := \epsilon(u)\epsilon(v)$ for all $\alpha, \beta \in Q_0^{sg}$, $a \in Q_1$ and $u, v \in \mathcal{P}(Q^{sg}, I^{sg})$.

**Lemma 4.** *Let $(Q, Sp, I)$ be a skewed-gentle triple and $w \in \mathcal{P}_{\ge 1}(Q^{sg}, I^{sg})$. Then*

- *if $s(w) \notin A_+$ and $e(w) \in A_+$, then $w = g\epsilon(w)$;*

- *if* $s(w) \in A_+$ *and* $e(w) \notin A_+$*, then* $w = \epsilon(w)g$*;*
- *if* $s(w) \notin A_+$ *and* $e(w) \notin A_+$*, then* $w = g\epsilon(w)g$*.*

*Proof.* Evident. □

**Corollary 2.** *Let* $(Q, Sp, I)$ *be a skewed-gentle triple and* $w \in \mathcal{P}_{\geq 1}(Q^{sg}, I^{sg})$*. Then the following hold:*

- *if* $w \in A_+$*, then there exist maximal path* $m = m(w) \in A_+$ *and paths* $\omega(w), w' \in A_+$ *such that* $m = \omega(w)ww'$*;*
- *if* $s(w) \notin A_+$ *and* $e(w) \in A_+$*, then there exist maximal path* $m = m(w) \in A_+$ *and paths* $\omega(w), w' \in A_+$ *such that* $m = \omega(w)gww'$*;*
- *if* $s(w) \in A_+$ *and* $e(w) \notin A_+$*, then there exist maximal path* $m = m(w) \in A_+$ *and paths* $\omega(w), w' \in A_+$ *such that* $m = \omega(w)wgw'$*;*
- *if* $s(w) \notin A_+$ *and* $e(w) \notin A_+$*, then there exist maximal path* $m = m(w) \in A_+$ *and paths* $\omega(w), w' \in A_+$ *such that* $m = \omega(w)gwgw'$*.*

*In each case, paths* $m(w)$*,* $\omega(w)$ *and* $w'$ *are uniquely determined by* $w$*.*

*Proof.* By Lemma 3, $A_+$ is gentle. Hence the first statement follows from [BeMe]. The others statements follow from first statement and Lemma 4. For example, if $s(w) \notin A_+$ and $e(w) \notin A_+$, then $m(w) = m(\epsilon(w))$, $\omega(w) = \omega(\epsilon(w))$ and $w'$ is the same as for $\epsilon(w)$. □

## 2.5 Skew-group algebras

(Compare with [GePe], section 4.) Geiss and de la Peña define skewed-gentle $k$-algebras as (non necessarily basic) algebras where certain loops - together with their corresponding vertices - are distinguished to allow for naturally determining a group action of a group of order two. We find preferrable to, say, *create* two vertices for each ordinary one, and two arrows for each arrow, and to add some natural relations, obtaining in this way a basic algebra which is Morita equivalent to the other one.)

Let $A$ be a $k$-algebra, and $G$ a finite group acting on $A$ via $k$-linear automorphisms. The *skew-group algebra* $AG$ is the vector space $\oplus_{g\in G} A[g]$ with multiplication induced by

$$a[g]b[h] := ag(b)[gh].$$

Let $(Q, Sp, I)$ be a skewed-gentle triple. For a given special (resp., ordinary) vertice $i$ let us denote by $Q_0[i]$ the set $\{i\}$ (resp., $\{(i,-),(i,+)\}$). Consider the pair $(Q^g, I^g)$, where $Q_0^g := \cup_{i\in Q_0} Q_0[i]$, $Q_1^g := \{(a,+),(a,-) \mid a \in Q_1\}$,

$$s((a,\pm)) := \begin{cases} (s(a),\pm) & \text{, if } s(a) \notin Sp; \\ s(a) & \text{, if } s(a) \in Sp, \end{cases} \quad e((a,\pm)) := \begin{cases} (e(a),\pm) & \text{, if } e(a) \notin Sp; \\ e(a) & \text{, if } e(a) \in Sp \end{cases}$$

and $I^g := \{(a,+)(b,+),(a,-)(b,-) \mid ab \in I, e(a) \notin Sp\} \cup \{(a,+)(b,-),(a,-)(b,+) \mid ab \in I, e(a) \in Sp\}$. It follows from **[GePe]** that algebra $B := kQ^g/\langle I^g\rangle$ is gentle. Consider the group $G = \{e, g \mid g^2 = e\}$ which acts on $B$ defined by the rule:

$$g((i,+)) := (i,-),\ g(j) := j,\ g((a,+)) := (a,-)$$

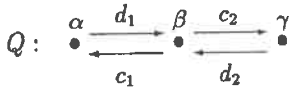
for all $i \in Q_0 \setminus Sp, j \in Sp$ and $a \in Q_1$. It follows from **[GePe]** that the skewed-gentle algebra $kQ^{sg}/\langle I^{sg}\rangle$ is Morita-equivalent to the skew-group algebra $BG$ in the case of $\operatorname{char} k \neq 2$.

Consider the group $G = \{e, g \mid g^2 = e\}$ whith acting on $A = kQ^{sg}/\langle I^{sg}\rangle$ defined by the rule:

$$g((i,+)) := (i,-),\ g(j) := j,\ g((\alpha, a, \beta)) := (g(\alpha), a, g(\beta)).$$

It is easy to see that the skew-group algebra $AG$ is gentle in the case of $\operatorname{char} k \neq 2$.

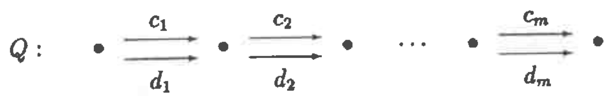
**Example 3.**




$I = \{c_i d_i, d_i c_i, i = 1, 2\}$

It is easy to see that $B = kQ/\langle I\rangle$ is a gentle algebra, Morita equivalent to $AG$, where $A$ is the skewed-gentle algebra of Example 1 (see Subsection 2.4).

**Example 4.**




$I = \{c_i d_{i+1}, d_i c_{i+1}\}$

It is easy to see that $B = kQ/\langle I\rangle$ is a gentle algebra, Morita equivalent to $AG$, where $A$ is the skewed-gentle algebra of Example 2 (see Subsection 2.4).

# 3 S-representations

(Cf. **[BeMe]**.)

We explain now the strategy we use from now on, and fix some notation. In the remaining of the paper we consider $A$ a skewed-gentle algebra and we want to describe the indecomposables of $\mathfrak{p}(A)$. We reduce this problem to a solved problem **[BoDr]**. We proceed as follows.

We consider $\mathcal{Y}$ a linearly ordered set and a fixed subset $\mathcal{Y}_*$, in this subset we assume we have defined an involution. We introduce a $k$-category $\mathcal{S}(\mathcal{Y}, k)$. This category is useful since the problem of finding the indecomposables of it can be reduced to a matrix problem solved in [Bo],[De]. In this paper we show that, when $A$ is skewed-gentle, we can construct a linearly ordered set $\mathcal{Y} = \mathcal{Y}(A)$ and subset $\mathcal{Y}_*$ with an involution defined on the subset, such that the problem of finding the indecomposables of $\mathfrak{p}(A)$ is equivalent to finding the indecomposables of a certain subcategory of $\mathcal{S}(\mathcal{Y}, k)$, which we describe completely.

Having described the strategy which we will use we start introducing the necessary preliminaries.

In this section $k$ is a field (as usual) and $\mathcal{Y}$ is a linearly ordered set (may be infinite) provided with an (fixed) involution $^*$ on some subset $\mathcal{Y}_* \subseteq \mathcal{Y}$ (*see* [BoDr]). For each $x \in \mathcal{Y}_*$ such that $x^* = x$ we introduce a new symbol $\bar{x}$ and let $\overline{\mathcal{Y}}$ denote the union of $\mathcal{Y}$ with the set of all $\bar{x}$. We extend the order to $\overline{\mathcal{Y}}$ assuming that the inequalities $\bar{x} < y$, $x < \bar{y}$ and $\bar{x} < \bar{y}$ (more precisely, those that have meaning) hold if and only if $x < y$.

Given two block matrices $B$ and $C$ (not necessarily square blocks), we say that the horizontal partition of $B$ is compatible with the vertical partition of $C$ if the number of rows in each $B_x$ is equal to the number of columns of each $C^x$ - so that we can multiply *$CB$ by blocks* -, and similarly we define what it means that the vertical partition of $B$ is compatible with the horizontal partition of $C$.

We define next the category $\mathcal{S}(\mathcal{Y}, k)$.

**Definition 5.** *The objects of $\mathcal{S}(\mathcal{Y}, k)$ are (finite) square block matrices, $B = B_x^y$ ($x, y \in \overline{\mathcal{Y}}$), called* S-representations *of $\mathcal{Y}$ or* $\mathcal{Y}$-matrices *with all the entries of all the blocks sitting in $k$, and verifying the following properties. (Notice that we represent the row $x$ of the blocks of $B$ by $B_x$, and the column $x$, by $B^x$. Notice also that some blocks may be empty.)*

- *The horizontal and vertical partitions of $B$ are compatible.*
- *If $x, y \in \mathcal{Y}_*$ are such that $y = x^*$, then all matrices in $B_x$ have the same number of rows as the matrices in $B_y$ (and, consequently, all matrices in $B^x$ have the same number of columns as the matrices in $B^y$).*
- $B^2 = 0$.

*A morphism of $\mathcal{S}(\mathcal{Y}, k)$ from $B$ to $C$ is a block matrix $T_x^y$ ($x, y \in \overline{\mathcal{Y}}$) with entries in $k$ such that the following are satisfied:*

- *the horizontal (resp., vertical) partition of $T$ is compatible with the vertical (resp., horizontal) partition of $B$ (resp., $C$);*
- $TC = BT$;
- *if $y \not\geq x$, then $T_x^y = 0$;*
- *if $x^* = y$, then $T_x^x = T_y^y$.*

It follows from the definition that $T$ is invertible if and only if all diagonal blocks $T_i^i$ are invertible.

It is clear that $\mathcal{S}(\mathcal{Y},k)$ is an additive $k$-category. It was shown in **[BoDr]** that finding the indecomposables of $\mathcal{S}(\mathcal{Y},k)$ can be reduced to finding the indecomposables of a matrix problem introduced and solved in **[NaRo]**. Presently, we show that, when $A$ is skewed-gentle, finding the indecomposables of $\mathfrak{p}(A)$ is equivalent to finding the indecomposables of certain subcategory of $\mathcal{S}(\mathcal{Y},k)$.

## 3.1 Bunches of semi-chains

We recall some definitions and results related to the bunches of semi-chains considered by Bondarenko in **[Bo]** and Deng in **[De]** in a form convenient for our purposes (see also **[CB]** for an alternative approach). We will use the classification of indecomposables representations of a bunch of semi-chains given in **[De]**.

**Definition 6.** *A bunch of semi-chains* $\mathbf{C} = \{\mathbf{I}, E_i, F_i, ^*\}$ *is defined by the following data:*

1. *A set* $\mathbf{I}$ *of indices;*
2. *Two chains (i.e., linearly ordered sets)* $E_i$ *and* $F_i$ *given for each* $i \in \mathbf{I}$*;*

   *Put* $\mathbf{E} := \cup_{i\in\mathbf{I}} E_i$, $\mathbf{F} := \cup_{i\in\mathbf{I}} F_i$ *and* $|\mathbf{C}| := \mathbf{E} \cup \mathbf{F}$.
3. *An involution* $^*$ *on some subset* $|\mathbf{C}|_* \subseteq |\mathbf{C}|$.

We consider the ordering on $|\mathbf{C}|$, which is just the union of all orderings on $E_i$ and $F_i$ (i.e., $a < b$ means that $a$ and $b$ belong to the same chain $E_i$ or $F_i$ and $a < b$ in this chain).

For each $x \in |\mathbf{C}|_*$ such that $x^* = x$ we introduce a new symbol $\bar{x}$ and let $\overline{E_i}$ (resp., $\overline{F_i}$) denote the union of $E_i$ (resp., $F_i$) with the set of all $\bar{x}$ for $x \in E_i$ (resp., $x \in F_i$ ). We extend the partial order to $\overline{E_i}$ and $\overline{F_i}$ as we did above for $\overline{\mathcal{Y}}$, and we put $\overline{\mathbf{E}} := \cup_{i\in\mathbf{I}} \overline{E_i}$, $\overline{\mathbf{F}} := \cup_{i\in\mathbf{I}} \overline{F_i}$ and $\overline{|\mathbf{C}|} = \overline{\mathbf{E}} \cup \overline{\mathbf{F}}$. We extend the partial order to $\overline{|\mathbf{C}|}$ as we did above for $|\mathbf{C}|$.

We recall the definition of the category $\operatorname{rep}\mathbf{C}$ of representations of $\mathbf{C}$ (see **[Bo]**).

**Definition 7.** *The objects of* $\operatorname{rep}\mathbf{C}$ *are sets* $\mathbf{A} = \{A(i) \mid i \in \mathbf{I}\}$ *of block matrices* $A(i) = (A_x^y)$ $(x \in \overline{F_i}, y \in \overline{E_i})$ *with entries in* $k$*, and verifying the following properties. (Notice that we represent the row* $x$ *of the blocks of* $A(i)$ *by* $A_x$*, and the column* $y$*, by* $A^y$*. Notice also that some blocks may be empty.)*

- *if* $x^* = y \neq x$*, where* $x, y \in \mathbf{F}$ *(resp.,* $x, y \in \mathbf{E}$*), then the number of rows in* $A_x$ *and* $A_y$ *(resp., the number of columns in* $A^x$ *and* $A^y$*) is the same;*
- *if* $x^* = y \neq x$*, where* $x \in \mathbf{F}, y \in \mathbf{E}$*, then the number of rows in* $A_x$ *is equal to the number of columns in* $A^y$*.*

*A morphism* $\mathbf{S} : \mathbf{A} \to \mathbf{B}$ *in* rep $\mathbf{C}$ *is a set of pairs of block matrices* $\mathbf{S} = \{(S(i,1), S(i,2) \mid i \in \mathbf{I}\}$ *with entries in* $k$*, where* $S(i,1) = (S_x^y)$ $(x, y \in \overline{F_i})$, $S(i,2) = (S_x^y)$ $(x, y \in \overline{E_i})$*, such that:*

- *the horizontal (resp., vertical) partition of* $S(i,1)$ *is compatible with the horizontal partition of* $A(i)$ *(resp.,* $B(i)$*);*
- *the vertical (resp., horizontal) partition of* $S(i,2)$ *is compatible with the vertical partition of* $B(i)$ *(resp.,* $A(i)$*);*
- $S(i,1)B(i) = A(i)S(i,2)$ *for all* $i \in \mathbf{I}$*;*
- *if* $y \not\geq x$ *in* $\overline{E_i}$ *(or* $\overline{F_i}$*), then* $S_x^y = 0$*;*
- *if* $x^* = y \neq x$ *for* $x, y \in |\mathbf{C}|_*$*, then* $S_x^x = S_y^y$*.*

¿From now onwards, we suppose that $\mathbf{C}$ is *complete*, i.e. that $|\mathbf{C}|_* = |\mathbf{C}|$.

**Definition 8.** *Let* $\mathbf{C} = \{\mathbf{I}, E_i, F_i, {}^*\}$ *be a complete bunch of semi-chains. We give now the following related definitions.*

- *WORDS.*
  - *A* $\mathbf{C}$-word *is a sequence* $w = w_1 w_2 \cdots w_m$ *of elements of* $|\mathbf{C}|$ *such that* $w_i^* | w_{i+1}$ *for all* $i < m$*, where, by definition* $u|v$ *if and only if* $u \in E_i$*,* $v \in F_i$ *for some* $i \in \mathbf{I}$ *or vice versa.*
  - *The* reverse *for a* $\mathbf{C}$*-word* $w = w_1 w_2 \cdots w_m$ *is the sequence* $w^* := w_m^* \cdots w_2^* w_1^*$*.*
  - *Call a* $\mathbf{C}$*-word* $w$ symmetric *if* $w^* = w$ *and* asymmetric *otherwise.*
- *PERIODIC WORDS.*
  - *A* periodic $\mathbf{C}$-word *is a sequence* $w = (w_i)_{i \in \mathbf{Z}}$ *of elements of* $|\mathbf{C}|$ *such that* $w_i^* | w_{i+1}$ *and* $w_{i+\pi} = w_i$ *for some* $\pi > 0$ *and all* $i \in \mathbf{Z}$*. The smallest* $\pi$ *satisfying these conditions is the* period *of* $w$*.*
  - *The* reverse *for a periodic* $\mathbf{C}$*-word* $w = (w_i)_{i \in \mathbf{Z}}$ *is the sequence* $w^* := (u_i)_{i \in \mathbf{Z}}$ *such that* $u_i := w_{-i}^*$ *for all* $i \in \mathbf{Z}$*.*
  - *The* $p$ translates *(*$p \in \mathbf{Z}$*) for a periodic* $\mathbf{C}$*-word* $w = (w_i)_{i \in \mathbf{Z}}$ *are the sequences* $w[p] := (w[p]_i)_{i \in \mathbf{Z}}$ *such that* $w[p]_i := w_{i+p}$ *for all* $i \in \mathbf{Z}$*.*
  - *Call a periodic* $\mathbf{C}$*-word* $w = (w_i)_{i \in \mathbf{Z}}$ symmetric *if* $w^* = w[p]$ *for some* $p$ *and* asymmetric *otherwise.*

*We remark that in general a periodic* $\mathbf{C}$*-word is not a* $\mathbf{C}$*-word.*

We will consider two equivalence relations on the set of C-words and periodic C-words, which will be denoted by $\cong_s$ and by $\cong_r$. By definition, $\cong_s$ is the equivalence relation on the set of all C-words, given by $u \cong_s w \Leftrightarrow u = w^*$; and $\cong_r$ is the equivalence relation on the set of all periodic C-words which identifies each periodic C-word with its translations and their reverses.

We denote by $\mathrm{Irr}_0\, k[x]$ the set of all irreducible polynomial $f(x) \neq x$ over a field $k$ with leading coefficient 1.

We set $\mathrm{Ind}_0\, k[x] := \{f^n \mid f \in \mathrm{Irr}_0\, k[x], n \in \mathbb{N}\}$.

We denote by $\mathcal{M}$ the set of the following matrices ($n \geq 0$):

$$\begin{pmatrix} 0\mathbf{1}_n & \mathbf{1}_n \\ \mathbf{1}_{n+1} & (\mathbf{1}_n 0)^T \end{pmatrix}, \begin{pmatrix} (\mathbf{1}_n 0)^T & \mathbf{1}_{n+1} \\ \mathbf{1}_n & 0\mathbf{1}_n \end{pmatrix}, \begin{pmatrix} \mathbf{1}_{n+1} & \mathbf{1}_{n+1} \\ \mathbf{1}_{n+1} & J_{n+1} \end{pmatrix}, \begin{pmatrix} J_{n+1} & \mathbf{1}_{n+1} \\ \mathbf{1}_{n+1} & \mathbf{1}_{n+1} \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{1}_{n+1} & (\mathbf{1}_n 0)^T \\ 0\mathbf{1}_n & \mathbf{1}_n \end{pmatrix}, \begin{pmatrix} \mathbf{1}_n & 0\mathbf{1}_n \\ (\mathbf{1}_n 0)^T & \mathbf{1}_{n+1} \end{pmatrix}, \begin{pmatrix} \mathbf{1}_{n+1} & \mathbf{1}_{n+1} \\ J_{n+1} & \mathbf{1}_{n+1} \end{pmatrix}, \begin{pmatrix} \mathbf{1}_{n+1} & J_{n+1} \\ \mathbf{1}_{n+1} & \mathbf{1}_{n+1} \end{pmatrix},$$

$$\begin{pmatrix} F_{f(x)} & \mathbf{1}_{n+1} \\ \mathbf{1}_{n+1} & \mathbf{1}_{n+1} \end{pmatrix},$$

where

$$J_{n+1} = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 1 \\ 0 & 0 & 0 & \cdot & 0 & 0 \end{pmatrix} \quad \text{and} \quad F_{f(x)} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & 1 \\ a_t & a_{t-1} & a_{t-2} & a_{t-3} & \cdot & a_2 & a_1 \end{pmatrix},$$

and where $f(x) = x^t - a_1 x^{t-1} - \cdots - a_t \in \mathrm{Ind}_0\, k[x], f(x) \neq (x-1)^t$.

For each asymmetric C-word $w$ (resp., pair $(w, i)$, where $w$ is a symmetric C-word and $i \in \{0, 1\}$ ), Deng constructed in [De] some indecomposable representation of C, which we will denote by $\mathbf{T}(w)$ (resp., $\mathbf{T}(w, i)$), and for each pair $(w, f(x))$ (resp., $(w, M)$), where $w$ is an assymetric (resp., symmetric) periodic C-word and $f(x)$ is an indecomposable polynomial from $\mathrm{Ind}_0\, k[x]$ (resp., $M$ is a matrix from the set $\mathcal{M}$), some indecomposable representation of C, which we will denote by $\mathbf{T}(w, f(x))$ (resp., $\mathbf{T}(w, M)$) (see [De] for details).

A representation of a bunch of semi-chains C, that is isomorphic to some $\mathbf{T}(w)$ (resp., $\mathbf{T}(w, i)$, $\mathbf{T}(w, f(x))$ or $\mathbf{T}(w, M)$) will be called an *assymetric string* (resp., *dimidiate string*, *assymetric band* or *dimidiate band*).

We denote by $\Omega$ (resp., $\Omega_p$) the set of all C-words (resp., all periodic C-words).

We denote by $\Omega_a$ (resp., $\Omega_s$) a fixed set of representatives of asymmetric (resp., symmetric) C-words over the equivalence relation $\cong_s$ and we denote by $\Omega_{ap}$ (resp., $\Omega_{sp}$) a fixed set of representatives of asymmetric (resp., symmetric) periodic C-words over the equivalence relation $\cong_r$.

**Theorem 1.** *[De],[Bo]. Let* **C** *be a complete bunch of semi-chains. Then each indecomposable representation of* **C** *is a string (asymmetric or dimidiate) or a band (asymmetric or dimidiate). The representations* $\mathrm{T}(\delta)$, *where*

$$\delta \in \Omega_a \amalg \Omega_s \times \{0,1\} \amalg \Omega_{ap} \times \mathrm{Ind}_0\, k[x] \amalg \Omega_{sp} \times \mathcal{M},$$

*constitute an exhaustive list of pairwise non-isomorphic indecomposable representations of the bunch of semi-chains* **C**.

# 4 The functor

Let $A = kQ^{sg}/\langle I^{sg}\rangle$ be a skewed-gentle algebra and let $(Q, Sp, I)$ be the corresponding skewed-gentle triple (see Section 2). Set $\mathcal{P} := \mathcal{P}(Q^{sg}, I^{sg})$ and $\mathcal{P}_{\geq 1} := \mathcal{P}_{\geq 1}(Q^{sg}, I^{sg})$. To begin with, we will fix a finite projective complex, $P^\bullet$, of length $m$, with the property that the images of all differential maps are contained in the radical of the corresponding module (in other words, $P^\bullet \in \mathfrak{p}(A)$)

$$P^\bullet : \quad P^n \overset{\partial^n}{\to} \cdots \overset{\partial^{n+m-2}}{\to} P^{n+m-1} \overset{\partial^{n+m-1}}{\to} P^{n+m} \quad , n, m \in \mathbf{Z}.$$

(Cf. for all what follows, [BeMe].)

We denote by $\mathbf{P}_i$ the indecomposable projective corresponding to the vertex $i \in Q_0^{sg}$ and by $p(w)$ the morphism between two indecomposable projectives corresponding to the path $w$ in $(Q^{sg}, I^{sg})$. Let us say that in each $P^j$ of the complex $P^\bullet$, the indecomposable $\mathbf{P}_i$ appears, $d_{i,j}$ times or, simplifying our notations, that $\mathbf{P}_i^{d_{i,j}}$ is the component of $P^j$ envolving the indecomposable $\mathbf{P}_i$. Thus, we can rewrite our complex as

$$\oplus_{i=1}^t \mathbf{P}_i^{d_{i,n}} \overset{\partial^n}{\to} \cdots \overset{\partial^{n+m-2}}{\to} \oplus_{i=1}^t \mathbf{P}_i^{d_{i,n+m-1}} \overset{\partial^{n+m-1}}{\to} \oplus_{i=1}^t \mathbf{P}_i^{d_{i,n+m}}.$$

As it is well known, each morphism between projectives (these being finite direct sums of indecomposables) is given by a block matrix, each block giving the morphism component that corresponds to each pair of indecomposables. In other words, each block matrix corresponds to a morphism $\mathbf{P}_r^{d_{r,j}} \to \mathbf{P}_s^{d_{s,j+1}}$. And, as we know, the paths $w \in \mathcal{P}, s(w) = r, e(w) = s$ form a basis of the morphisms space, Hom $(\mathbf{P}_r, \mathbf{P}_s)$, but in our particular case of the category $\mathfrak{p}(A)$ we can assume that only paths $w \in \mathcal{P}_{\geq 1}$ are involved.

(If $w$ is as indicated, it defines the morphism $p(w)$ from $\mathbf{P}_r$ to $\mathbf{P}_s$ consisting in multiplication times $w$ *on the right*: $u \mapsto v = uw$. Any homomorphism from $\mathbf{P}_r$ to $\mathbf{P}_s$ is associated then to a linear combination of paths like $w$.)

Hence, in order to represent our complex, we need to give a matrix, say, $X = (X_{r,j}^{s,j+1})$ determining the sequence of morphisms $\partial^j, (j = n, ..., n+m-1)$ which in turn determine our complex. In particular, we have to represent the family of morphisms $p(w)$ which appear in $\partial^j : P^j \to P^{j+1}$. To facilitate to remember, it is now convenient that we use a formal sum

$$\partial^j : \sum_{w \in \mathcal{P}_{\geq 1}} X_{w,j} p(w),$$

where $X_{w,j}$ denotes the matrix block that expresses the "multiplicities" of the morphism $p(w)$ in the component corresponding to $\mathbf{P}_s^{d_s,j+1}$ of the restriction of $\partial^j$ to $\mathbf{P}_r^{d_r,j}$. Let us explain this in a detailed way:

Fixed the place $j$ the component of $\partial^j$ going from $\mathbf{P}_r^{d_r,j}$ to $\mathbf{P}_s^{d_s,j+1}$ is represented by a matrix $X_{r,j}^{s,j+1} \in \mathrm{Mat}(d_r \times d_s; k\langle p(w_1), \ldots, p(w_t)\rangle)$, where $w_i$'s are the parallel non trivial paths from $r$ to $s$ and $k\langle p(w_1), \ldots, p(w_t)\rangle$ is the $k$-vector space with basis $\{p(w_1), \ldots, p(w_t)\}$. It is clear that $X_{r,j}^{s,j+1}$ can be writing uniquely as

$$X_{r,j}^{s,j+1} = \sum_{i=1}^{t} X_{w_i,j} p(w_i),$$

where $X_{w_i,j} \in \mathrm{Mat}(d_r \times d_s; k)$.

(It should be kept in mind that our convention is that the indecomposable projectives appearing in the domain of our $\partial^j$, say, correspond to rows, whereas the indecomposable appearing in the co-domain (target) correspond to columns.)

The condition $\partial^i \partial^{i+1} = 0$ is equivalent to:

$$\sum_{w_1 \in \mathcal{P}_{\geq 1}, w_2 \in \mathcal{P}_{\geq 1} : w = w_1 w_2} X_{w_1,j} X_{w_2,j+1} = 0 \tag{1}$$

for all $w \in \mathcal{P}_{\geq 2}$ and all $j \in \mathbb{Z}$.

Now, let us consider a morphism $\varphi^\bullet : P^\bullet \to P'^\bullet$ between two complexes in $\mathfrak{p}(A)$. At each place, the morphism $\varphi^\bullet$ is a homomorphism from the projective, $P^j$ to the projective $P'^j$; that is, a block matrix between the direct sums of indecomposable projectives. By representing the blocks of $\varphi^j$ similarly as how we did with the differentials maps, by $\phi_{w,j}$, and the blocks of the differential maps of $P'^\bullet$ by $X'_{w,j}$, we must have that

$$\sum_{w_1 \in \mathcal{P}, w_2 \in \mathcal{P}_{\geq 1} : w = w_1 w_2} \phi_{w_1,j} X'_{w_2,j} = \sum_{w_3 \in \mathcal{P}_{\geq 1}, w_4 \in \mathcal{P} : w = w_3 w_4} X_{w_3,j} \phi_{w_4,j+1} \tag{2}$$

The preceding ideas and Corollary 2 lead us to the definition of our poset $\mathcal{Y} = \mathcal{Y}(A)$. It has to be a product of two posets: the first corresponding to the paths and the second to the places $j$. As a matter of fact, we introduce a poset $\mathcal{Y}_w = \mathcal{Y}_w(A)$ for each path $w \in \mathrm{M}_+ = \mathrm{M}_+(A)$. It is the set of the subpaths $u$ of $w$ such that $s(u) = s(w)$, ordered by the lenght of them. Then our definitions are the following.

$$\mathcal{Y} = (\dot{\cup}_{w \in \mathrm{M}_+} \mathcal{Y}_w) \times \mathbb{Z},$$

where the first component is an ordered disjoint union and the second one is the set of the integers and where we order the two products antilexicographically. (As for this, we assume to have given some linear ordering, fixed, to $M_+$.) This means that $[u,i] < [v,j]$ if and only if $i < j$ or ($i = j$ and $m(u) < m(v)$) or ($i = j$, $m(u) = m(v)$ and $l(u) < l(v)$) (see Corollary 2 for the meaning of the notation $m(w)$). It should be observed that it is possible that a (trivial) path $u$ belongs to two differen paths from $M_+$. If it is so, the two occurrencies of $u$ must be regarded, obviously, as *different*.

Next, we indicate how to define the subset $\mathcal{Y}_* \subseteq \mathcal{Y}$ and the involution $^*$ on $\mathcal{Y}_*$. For given $u, v \in \dot{\bigcup}_{w \in M_+} \mathcal{Y}_w$, we state that $u^* = v$ if and only if either $u \neq v$ and $e(u) = e(v)$ or $u = v$ and $e(u) \in Sp$. Then we state that $[u,i]^* = [v,j]$ if and only if $i = j$ and $u^* = v$.

Next, to facilitate our exposition, let us introduce maps $\gamma_1 : \overline{\mathcal{Y}} \to \mathcal{P}$ and $\gamma_2 : \overline{\mathcal{Y}} \to \mathbb{Z}$, which we define according to the following rules:

- $\gamma_1([u,i]) := u$ and $\gamma_1(\overline{[u,i]}) := ug$;
- $\gamma_2([u,i]) := i$ and $\gamma_2(\overline{[u,i]}) := i$.

In order to fulfill all our promises, it will be enough to define a functor, $\mathbf{F}$, from the category $\mathfrak{p}(A)$ to the category $\mathcal{S}(\mathcal{Y}, k)$ and show that it respects and preserves indecomposable objects. This is what we do next.

In objects,

$$\mathbf{F}(P^\bullet)_x^y = \begin{cases} X_{w,\gamma_2(x)} & \text{, if } \gamma_2(y) = \gamma_2(x) + 1, \gamma_1(y) = \gamma_1(x)w, w \in \mathcal{P}_{\geq 1}; \\ 0 & \text{, otherwise,} \end{cases}$$

where the block $F(P^\bullet)_x$ (resp., $F(P^\bullet)^x$) has $d_{\gamma_1(x),\gamma_2(x)}$ rows (resp., columns).

In morphisms,

$$\mathbf{F}(\varphi^\bullet)_x^y = \begin{cases} \phi_{w,\gamma_2(x)} & \text{, if } \gamma_2(y) = \gamma_2(x), \gamma_1(y) = \gamma_1(x)w, w \in \mathcal{P}; \\ 0 & \text{, otherwise.} \end{cases}$$

It follows easily that $(\mathbf{F}(P^\bullet))^2 = 0$ and that $\mathbf{F}(\varphi^\bullet)$ is a morphism of $\mathcal{S}(\mathcal{Y}, k)$ for all $P^\bullet \in \operatorname{Ob} \mathfrak{p}(A)$ and all $\varphi^\bullet \in \operatorname{Mor} \mathfrak{p}(A)$.

**Example 5.**

*Let $(Q, I, Sp)$ be the skewed-gentle triple of Example 1 (see Subsection 2.4) and $A = kQ^{sg}/\langle I^{sg}\rangle$ be the corresponding skewed-gentle algebra. We will use the notations of Example 1. Firstly, we look for the set $M_+$ of maximal paths in the algebra $A_+$. We see that there is only one maximal path, so that $M_+ = \{a_1 b_1\}$, and $\mathcal{P}_{\geq 1}(Q^{sg}, I^{sg}) = \{a_1, a_2, b_1, b_2, a_1 b_1\}$. Hence, the poset $\mathcal{Y}$ will be*

$$\{1_\beta < a_1 < a_1 b_1\} \times \mathbb{Z},$$

*and the involution* $^*$ *will be given by* $[1_\beta, j]^* = [a_1b_1, j]$ *and* $[a_1, j]^* = [a_1, j]$. *The poset* $\overline{\mathcal{Y}}$ *will be*

$$\{1_\beta < \{a_1, \overline{a_1}\} < a_1b_1\} \times \mathbb{Z}.$$

*We see that the differential maps correspond to the formal sums*

$$\partial^j = X_{a_1,j}p(a_1) + X_{a_2,j}p(a_2) + X_{b_1,j}p(b_1) + X_{b_2,j}p(b_2) + X_{a_2b_2,j}p(a_2b_2).$$

*Now, let us consider the following projective complex* $P^\bullet$*:*

$$\cdots \to 0 \to P^1 = \mathbf{P}_\beta \xrightarrow{\partial^1} P^2 = \mathbf{P}_\alpha^2 \oplus \mathbf{P}_\beta \oplus \mathbf{P}_\gamma \to 0 \to \cdots,$$

*where*

$$\partial^1 = \begin{pmatrix} 2p(a_1) & 0 & p(a_1b_1) & p(a_2) \end{pmatrix}.$$

*Then we have* $X_{a_1,1} = (2\ 0)$, $X_{a_1b_1,1} = (1)$, $X_{a_2,1} = (1)$, $X_{b_1,1} = 0$, $X_{b_2,1} = 0$, $\mathbf{F}(P^\bullet)_{[1_\beta,1]}^{[a_1,2]} = X_{a_1,1}$, $\mathbf{F}(P^\bullet)_{[1_\beta,1]}^{[a_1b_1,2]} = X_{a_1b_1,1}$, $\mathbf{F}(P^\bullet)_{[1_\beta,1]}^{\overline{[a_1,2]}} = X_{a_2,1}$ *and* $\mathbf{F}(P^\bullet)_x^y$ *is a zero or the empty matrix in other cases.*

Let $\mathcal{U}$ be the full subcategory of $\mathcal{S}(\mathcal{Y}, k)$ defined by the objects of $\operatorname{Im}\mathbf{F}$. We can prove the following lemma which has, clearly, the corollary that follows it. Remember that we use the symbol **Ver** to denote the set of objects of a skeleton of a Krull-Schmidt category.

**Lemma 5.** *Let* $\mathbf{F}$ *and* $\mathcal{U}$ *be as above. Then*

- $\operatorname{Ker}\mathbf{F} = 0$*;*
- $X \cong Y$ *in* $\mathcal{U}$ *if and only if* $X \cong Y$ *in* $\operatorname{Im}\mathbf{F}$.

*Proof.* The proof is similar to the proof of the Lemma 3 in [BeMe]. □

**Corollary 3.** $\mathbf{Ver}\operatorname{Im}\mathbf{F} = (\mathbf{Ver}\,\mathcal{S}(\mathcal{Y}, k)) \cap \operatorname{Im}\mathbf{F} = \mathbf{Ver}\,\mathcal{U}$.

# 5 Description of the indecomposables

In this Section, $k$ is a field and $A$ denotes a finite-dimensional skewed-gentle algebra of the form $kQ^{sg}/\langle I^{sg}\rangle$, where $(Q, Sp, I)$ is the corresponding skewed-gentle triple (see Subsection 2.4 for the definition). We will identify the algebra $A_+$ with the algebra $kQ/\langle J\rangle$, where $J = I \setminus \{ab \mid ab \in I, e(a) \in Sp\}$.

## 5.1 Generalized strings and bands

In this Subsection we use the following notations: $\mathcal{P} := \mathcal{P}(Q,J)$ and $\mathcal{P}_{\geq 1} := \mathcal{P}_{\geq 1}(Q,J)$. Note that this notations are distinct from the same given in Section 4. Given an arrow $a$ of $Q$, let us denote by $a^{-1}$ a formal inverse of $a$, and let us set $s(a^{-1}) = e(a)$ and $e(a^{-1}) = s(a)$, and let us extend it, as usual, writing $(a^{-1})^{-1} = a$. For each path $p = a_1 \ldots a_n$ we define $(a_1 \ldots a_n)^{-1} = a_n^{-1} \ldots a_1^{-1}$, $s(p^{-1}) = e(p)$ and $e(p^{-1}) = s(p)$.

By a *walk* $w$ (resp., a *generalized walk*) of length $n > 0$ we mean a sequence $w_1 \cdots w_n$ where each $w_i$ is either of the form $p$ or $p^{-1}$, $p$ being a path of length 1 (i. e. an arrow) (resp., a path of length $> 0$ ) in $(Q,J)$ ($=$ in $A_+$) and where $s(w_{i+1}) = e(w_i)$ for $1 \leq i < n$. Again, $s(w) = s(w_1)$ and $e(w) = e(w_n)$. As usual, we consider *inverses* of a walk (resp., generalized walk). It is clear that passage to inverse is an involutory transformation.

If we have a closed walk (resp., closed generalized walk), *i.e.* it happens that $s(w) = e(w)$ we consider also its , *rotations*, $w[j]$, which are the walks (generalized walks) $w_{j+1} \cdots w_n w_1 \cdots w_j$ $(j = 1, \cdots, n-1)$.

The product ($=$ concatenation) of two walks (resp., generalized walks) $w = w_1 \cdots w_n$ and $w' = w'_1 \cdots w'_{n'}$ is defined as the walk (resp., generalized walk) $ww' = w_1 \cdots w_n w'_1 \cdots w'_{n'}$ provided that $e(w_n) = s(w'_1)$.

We will consider two equivalence relations on the set of generalized walks, which will be denoted by $\cong_s$ and by $\cong_r$. By definition, $\cong_s$ is the equivalence relation on the set of all generalized walks, generated by stablishing that $u \cong_s w \Leftrightarrow u = w^{-1}$; and $\cong_r$ is the equivalence relation on the set of all closed generalized walks which identifies each generalized walk with its rotations and their inverses.

By definition, a *string* is a walk $w = w_1 \cdots w_n$ such that $w_{i+1} \neq w_i^{-1}$, for $1 \leq i < n$ and such that no subword of $w$ or of $w^{-1}$ is in $J$. The set of all strings will be denoted by $St$.

With $\overline{GSt}$ let us denote the set of all generalized walks $w = w_1 \cdots w_n$ satisfying

- if $w_i, w_{i+1} \in \mathcal{P}_{\geq 1}$ and $e(w_i) \notin Sp$, then $w_i w_{i+1} \in J$;
- if $w_i^{-1}, w_{i+1}^{-1} \in \mathcal{P}_{\geq 1}$ and $e(w_i) \notin Sp$, then $w_{i+1}^{-1} w_i^{-1} \in J$;
- if $w_i, w_{i+1}^{-1} \in \mathcal{P}_{\geq 1}$ or $w_i^{-1}, w_{i+1} \in \mathcal{P}_{\geq 1}$, and $e(w_i) \notin Sp$, then $w_i w_{i+1} \in St$.

We denote by $GSt$ a fixed set of representatives $w$ of $\overline{GSt}$ over the equivalence relation $\cong_s$ and all trivial paths, and its elements will be called *generalized strings*.

Call a nontrivial generalized string $w$ *symmetric* if $w = w^{-1}$ and *asymmetric* otherwise. Call a trivial generalized string $1_i$ *symmetric* if $i \in Sp$ and *assymetric* otherwise We denote by $GSt_s$ the subset of all symmetric generalized strings and put $GSt_a := GSt \setminus GSt_s$.

Similarly, we define the *generalized bands* in the following way.

Given the generalized walk $w = w_1 \cdots w_n$, we put, for $1 \leq i \leq n$,

$$\mu_w(0) = 0, \mu_w(i) = \mu_w(i-1) + 1 \text{ (if } w_i \in \mathcal{P}_{\geq 1}) \text{ or } \mu_w(i) = \mu_w(i-1) - 1 \text{ (otherwise)}.$$

After this, let us consider the set $\overline{GBa}$ of all closed generalized walks $w = w_1 \cdots w_n$ (*i.e.* $e(w_n) = s(w_1)$) such that $w^2 \in \overline{GSt}$, such that $\mu_w(n) = \mu_w(0)$ and such that they are not themselves powers. We use $GBa$ for a fixed set of representatives of the quotient set of $\overline{GBa}$ over the equivalence relation $\cong_r$ and we call its elements *generalized bands.* Call a generalized band $w$ *symmetric* if $w = w^{-1}[t]$ for some $t$ and *asymmetric* otherwise. We denote by $GBa_s$ the subset of all symmetric generalized bands and put $GBa_a := GBa \setminus GBa_s$.

**Remark 3.** *In practice, we assume that $\mu_w(0) \leq \mu_w(n)$. One is allowed to do this (inverting $w$ if necessary) because if $\mu_w(0) \geq \mu_w(n)$, then $\mu_{w^{-1}}(0) \leq \mu_{w^{-1}}(n)$.*

We will need the following

**Lemma 6.** *Let $w = w_1 \cdots w_n$ be a generalized walk. Then*

$$\mu_w(i) - \mu_{w^{-1}}(n-i) = \mu_w(n) \text{ for any } 0 \leq i \leq n.$$

*Proof.* Straightforward. □

We define an order relation on the set $\overline{GSt}$ by the following rule: $u = u_1 \cdots u_m < v = v_1 \cdots v_n$ if and only if one of the following conditions hold:

- $v_i = u_i$ for $1 \leq i \leq m$ and $v_{m+1}^{-1} \in \mathcal{P}_{\geq 1}$;
- $u_i = v_i$ for $1 \leq i \leq n$ and $u_{n+1} \in \mathcal{P}_{\geq 1}$;
- $v_i = u_i$ for $i < s$ (it is always true if $s = 1$), $u_s, v_s \in \mathcal{P}_{\geq 1}$ and $l(\omega(u_s)u_s) < l(\omega(v_s)v_s)$ (see Corollary 2 for the meaning of the notation $\omega(u_i)$);
- $v_i = u_i$ for $i < s$, $u_s^{-1}, v_s^{-1} \in \mathcal{P}_{\geq 1}$ and $l(\omega(u_s^{-1})) < l(\omega(v_s^{-1}))$.

## 5.2 String and band complexes

In this subsection we associate to generalized strings and bands certain finite projective complexes which, as we shall see, give all the indecomposables in the category $\mathfrak{p}(A)$.

For each $i \in Q_0$ we put

$$P(i) := \begin{cases} \mathbf{P}_i \oplus \mathbf{P}_{gi} & \text{, if } i \in Sp; \\ \mathbf{P}_i & \text{, otherwise.} \end{cases}$$

We consider the following left and right action of the group $G = \{e, g \,|\, g^2 = e\}$ (see Subsection 2.4) on $A-\text{pro}$ : $gp(w) := p(gw)$, $p(w)g := p(wg)$ for all $w \in \mathcal{P}(Q^{sg}, I^{sg})$.

For given $w = w_1 \cdots w_m \in \overline{GSt}$ and $i = 1, \cdots, m$, we define two matrices $G(w,i)$ and $H(w,i)$ over $kG$ by the following rule:

- *Case $w_i \in \mathcal{P}_{\geq 1}$. Construction of $G(w,i)$:*
  If $s(w_i) \notin Sp$, we set $G(w,i) := (e)$.
  If $s(w_i) \in Sp$ and $i = 1$, we set $G(w,i) := \begin{pmatrix} e & g \end{pmatrix}^T$.
  If $s(w_i) \in Sp$, $i > 1$ and $w_{i-1} \in \mathcal{P}_{\geq 1}$, we set $G(w,i) := \begin{pmatrix} e & -g \end{pmatrix}^T$.
  If $s(w_i) \in Sp$, $i > 1$ and $w_{i-1}^{-1} \in \mathcal{P}_{\geq 1}$, we set $G(w,i) := \begin{pmatrix} 0 & g \end{pmatrix}^T$ provided $w_{i-1}^{-1} \cdots w_1^{-1} \leq w_i \cdots w_m$ and $G(w,i) := \begin{pmatrix} e & g \end{pmatrix}^T$ otherwise.
- *Case $w_i \in \mathcal{P}_{\geq 1}$. Construction of $H(w,i)$:*
  If $e(w_i) \notin Sp$, we set $H(w,i) := (e)$.
  If $e(w_i) \in Sp$, we set $H(w,i) := \begin{pmatrix} e & 0 \end{pmatrix}$ provided $i < m$, $w_{i+1}^{-1} \in \mathcal{P}_{\geq 1}$ and $w_i^{-1} \cdots w_1^{-1} \leq w_{i+1} \cdots w_m$, and $H(w,i) := \begin{pmatrix} e & g \end{pmatrix}$ otherwise.
- *Case $w_i^{-1} \in \mathcal{P}_{\geq 1}$. Construction of $G(w,i)$:*
  If $e(w_i) \notin Sp$, we set $G(w,i) := (e)$.
  If $e(w_i) \in Sp$ and $i = m$, we set $G(w,i) := \begin{pmatrix} e & g \end{pmatrix}^T$.
  If $e(w_i) \in Sp$, $i < m$ and $w_{i+1}^{-1} \in \mathcal{P}_{\geq 1}$, we set $G(w,i) := \begin{pmatrix} e & -g \end{pmatrix}^T$.
  If $e(w_i) \in Sp$, $i < m$ and $w_{i+1} \in \mathcal{P}_{\geq 1}$, we set $G(w,i) := \begin{pmatrix} e & 0 \end{pmatrix}^T$ provided $w_i^{-1} \cdots w_1^{-1} > w_{i+1} \cdots w_m$, and $G(w,i) := \begin{pmatrix} e & g \end{pmatrix}^T$ otherwise.
- *Case $w_i^{-1} \in \mathcal{P}_{\geq 1}$. Construction of $H(w,i)$:*
  If $s(w_i) \notin Sp$, we set $H(w,i) := (e)$.
  If $s(w_i) \in Sp$, we set $H(w,i) := \begin{pmatrix} 0 & g \end{pmatrix}$ provided $i > 1$, $w_{i-1} \in \mathcal{P}_{\geq 1}$ and $w_{i-1}^{-1} \cdots w_1^{-1} > w_i \cdots w_m$, and $H(w,i) := \begin{pmatrix} e & g \end{pmatrix}$ otherwise.

For given $i = 1, \cdots, m$ we define a matrix $F(w,i)$, defining a morphism in $A-$pro, by the following rule:

- $F(w,i) := G(w,i)p(w_i)H(w,i)$ if $w_i \in \mathcal{P}_{\geq 1}$;
- $F(w,i) := G(w,i)p(w_i^{-1})H(w,i)$ if $w_i^{-1} \in \mathcal{P}_{\geq 1}$.

**Definition 9.** • *For each nontrivial generalized string $w = w_1 \ldots w_n$ let us define a projective complex $P_w^\bullet$ as follows. For each $i \in \mathbb{Z}$, let us define the projective module at the place $i$ by*

$$P_w^i = \oplus_{j=0}^n \delta(\mu_w(j), i) P(c(j)),$$

*where $c(j) := e(w_j)$ for $1 \leq j \leq n$ and $c(0) = s(w)$, and where $\delta$ is the Kronecker-delta.*

*And, for each $i \in \mathbb{Z}$, let us define the differential map from the module at $i$ to the module at $i+1$ by $\partial_w^i = (\partial_{jk}^i)_{0 \le j,k \le n}$, where*

$$\partial_{jk}^i = \begin{cases} F(w, j+1) & , \text{ if } w_{j+1} \in \mathcal{P}_{\ge 1}, \mu_w(j) = i \text{ and } k = j+1; \\ F(w, j) & , \text{ if } w_j^{-1} \in \mathcal{P}_{\ge 1}, \mu_w(j) = i \text{ and } k = j-1; \\ 0 & , \text{ otherwise.} \end{cases}$$

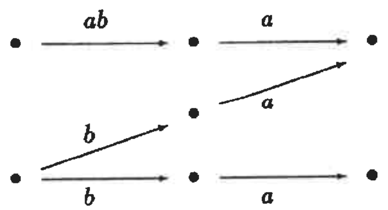
- *For each trivial generalized string $1_\alpha$ let us denote by $P_{1_\alpha}^\bullet$ the following projective complex*

$$\cdots \to 0 \to P^0 = \mathbf{P}_\alpha \overset{\partial^0}{\to} 0 \to \cdots$$

**Example 6.** *Let $(Q, I, Sp)$ be the skewed-gentle triple from Example 1 (see Subsection 2.4) and $A = kQ^{sg}/\langle I^{sg}\rangle$ the corresponding skewed-gentle algebra. We will use for $A$ the notations from Example 1. As a typical example, we consider the generalized string*

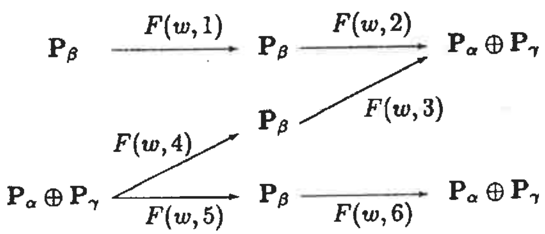
$$w = w_1 w_2 \cdots w_6 = (ab)(a)(a)^{-1}(b)^{-1}(b)(a),$$

*which is visualized by the following diagram:*




*The corresponding projective complex $P_w^\bullet$ is then visualized by the following diagram:*

$$\cdots \to 0 \overset{\partial_w^{-1}}{\to} P_w^0 \overset{\partial_w^0}{\to} P_w^1 \overset{\partial_w^1}{\to} P_w^2 \overset{\partial_w^2}{\to} 0 \overset{\partial_w^3}{\to} \cdots$$



where $F(w,1) = p(a_1b_1)$, $F(w,2) = \begin{pmatrix} p(a_1) & 0 \end{pmatrix}$, $F(w,3) = \begin{pmatrix} p(a_1) & p(a_2) \end{pmatrix}$, $F(w,4) = \begin{pmatrix} p(b_1) & 0 \end{pmatrix}^T$, $F(w,5) = \begin{pmatrix} p(b_1) & p(b_2) \end{pmatrix}^T$ and $F(w,6) = \begin{pmatrix} p(a_1) & p(a_2) \end{pmatrix}$.

It follows from Subsection 5.2 that, if $w$ is an asymmetric generalized string, then the projective complex $P^\bullet_w$ is indecomposable. Projective complexes isomorphic to such a $P^\bullet_w$ will be called *asymmetric strings*. If $w$ is a symmetric generalized string, then the projective complex $P^\bullet_w$ decomposes into the direct sum of two indecomposables projective complexes $P^\bullet_{w,0}$ and $P^\bullet_{w,1}$. Projective complexes isomorphic to a $P^\bullet_{w,i}$ $(i = 0, 1)$ will be called *dimidiate strings*.

Next, we consider the case of generalized asymmetric bands.

**Definition 10.** *For each asymmetric generalized band $w = w_1 \dots w_n$ and each $f(x) \in \mathrm{Ind}_0\, k[x]$ we define a projective complex $P^\bullet_{w,f}$ as follows.*

*First, we consider the matrices $G(u,i)$, $H(u,i)$ and $F(u,i)$ which correspond to the generalized string*

$$u = u_1 \cdots u_n u_{n+1} \cdots u_{2n} u_{2n+1} \cdots u_{3n} = w_1 \cdots w_n w_1 \cdots w_n w_1 \cdots w_n.$$

*For each $i \in \mathbb{Z}$, let*

$$P^i_{w,f} = \oplus_{j=0}^{n-1} \delta(\mu_w(j), i) P(c(j)) \otimes_k k^{\deg f(x)},$$

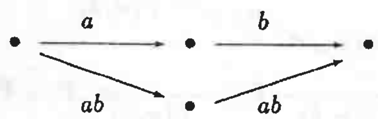
*and, also, for each $i \in \mathbb{Z}$, $\partial^i_{w,f} = (\partial^i_{jk})_{0 \le j,k \le n-1}$, where*

$$\partial^i_{jk} = \begin{cases} F(u, j+n+1) \otimes 1_{\deg f(x)} & \text{, if } w_{j+1} \in \mathcal{P}_{\ge 1}, \mu_w(j) = i \text{ and } k = j+1; \\ F(u, j+n) \otimes 1_{\deg f(x)} & \text{, if } w_j^{-1} \in \mathcal{P}_{\ge 1}, \mu_w(j) = i \text{ and } k = j-1; \\ (G(u,2n)p(w_n)H(u,n)) \otimes F_{f(x)} & \text{, if } w_n \in \mathcal{P}_{\ge 1}, \mu_w(j) = i,\ j = n-1 \text{ and } k = 0; \\ (G(u,n)p(w_n^{-1})H(u,2n)) \otimes F_{f(x)} & \text{, if } w_n^{-1} \in \mathcal{P}_{\ge 1}, \mu_w(j) = i,\ j = 0 \text{ and } k = n-1; \\ 0 & \text{, otherwise.} \end{cases}$$

**Example 7.** *Let $(Q, I, Sp)$ be the skewed-gentle triple from Example 1 (see Subsection 2.4), $k$ the field $\mathbb{Z}_2$ and $A = kQ^{sg}/\langle I^{sg} \rangle$ the corresponding skewed-gentle algebra. We will use for $A$ the notations from Example 1. As a typical example, we consider the asymmetric generalized band*

$$w = w_1 \cdots w_4 = (a)(b)(ab)^{-1}(ab)^{-1},$$

*which is visualized by the following diagram:*



*Given the indecomposable polynomial $f(x) = x^2 + x + 1$, the corresponding projective complex $P^{\bullet}_{w,f}$ is then visualized by the following diagram:*

$$\cdots \to 0 \xrightarrow{\partial^{-1}_{w,f}} P^0_{w,f} \xrightarrow{\partial^{0}_{w,f}} P^1_{w,f} \xrightarrow{\partial^{1}_{w,f}} P^2_{w,f} \xrightarrow{\partial^{2}_{w,f}} 0 \xrightarrow{\partial^{3}_{w,f}} \cdots$$

$$\begin{array}{ccccc} \mathbf{P}^2_\beta & \xrightarrow{M_1} & (\mathbf{P}_\alpha \oplus \mathbf{P}_\gamma)^2 & \xrightarrow{M_2} & \mathbf{P}^2_\beta \\ & \searrow^{M_4} & & \nearrow_{M_3} & \\ & & \mathbf{P}^2_\beta & & \end{array}$$

*where*

$$M_1 = \begin{pmatrix} p(a_1) & 0 & p(a_2) & 0 \\ 0 & p(a_1) & 0 & p(a_2) \end{pmatrix}, \quad M_2 = \begin{pmatrix} p(b_1) & 0 & -p(b_2) & 0 \\ 0 & p(b_1) & 0 & -p(b_2) \end{pmatrix}^T,$$

$$M_3 = \begin{pmatrix} p(a_1b_1) & 0 \\ 0 & p(a_1b_1) \end{pmatrix}, \quad M_4 = \begin{pmatrix} p(a_1b_1) & p(a_1b_1) \\ p(a_1b_1) & 0 \end{pmatrix}.$$

It follows from Subsection 5.2 that if $w$ is an asymmetric generalized band and $f(x) \in \operatorname{Ind}_0 k[x]$ , then the projective complex $P^{\bullet}_{w,f(x)}$ is indecomposable. Projective complexes isomorphic to such a $P^{\bullet}_{w,f(x)}$ will be called *asymmetric bands.*

Finally, we consider the case of a generalized symmetric band $w = w_1 \cdots w_n$. It is easy to see that $w$ then has the form

$$w = w_1 \cdots w_n = a_1 \cdots a_r a_r^{-1} \cdots a_1^{-1} b_1 \cdots b_s b_s^{-1} \cdots b_1^{-1},$$

where $e(a_r), e(b_s) \in Sp$.

**Definition 11.** *For each symmetric generalized band*

$$w = w_1 \cdots w_n = a_1 \cdots a_r a_r^{-1} \cdots a_1^{-1} b_1 \cdots b_s b_s^{-1} \cdots b_1^{-1},$$

*where $e(a_r), e(b_s) \in Sp$, and each matrix (see Subsection 3.1)*

$$M \in \mathcal{M}, \quad M = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \operatorname{Mat}(k, (l + l') \times (m + m')),$$

*we define the projective complex $P^{\bullet}_{w,M}$ as follows.*

*First we consider the matrices* $G(u,i)$, $H(u,i)$ *and* $F(u,i)$ *which correspond to the generalized string*

$$u = u_1 \cdots u_n u_{n+1} \cdots u_{2n} u_{2n+1} \cdots u_{3n} = w_1 \cdots w_n w_1 \cdots w_n w_1 \cdots w_n.$$

*For each* $0 \le j < n$ *we set*

$$d(j) := \begin{cases} l & , \text{if } 0 \le j \le r-1; \\ l' & , \text{if } r+1 \le j \le 2r; \\ m' & , \text{if } 2r+1 \le j \le 2r+s-1; \\ m & , \text{if } 2r+s+1 \le j \le n-1. \end{cases}$$

*For each* $i \in \mathbf{Z}$, *let* $P^i := \oplus_{j=0}^{n-1} P(i,j)$, *where*

- $P(i,j) := \delta(\mu_w(j), i) k^{d(j)} \otimes_k P(c(j))$ *if* $j \notin \{r, 2r+s\}$;
- $P(i,j) := \delta(\mu_w(r), i)((k^l \otimes \mathcal{P}_{c(r)}) \oplus (k^{l''} \otimes_k \mathcal{P}_{gc(r)}))$ *if* $j = r$;
- $P(i,j) := \delta(\mu_w(2r+s), i)((k^{m'} \otimes \mathcal{P}_{c(2r+s)}) \oplus (k^m \otimes_k \mathcal{P}_{gc(2r+s)}))$ *if* $j = 2r+s$,

*and, also, for each* $i \in \mathbf{Z}$, $\partial^i_{w,f} = (\partial^i_{jk})_{0 \le j,k \le n-1}$, *where*

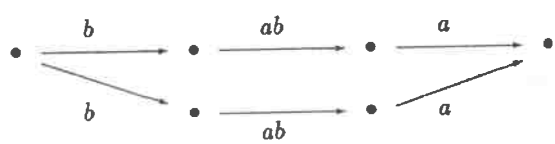
- $\partial^i_{jk} := 1_{d(j)} \otimes F(u, n+j+1)$ *if* $w_{j+1} \in \mathcal{P}_{\ge 1}$, $\mu_w(j) = i$, $k = j+1$ *and* $j \notin \{r-1, r, 2r, 2r+s-1, 2r+s\}$;
- $\partial^i_{jk} := 1_{d(j)} \otimes F(u, n+j)$ *if* $w_j^{-1} \in \mathcal{P}_{\ge 1}$, $\mu_w(j) = i$, $k = j-1$ *and* $j \notin \{r, r+1, 2r+1, 2r+s, 2r+s+1\}$;
- $\partial^i_{jk} := (1_l \otimes G(u, n+r) p(w_r) \quad 0)$ *if* $w_r \in \mathcal{P}_{\ge 1}$, $\mu_w(j) = i$, $j = r-1$ *and* $k = j+1$;
- $\partial^i_{jk} := (1_l \otimes p(w_r^{-1}) H(u, n+r) \quad 0)^T$ *if* $w_r^{-1} \in \mathcal{P}_{\ge 1}$, $\mu_w(j) = i$, $j = r$ *and* $k = j-1$;
- $\partial^i_{jk} := (0 \quad 1_{l'} \otimes p(g w_{r+1}) H(u, n+r+1))^T$ *if* $w_{r+1} \in \mathcal{P}_{\ge 1}$, $\mu_w(j) = i$, $j = r$ *and* $k = j+1$;
- $\partial^i_{jk} := (0 \quad 1_{l'} \otimes G(u, n+r+1) p(w_{r+1}^{-1} g))$ *if* $w_{r+1}^{-1} \in \mathcal{P}_{\ge 1}$, $\mu_w(j) = i$, $j = r+1$ *and* $k = j-1$;
- $\partial^i_{jk} := (1_{m'} \otimes G(u, n+2r+s) p(w_{2r+s}) \quad 0)$ *if* $w_{2r+s} \in \mathcal{P}_{\ge 1}$, $\mu_w(j) = i$, $j = 2r+s-1$ *and* $k = j+1$;
- $\partial^i_{jk} := (1_{m'} \otimes p(w_{2r+s}^{-1}) H(u, n+2r+s) \quad 0)^T$ *if* $w_{2r+s}^{-1} \in \mathcal{P}_{\ge 1}$, $\mu_w(j) = i$, $j = 2r+s$ *and* $k = j-1$;
- $\partial^i_{jk} := (0 \quad 1_m \otimes p(g w_{2r+s+1}) H(u, n+2r+s+1))^T$ *if* $w_{2r+s+1} \in \mathcal{P}_{\ge 1}$, $\mu_w(j) = i$, $j = 2r+s$ *and* $k = j+1$;

- $\partial^i_{jk} := (0 \quad \mathbf{1}_m \otimes G(u, n+2r+s+1)p(w^{-1}_{2r+s+1}g))$ *if* $w^{-1}_{2r+s+1} \in \mathcal{P}_{\geq 1}$, $\mu_w(j) = i$, $j = 2r+s+1$ *and* $k = j-1$;
- $\partial^i_{jk} := (\mathbf{1}_l \otimes G(u,n))Ap(b_1)(\mathbf{1}_m \otimes H(u, n+2r+1))$ *if* $b_1 \in \mathcal{P}_{\geq 1}$, $\mu_w(j) = i$, $j = 0$ *and* $k = n-1$;
- $\partial^i_{jk} := (\mathbf{1}_m \otimes G(u, n+2r+1))A^T p(b_1^{-1})(\mathbf{1}_l \otimes H(u,n))$ *if* $b_1^{-1} \in \mathcal{P}_{\geq 1}$, $\mu_w(j) = i$, $j = n-1$ *and* $k = 0$;
- $\partial^i_{jk} := (\mathbf{1}_l \otimes G(u,n))Cp(b_1)(\mathbf{1}_{m'} \otimes H(u, 2n))$ *if* $b_1 \in \mathcal{P}_{\geq 1}$, $\mu_w(j) = i$, $j = 0$ *and* $k = 2r+1$;
- $\partial^i_{jk} := (\mathbf{1}_{m'} \otimes G(u, 2n))C^T p(b_1^{-1})(\mathbf{1}_l \otimes H(u,n))$ *if* $b_1^{-1} \in \mathcal{P}_{\geq 1}$, $\mu_w(j) = i$, $j = 2r+1$ *and* $k = 0$;
- $\partial^i_{jk} := (\mathbf{1}_{l'} \otimes G(u, n+2r+1))Bp(b_1)(\mathbf{1}_m \otimes H(u, n+2r+1))$ *if* $b_1 \in \mathcal{P}_{\geq 1}$, $\mu_w(j) = i$, $j = 2r$ *and* $k = n-1$;
- $\partial^i_{jk} := (\mathbf{1}_m \otimes G(u, n+2r+1))B^T p(b_1^{-1})(\mathbf{1}_{l'} \otimes H(u, n+2r+1))$ *if* $b_1^{-1} \in \mathcal{P}_{\geq 1}$, $\mu_w(j) = i$, $j = n-1$ *and* $k = 2r$;
- $\partial^i_{jk} := (\mathbf{1}_{l'} \otimes G(u, n+2r+1))Dp(b_1)(\mathbf{1}_{m'} \otimes H(u, 2n))$ *if* $b_1 \in \mathcal{P}_{\geq 1}$, $\mu_w(j) = i$, $j = 2r$ *and* $k = 2r+1$;
- $\partial^i_{jk} := (\mathbf{1}_{m'} \otimes G(u, 2n))D^T p(b_1^{-1})(\mathbf{1}_{l'} \otimes H(u, n+2r+1))$ *if* $b_1^{-1} \in \mathcal{P}_{\geq 1}$, $\mu_w(j) = i$, $j = 2r+1$ *and* $k = 2r$;
- $\partial^i_{jk} := 0$ *in the others cases.*

**Example 8.** *Let $(Q, I, Sp)$ be the skewed-gentle triple of Example 1 (see Subsection 2.4) and $A = kQ^{sg}/\langle I^{sg}\rangle$ be the corresponding skewed-gentle algebra. We will use for A the notations from Example 1. As a typical example, we consider the symmetric generalize band*

$$w = w_1 \cdots w_4 = (a)(a)^{-1}(ab)^{-1}(b)^{-1}(b)(ab),$$

*which is visualized by the following diagram:*

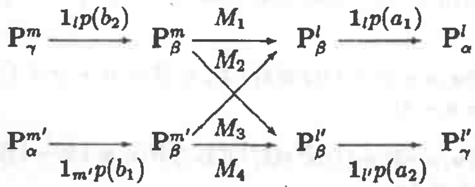


*Given a matrix*

$$M \in \mathcal{M}, \quad M = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \mathrm{Mat}\,(k, (l + l') \times (m + m')),$$

*the corresponding projective complex $P^{\bullet}_{w,M}$ is then visualized by the following diagram:*

$$\cdots \to 0 \xrightarrow{\partial^{-3}_{w,M}} P^{-2}_{w,M} \xrightarrow{\partial^{-2}_{w,M}} P^{-1}_{w,M} \xrightarrow{\partial^{-1}_{w,M}} P^{0}_{w,M} \xrightarrow{\partial^{0}_{w,M}} P^{1}_{w,M} \xrightarrow{\partial^{1}_{w,M}} 0 \xrightarrow{\partial^{2}_{w,M}} \cdots$$




*where $M_1 = A^T p(a_1 b_1)$, $M_2 = B^T p(a_1 b_1)$, $M_3 = C^T p(a_1 b_1)$, $M_4 = D^T p(a_1 b_1)$.*

It follows from Subsection 5.2 that if $w$ is a symmetric generalized band and $M \in \mathcal{M}$ (see Subsection 3.1), then the projective complex $P^{\bullet}_{w,M}$ is indecomposable. Projective complexes isomorphic to such a $P^{\bullet}_{w,M}$ will be called *dimidiate bands.*

## 5.3 Bunch of semi-chains $\mathbf{C}(A)$

In this subsection we associate to a skewed-gentle algebra $A$ a bunch of semi-chains.

Initially, we recall some definitions and notations from **[BoDr]** in a form convenient for our purposes.

We set

$$\mathcal{Y}(1) := \mathcal{Y} \setminus \mathcal{Y}_*, \mathcal{Y}(2) := \{x \in \mathcal{Y}_* \,|\, x^* = x\}, \overline{\mathcal{Y}(2)} := \{\bar{x} \,|\, x \in \mathcal{Y}(2)\},$$

$$\mathcal{Y}(3) := \{x \in \mathcal{Y}_* \,|\, x < x^*\}, \mathcal{Y}(4) := \{x \in \mathcal{Y}_* \,|\, x > x^*\},$$

$$\mathcal{Y}_w(i) := \{\gamma_1(x) \,|\, x \in \mathcal{Y}(i)\} \cap \mathcal{Y}_w,$$

where $\gamma_1$ is as in Subsection 4.

**Definition 12.** *We associate to the given skewed-gentle algebra $A$ a complete bunch of semi-chains $\mathbf{C} = \mathbf{C}(A) := \{\mathbf{I}, E_i, F_i, ^*\}$, where*

- $\mathbf{I} = (\mathbf{M}_+ \times \mathbb{Z}) \cup (\mathcal{Y}(1) \times \{-, +\})$;

- $F_{(w,i)} = \{u[i]^1_-, v[i]^1_-, v[i]^0_- \mid u \in \mathcal{Y}_w(1), v \in \cup_{j=2}^4 \mathcal{Y}_w(j)\}$;
- $E_{(w,i)} = \{u[i+1]^1_+, v[i+1]^1_+, v[i+1]^0_+ \mid u \in \mathcal{Y}_w(1), v \in \cup_{j=2}^4 \mathcal{Y}_w(j)\}$;
- $F_{x[j],-} = \{x[j]^0_-\}, F_{x[j],+} = \emptyset$;
- $E_{x[j],+} = \{x[j]^0_+\}, E_{x[j],-} = \emptyset$;
- $u[i]^j_- < v[i]^k_-$ *if and only if either* $u < v$ *or* $u = v$ *and* $j > k$;
- $u[i]^j_+ < v[i]^k_+$ *if and only if either* $u < v$ *or* $u = v$ *and* $j < k$;
- $(u[i]^1_+)^* = (u[i]^*)^1_+$, $(u[i]^1_-)^* = (u[i]^*)^1_-$ *and* $(u[i]^0_+)^* = (u[i]^*)^0_-$ *for all* $u[i] \in \mathcal{Y}_*$;
- $(u[i]^1_-)^* = u[i]^0_-$ *and* $(u[i]^1_+)^* = u[i]^0_+$ *for all* $u[i] \in \mathcal{Y}(1)$.

Let us define a map $f : \overline{|\mathbf{C}(A)|} \to \overline{\mathcal{Y}} \times \mathbf{N}$ by the following rules:

$$f(x[i]^1_+) = \begin{cases} (x[i],2) & \text{, if } x[i] \in \mathcal{Y}(1); \\ (x[i],3) & \text{, if } x[i] \in \mathcal{Y}(2); \\ (x[i],4) & \text{, if } x[i] \in \mathcal{Y}(3) \cup \mathcal{Y}(4); \end{cases}$$

$$f(x[i]^0_+) = \begin{cases} (x[i],2) & \text{, if } x[i] \in \mathcal{Y}(2) \cup \mathcal{Y}(4); \\ (x[i],3) & \text{, if } x[i] \in \mathcal{Y}(3); \end{cases}$$

$$f(x[i]^0_-) = \begin{cases} (x[i],2) & \text{, if } x[i] \in \mathcal{Y}(2) \cup \mathcal{Y}(3); \\ (x[i],3) & \text{, if } x[i] \in \mathcal{Y}(4); \end{cases}$$

$$f(x[i]^1_-) = (x[i],1),\ f(\overline{x[i]^1_+}) = (\overline{x[i]},3),\ f(\overline{x[i]^1_-}) = (\overline{x[i]},1).$$

Now, if $\mathbf{T}$ is an indecomposable representation of the bunch of semi-chains $\mathbf{C}(A)$, we define an $\mathcal{Y}$-matrix $B = B(\mathbf{T})$ as follows.

The horizontal and vertical bands of $B$ will be partitioned in a compatible way into narrower bands $B_{xk}$ and $B^{xk}$, where $x \in \overline{\mathcal{Y}}$ and $k$ assumes the following values: $k = 1,2$ if $x \in \mathcal{Y}(1)$, $k = 1,2,3$ if $x \in \mathcal{Y}(2) \cup \overline{\mathcal{Y}}(2)$ and $k = 1,2,3,4$ if $x \in \mathcal{Y}(3) \cup \mathcal{Y}(4)$.

The blocks $B^{ys}_{xk}$ are defined as follows:

$$B^{ys}_{xk} = T^t_r, \text{ if } (x,k) = f(r) \text{ and } (y,s) = f(t),$$

$$B^{ys}_{\overline{x}k} = -B^{ys}_{xk}, \text{ if } x \in \mathcal{Y}(2),$$

$$B^{\overline{y}s}_{xk} = B^{ys}_{xk}, \text{ if } y \in \mathcal{Y}(2),$$

$$\text{and } B^{ys}_{xk} = 0 \text{ otherwise,}$$

where the sizes of zero blocks are the smallest such that conditions in the definition of S-representations are satisfied.

We define a map $\lambda : |\mathbf{C}(A)| \to \mathcal{P}$ by

$$\lambda(x[i]^{\beta}_{\alpha}) := x$$

for any $\alpha \in \{-, +\}$ and any $\beta \in \{0, 1\}$.

**Definition 13.** *We denote by $\Omega^{im}$ the set of all $\mathbf{C}(A)$-words $w = w_1 w_2 \cdots w_n$ such that the following conditions hold:*

- $n \geq 2$;
- *if* $w_1 \in \mathbf{E}$, *then* $\lambda(w_1)[1] \in \mathcal{Y}(1)$;
- *if* $w_n^* \in \mathbf{E}$, *then* $\lambda(w_n)[1] \in \mathcal{Y}(1)$;
- *if* $w_i \in \mathbf{E}$, *then* $\lambda(w_{i-1}^*) < \lambda(w_i)$, *where* $1 < i \leq n$;
- *if* $w_i \in \mathbf{F}$, *then* $\lambda(w_{i-1}^*) > \lambda(w_i)$, *where* $1 < i \leq n$.

*We set* $\Omega^{im}_a := \Omega^{im} \cap \Omega_a$ *and* $\Omega^{im}_s := \Omega^{im} \cap \Omega_s$.

**Definition 14.** *We denote by $\Omega^{im}_p$ the set of all periodic $\mathbf{C}(A)$-words $w = (w_i)_{i \in \mathbf{Z}}$ such that the following conditions hold:*

- *if* $w_i \in \mathbf{E}$, *then* $\lambda(w_{i-1}^*) < \lambda(w_i)$ *for all* $i \in \mathbf{Z}$;
- *if* $w_i \in \mathbf{F}$, *then* $\lambda(w_{i-1}^*) > \lambda(w_i)$ *for all* $i \in \mathbf{Z}$.

*We set* $\Omega^{im}_{ap} := \Omega^{im}_p \cap \Omega_{ap}$ *and* $\Omega^{im}_{sp} := \Omega^{im}_p \cap \Omega_{sp}$.

The next theorem follows from Theorem 3 in [**BoDr**] and Theorem 1.

**Theorem 2.** *The $\mathcal{Y}$-matrices $B(\mathbf{T}(\delta))$, where*

$$\delta \in \Omega^{im}_a \amalg \Omega^{im}_s \times \{0,1\} \amalg \Omega^{im}_{ap} \times \mathrm{Ind}_0\, k[x] \amalg \Omega^{im}_{sp} \times \mathcal{M},$$

*constitute an exhaustive list of pairwise non-isomorphic non-zero indecomposable $\mathcal{Y}$-matrices in* $\mathrm{Im}\,\mathbf{F}$.

For a given generalized walk $u = u_1 \cdots u_n$ in $\overline{GSt}$ and $m \in \mathbf{Z}$, we define a sequence $\tau((u,m)) := w_1 \cdots w_{n+1}$, where $w_i \in |\mathbf{C}|$, by putting

- $w_i := \omega(u_{i-1})u_{i-1}[\mu_u(i-1)+m]^0_+$ if $1 < i < n+1$ and $u_{i-1}, u_i \in \mathcal{P}_{\geq 1}$;
- $w_i := \omega(u_{i-1}^{-1})[\mu_u(i-1)+m]^0_-$ if $1 < i < n+1$ and $u_{i-1}^{-1}, u_i^{-1} \in \mathcal{P}_{\geq 1}$;
- $w_i := \omega(u_{i-1}^{-1})[\mu_u(i-1)+m]^1_-$ if $1 < i < n+1$ and $u_{i-1}^{-1}, u_i \in \mathcal{P}_{\geq 1}$;

- $w_i := \omega(u_{i-1})u_{i-1}[\mu_u(i-1)+m]^1_+$ if $1 < i < n+1$ and $u_{i-1}, u_i^{-1} \in \mathcal{P}_{\geq 1}$;
- $w_i := (\omega(u_1))^*[m]^1_-$ if $i = 1$, $u_1 \in \mathcal{P}_{\geq 1}$ and $\omega(u_1)[1] \in \mathcal{Y}_*$;
- $w_i := (\omega(u_1^{-1})u_1^{-1})^*[m]^0_-$ if $i = 1, u_1^{-1} \in \mathcal{P}_{\geq 1}$ and $\omega(u_1^{-1})u_1^{-1}[1] \in \mathcal{Y}_*$;
- $w_i := \omega(u_1)[m]^0_-$ if $i = 1$, $u_1 \in \mathcal{P}_{\geq 1}$ and $\omega(u_1)[1] \in \mathcal{Y}(1)$;
- $w_i := \omega(u_1^{-1})u_1^{-1}[m]^0_+$ if $i = 1, u_1^{-1} \in \mathcal{P}_{\geq 1}$ and $\omega(u_1^{-1})u_1^{-1}[1] \in \mathcal{Y}(1)$;
- $w_i := \omega(u_n)u_n[\mu_u(n)+m]^0_+$ if $i = n+1$, $u_n \in \mathcal{P}_{\geq 1}$ and $\omega(u_n)u_n[1] \in \mathcal{Y}_*$;
- $w_i := \omega(u_n^{-1})[\mu_u(n)+m]^1_-$ if $i = n+1, u_n^{-1} \in \mathcal{P}_{\geq 1}$ and $\omega(u_n^{-1})[1] \in \mathcal{Y}_*$;
- $w_i := \omega(u_n)u_n[\mu_u(n)+m]^1_+$ if $i = n+1$, $u_n \in \mathcal{P}_{\geq 1}$ and $\omega(u_n)u_n[1] \in \mathcal{Y}(1)$;
- $w_i := \omega(u_n^{-1})[\mu_u(n)+m]^1_-$ if $i = n+1, u_n^{-1} \in \mathcal{P}_{\geq 1}$ and $\omega(u_n^{-1})[1] \in \mathcal{Y}(1)$.

And, for a given closed generalized walk $u = u_1 \cdots u_n$ in $\overline{GBa}$ and $m \in \mathbf{Z}$, we define a $\mathbf{Z}$-sequence $\tau_c((u,m)) := (v_i)_{i\in\mathbf{Z}}$, where $v_i \in |\mathbf{C}|$, by putting $v_{ln+i} := w_i$ for any $l \in \mathbf{N}$ and any $0 < i \leq n$, where $w_i$ are as above.

**Lemma 7.** *Let $u = u_1 \cdots u_n \in \overline{GSt}$. Then*

1. *$\tau((u,m))$ is a $\mathbf{C}(A)$-word for any $m \in \mathbf{Z}$;*
2. *If $u \in \overline{GBa}$, then $\tau_c((u,m))$ is a periodic $\mathbf{C}(A)$-word for any $m \in \mathbf{Z}$;*
3. *$\tau((u^{-1},m)) = (\tau((u,m+\mu_u(n))))^*$ and $\tau_c((v^{-1},m)) = (\tau_c((v,m+\mu_v(n))))^*$ for any $u \in \overline{GSt}$, $v \in \overline{GBa}$ and any $m \in \mathbf{Z}$ ;*
4. *$\tau_c((u[i],m)) = \tau((u,m+\mu_u(i)))[i]$ for any $u \in \overline{GBa}$ and any $m \in \mathbf{Z}$;*
5. *If $u \in \overline{GSt}$ is symmetric (resp., asymmetric), then $\tau((u,m))$ is symmetric (resp., asymmetric) for any $m \in \mathbf{Z}$;*
6. *If $u \in \overline{GBa}$ is symmetric (resp., asymmetric), then $\tau_c((u,m))$ is symmetric (resp., asymmetric) for any $m \in \mathbf{Z}$.*

*Proof.* 1. Let $\tau((u,m)) = w_1 \cdots w_{n+1}$. We are going to show that $w_i^*|w_{i+1}$ for any $1 \leq i \leq n$. We distinguish the following cases.

(a) $i = 1$, $u_1 \in \mathcal{P}_{\geq 1}$ and $\omega(u_1)[1] \in \mathcal{Y}_*$.
Then $w_1 = (\omega(u_1))^*[m]^1_-$, $w_1^* = \omega(u_1)[m]^1_-$ and $w_2 = \omega(u_1)u_1[m+1]^s_+$ for some $s \in \{0,1\}$. Therefore $w_1^* \in F_{(m(u_1),m)}$ and $w_2 \in E_{(m(u_1),m)}$, hence $w_1^*|w_2$.

(b) $i = 1$, $u_1^{-1} \in \mathcal{P}_{\geq 1}$ and $\omega(u_1^{-1})u_1^{-1}[1] \in \mathcal{Y}_*$.

Then $w_1 = (\omega(u_1^{-1})u_1^{-1})^*[m]_-^0$, $w_1^* = \omega(u_1^{-1})u_1^{-1}[m]_+^0$ and $w_2 = \omega(u_1^{-1})[m-1]_-^s$ for some $s \in \{0,1\}$. Therefore $w_1^* \in E_{(m(u_1^{-1}),m-1)}$ and $w_2 \in F_{(m(u_1^{-1}),m-1)}$, hence $w_1^*|w_2$.

(c) $i = 1$, $u_1 \in \mathcal{P}_{\geq 1}$ and $\omega(u_1)[1] \in \mathcal{Y}(1)$.

Then $w_1 = \omega(u_1)[m]_-^0$, $w_1^* = \omega(u_1)[m]_-^1$ and $w_2 = \omega(u_1)u_1[m+1]_+^s$ for some $s \in \{0,1\}$. Therefore $w_1^* \in F_{(m(u_1),m)}$ and $w_2 \in E_{(m(u_1),m)}$, hence $w_1^*|w_2$.

(d) $i = 1$, $u_1^{-1} \in \mathcal{P}_{\geq 1}$ and $\omega(u_1^{-1})u_1^{-1}[1] \in \mathcal{Y}(1)$.

Then $w_1 = \omega(u_1^{-1})u_1^{-1}[m]_+^0$, $w_1^* = \omega(u_1^{-1})u_1^{-1}[m]_+^1$ and $w_2 = \omega(u_1^{-1})[m-1]_-^s$ for some $s \in \{0,1\}$. Therefore $w_1^* \in E_{(m(u_1^{-1}),m-1)}$ and $w_2 \in F_{(m(u_1^{-1}),m-1)}$, hence $w_1^*|w_2$.

(e) $1 < i \leq n$ and $u_{i-1}, u_i \in \mathcal{P}_{\geq 1}$.

Then $w_i = \omega(u_{i-1})u_{i-1}[\mu_u(i-1)+m]_+^0$, $w_i^* = (\omega(u_{i-1})u_{i-1})^*[\mu_u(i-1)+m]_-^0$ and $w_{i+1} = \omega(u_i)u_i[\mu_u(i)+m]_+^s$ for some $s \in \{0,1\}$. Since $u_{i-1}u_i \in \overline{GSt}$, we have $m((\omega(u_{i-1})u_{i-1})^*) = m(\omega(u_i)u_i) = m(u_i)$ and therefore $w_i^* \in F_{(m(u_i),\mu_u(i-1)+m)}$ and $w_{i+1} \in E_{(m(u_i),\mu_u(i-1)+m)}$, hence $w_i^*|w_{i+1}$.

(f) $1 < i \leq n$ and $u_{i-1}, u_i^{-1} \in \mathcal{P}_{\geq 1}$.

Then $w_i = \omega(u_{i-1})u_{i-1}[\mu_u(i-1)+m]_+^1$, $w_i^* = (\omega(u_{i-1})u_{i-1})^*[\mu_u(i-1)+m]_+^1$ and $w_{i+1} = \omega(u_i^{-1})[\mu_u(i)+m]_-^s$ for some $s \in \{0,1\}$. Since $u_{i-1}u_i \in \overline{GSt}$, we have $m((\omega(u_{i-1})u_{i-1})^*) = m(\omega(u_i^{-1})) = m(u_i^{-1})$ and therefore $w_i^* \in E_{(m(u_i^{-1}),\mu_u(i)+m)}$ and $w_{i+1} \in F_{(m(u_i^{-1}),\mu_u(i)+m)}$, hence $w_i^*|w_{i+1}$.

(g) $1 < i \leq n$ and $u_{i-1}^{-1}, u_i^{-1} \in \mathcal{P}_{\geq 1}$.

This, in a sense, is the dual of case (e).

(h) $1 < i \leq n$ and $u_{i-1}^{-1}, u_i \in \mathcal{P}_{\geq 1}$.

This, in a sense, is the dual of case (f).

2. Since $u, u[1] \in \overline{GSt}$, this statement follows from the statement 1.

3. Let $\tau((u,m)) = w_1 \cdots w_{n+1}$, $(\tau((u,m)))^* = w_{n+1}^* \cdots w_1^* = x_1 \cdots x_{n+1}$ and $\tau((u^{-1}, m + \mu_u(n))) = y_1 \cdots y_{n+1}$. It is necessary to prove that $x_i = y_i$ for all $i$. We distinguish the following cases.

(a) $u_n \in \mathcal{P}_{\geq 1}, \omega(u_n)u_n[1] \in \mathcal{Y}_*$.

Then $y_1 = (\omega(u_n)u_n)^*[m + \mu_u(n)]_-^0$ and $x_1 = (w_{n+1})^* = (\omega(u_n)u_n[\mu_u(n) + m]_+^0)^* = (\omega(u_n)u_n)^*[m + \mu_u(n)]_-^0$, hence $x_1 = y_1$.

(b) $u_n^{-1} \in \mathcal{P}_{\geq 1}, \omega(u_n^{-1})[1] \in \mathcal{Y}_*$.

Then $y_1 = (\omega(u_n^{-1}))^*[m + \mu_u(n)]_-^1$ and $x_1 = (w_{n+1})^* = (\omega(u_n^{-1})[\mu_u(n) + m]_-^1)^* = (\omega(u_n^{-1}))^*[m + \mu_u(n)]_-^1$, hence $x_1 = y_1$.

(c) $u_n \in \mathcal{P}_{\geq 1}, \omega(u_n)u_n[1] \in \mathcal{Y}(1)$.

Then $y_1 = \omega(u_n)u_n[m + \mu_u(n)]_+^0$ and $x_1 = (w_{n+1})^* = (\omega(u_n)u_n[\mu_u(n) + m]_+^1)^* = \omega(u_n)u_n[m + \mu_u(n)]_+^0$, hence $x_1 = y_1$.

(d) $u_n^{-1} \in \mathcal{P}_{\geq 1}, \omega(u_n^{-1})[1] \in \mathcal{Y}(1)$.

Then $y_1 = \omega(u_n^{-1})[m+\mu_u(n)]^0_-$ and $x_1 = (w_{n+1})^* = (\omega(u_n^{-1})[\mu_u(n)+m]^1_-)^* = \omega(u_n^{-1})[m+\mu_u(n)]^0_-$, hence $x_1 = y_1$.

(e) $u_1 \in \mathcal{P}_{\geq 1}, \omega(u_1)[1] \in \mathcal{Y}_*$.
Then $y_{n+1} = \omega(u_1)[m+\mu_u(n)+\mu_{u^{-1}}(n)]^1_-$ and $x_{n+1} = (w_1)^* = ((\omega(u_1))^*[m]^1_-)^* = \omega(u_1)[m]^1_-$, hence $x_{n+1} = y_{n+1}$ by lemma 6.

(f) $u_1^{-1} \in \mathcal{P}_{\geq 1}, \omega(u_1^{-1})u_1^{-1}[1] \in \mathcal{Y}_*$.
Then $y_{n+1} = \omega(u_1^{-1})u_1^{-1}[m+\mu_u(n)+\mu_{u^{-1}}(n)]^0_+$ and $x_1 = (w_1)^* = ((\omega(u_1^{-1})u_1^{-1})^*[m]^0_-)^* = \omega(u_n^{-1})u_1^{-1}[m]^0_+$, hence $x_{n+1} = y_{n+1}$ by lemma 6.

(g) $u_1 \in \mathcal{P}_{\geq 1}, \omega(u_1)[1] \in \mathcal{Y}(1)$.
Then $y_{n+1} = \omega(u_1)[m+\mu_u(n)+\mu_{u^{-1}}(n)]^1_-$ and $x_{n+1} = (w_1)^* = (\omega(u_1)[m]^0_-)^* = \omega(u_1)[m]^1_-$, hence $x_{n+1} = y_{n+1}$ by lemma 6.

(h) $u_1^{-1} \in \mathcal{P}_{\geq 1}, \omega(u_1^{-1})u_1^{-1}[1] \in \mathcal{Y}(1)$.
Then $y_{n+1} = \omega(u_1^{-1})u_1^{-1}[m+\mu_u(n)+\mu_{u^{-1}}(n)]^1_+$ and $x_{n+1} = (w_1)^* = (\omega(u_1^{-1})u_1^{-1}[m]^0_+)^* = \omega(u_1^{-1})u_1^{-1}[m]^1_+$, hence $x_{n+1} = y_{n+1}$ by lemma 6.

(i) $u_{n-i+2}^{-1}, u_{n-i+1}^{-1} \in \mathcal{P}_{\geq 1}$, where $1 < i < n+1$.
Then we have $y_i = \omega(u_{n-i+2}^{-1})u_{n-i+2}^{-1}[\mu_u(n)+m+\mu_{u^{-1}}(i-1)]^0_+$ and $x_i = (w_{n-i+2})^* = (\omega(u_{n-i+1}^{-1})[\mu_u(n-i+1)+m]^0_-)^* = \omega(u_{n-i+2}^{-1})u_{n-i+2}^{-1}[\mu_u(n-i+1)+m]^0_+)$, hence $x_i = y_i$ by lemma 6.

(j) $u_{n-i+2}^{-1}, u_{n-i+1} \in \mathcal{P}_{\geq 1}$, where $1 < i < n+1$.
Then we have $y_i = \omega(u_{n-i+2}^{-1})u_{n-i+2}^{-1}[\mu_u(n)+m+\mu_{u^{-1}}(i-1)]^1_+$ and $x_i = (w_{n-i+2})^* = (\omega(u_{n-i+1})u_{n-i+1}[\mu_u(n-i+1)+m]^1_+)^* = \omega(u_{n-i+2}^{-1})u_{n-i+2}^{-1}[\mu_u(n-i+1)+m]^1_+)$, hence $x_i = y_i$ by lemma 6.

(k) $u_{n-i+2}, u_{n-i+1} \in \mathcal{P}_{\geq 1}$, where $1 < i < n+1$.
This, in a sense, is the dual of case (i).

(l) $u_{n-i+2}, u_{n-i+1}^{-1} \in \mathcal{P}_{\geq 1}$, where $1 < i < n+1$.
This, in a sense, is the dual of case (j).

4. Evident.

5. The statement follows from 3.

6. The statement follows from 4. □

**Corollary 4.** *The map $\tau$ (resp., $\tau_c$) is a map from $\overline{GSt} \times \mathbf{Z}$ (resp., $\overline{GBa} \times \mathbf{Z}$) to $\Omega$ (resp., $\Omega_p$).*

We denote by $GSt_a^{nt}$ (resp., $GSt_s^{nt}$) the set of all nontrivial asymmetric (resp., symmetric) generalized strings.

**Lemma 8.** *1. The maps $\tau : \overline{GSt} \times \mathbf{Z} \to \Omega^{im}$ and $\tau_c : \overline{GBa} \times \mathbf{Z} \to \Omega_p^{im}$ are bijections;*

2. *We can choose* $GSt$ *and* $\Omega_a$ *such that* $\tau(GSt_a^{nt} \times \mathbf{Z}) = \Omega_a^{im}$;

3. *We can choose* $GSt$ *and* $\Omega_s$ *such that* $\tau(GSt_s^{nt} \times \mathbf{Z}) = \Omega_s^{im}$;

4. *We can choose* $GBa$ *and* $\Omega_{ap}$ *such that* $\tau_c(GBa_a \times \mathbf{Z}) = \Omega_{ap}^{im}$;

5. *We can choose* $GBa$ *and* $\Omega_{sp}$ *such that* $\tau_c(GBa_s \times \mathbf{Z}) = \Omega_{sp}^{im}$.

*Proof.* 1. We define maps $\tau' : \Omega^{im} \to \overline{GSt} \times \mathbf{Z}$ and $\tau_c' : \Omega_p^{im} \to \overline{GBa} \times \mathbf{Z}$ by the following rule.

Let $w = w_1 \cdots w_{n+1} \in \Omega^{im}$, where $n \geq 1$. Suppose first that $w_{i+1} \in \mathbf{E}$, where $1 \leq i \leq n$. Since $w_i^* | w_{i+1}$ and $\lambda(w_i^*) < \lambda(w_{i+1})$, we have $w_i^* = x_i[m_i]_-^{s_i}$ and $w_{i+1} = x_i y_i [m_i + 1]_+^{t_i}$ for some $x_i \in \mathcal{P}$, $y_i \in \mathcal{P}_{\geq 1}$, $m_i \in \mathbf{Z}$ and $s_i, t_i \in \{0,1\}$. Then we set $u_i := y_i$.

Suppose finally that $w_{i+1} \in \mathbf{F}$. Since $w_i^* | w_{i+1}$ and $\lambda(w_i^*) > \lambda(w_{i+1})$, we have $w_i^* = x_i y_i [m_i]_+^{s_i}$ and $w_{i+1} = x_i[m_i - 1]_-^{t_i}$ for some $x_i \in \mathcal{P}$, $y_i \in \mathcal{P}_{\geq 1}$, $m_i \in \mathbf{Z}$ and $s_i, t_i \in \{0,1\}$. Then we set $u_i := y_i^{-1}$.

We will show that $u = u_1 \cdots u_n \in \overline{GSt}$, that is that $u_i u_{i+1} \in \overline{GSt}$ for $1 \leq i < n$. As for this, we distinguish four cases.

(a) $w_{i+1}, w_{i+2} \in \mathbf{E}$, where $1 \leq i < n$.

Then $u_i, u_{i+1} \in \mathcal{P}_{\geq 1}$. Since $w_{i+1} = x_i u_i [m_i + 1]_+^{t_i}$, $w_{i+1}^* = x_{i+1}[m_{i+1}]_-^{s_{i+1}}$ and $w_{i+2} = x_{i+1} u_{i+1} [m_{i+1} + 1]_+^{t_{i+1}}$ (see above), we have $(x_i u_i)^* = x_{i+1}$ and hence $e(u_i) = s(u_{i+1})$. If $e(u_i) \notin Sp$, then $x_i u_i \neq x_{i+1}$ and therefore $u_i u_{i+1} \in J$ becose of $x_{i+1} u_{i+1} \notin J$ and $A$ is skewed-gentle. Hence $u_i u_{i+1} \in \overline{GSt}$.

(b) $w_{i+1} \in \mathbf{E}$, $w_{i+2} \in \mathbf{F}$, where $1 \leq i < n$.

Then $u_i, u_{i+1}^{-1} \in \mathcal{P}_{\geq 1}$. Since $w_{i+1} = x_i u_i [m_i + 1]_+^{t_i}$, $w_{i+1}^* = x_{i+1} u_{i+1}^{-1} [m_{i+1}]_+^{s_{i+1}}$ and $w_{i+2} = x_{i+1}[m_{i+1} - 1]_-^{t_{i+1}}$ (see above), we have $(x_i u_i)^* = x_{i+1} u_{i+1}^{-1}$ and hence $e(u_i) = e(u_{i+1}^{-1}) = s(u_{i+1})$. If $e(u_i) \notin Sp$, then $x_i u_i \neq x_{i+1} u_{i+1}^{-1}$ and therefore $u_i u_{i+1} \in St$. Hence $u_i u_{i+1} \in \overline{GSt}$.

(c) $w_{i+1}, w_{i+2} \in \mathbf{F}$, where $1 \leq i < n$.

This, in a sense, is the dual of case (a).

(d) $w_{i+1} \in \mathbf{F}$, $w_{i+2} \in \mathbf{E}$, where $1 \leq i < n$.

This, in a sense, is the dual of case (b).

Then we set $\tau'(w) := (u, m_1)$.

Let $v = (v_i)_{i \in \mathbf{Z}}$ be a periodic $\mathrm{C}(A)$-word from $\Omega_p^{im}$ and $n+1$ be the period of it. Then $w = v_1 \cdots v_{n+1} \in \Omega^{im}$ and we set $\tau_c'(v) := \tau'(w)$. Since $\tau'(ww) \in \overline{GSt}$, we have $\tau_c'(v) \in \overline{GBa}$.

It is easy to see that $\tau\tau'$, $\tau'\tau$, $\tau_c\tau_c'$ and $\tau_c'\tau_c$ are identity maps.

The others statements follow from the statement 1 and lemma 7. □

## 5.4 The main theorem

For a walk $w = w_1 \cdots w_n$ let us write $\mu(w)$ for the minimum of the $\mu_w(i)$, $i = 0, \cdots, n$ and let us introduce also the following additional notations.

- $Q_c = \{a \in Q_1 \mid \exists\, a_1, \ldots, a_m \in Q_1$ such that $s(a_{i+1}) = e(a_i), s(a_1) = e(a_m), a_1 = a,$ $a_i a_{i+1}, a_m a_1 \in I\}$;
- $\overline{GSt}_c = \{w \in GSt \mid l(w) > 0$ and $\exists\, a \in Q_c$ such that $aw \in \overline{GSt}$, and $\mu(w) = 0\}$;
- $\overline{GSt}^c = \{w \in GSt \mid l(w) = n > 0$ and $\exists\, a \in Q_c$ such that $wa^{-1} \in \overline{GSt}$, and $\mu(w) = \mu_w(n)\}$;
- $GSt_c = \{w = w_1 \ldots w_n \in \overline{GSt}_c \mid$ if $w_1 \in Q_c$ then $w_2 \ldots w_n \notin \overline{GSt}_c\}$;
- $GSt^c = \{w = w_1 \ldots w_n \in \overline{GSt}^c \mid$ if $w_n^{-1} \in Q_c$ then $w_1 \ldots w_{n-1} \notin \overline{GSt}^c\}$;
- $GSt_c^c = GSt_c \cap GSt^c$.

Given $w = w_1 \cdots w_n \in \mathcal{P}_{\geq 1}(Q, J)$, we set

$$\breve{w} := \begin{cases} a, & \text{if } \exists\, a \in Q_1 \text{ such that } aw_1 \in I; \\ 0, & \text{otherwise.} \end{cases}$$

**Lemma 9.** *Let $A$ be a skewed-gentle algebra and let $w \in \mathcal{P}_{\geq 1}(Q, J)$. Then we have*

1. *If $s(w) \in Sp$, then* $\operatorname{Ker} \begin{pmatrix} p(w) & p(gw) \end{pmatrix}^T = A(\breve{w}, -\breve{w}g) + A(g\breve{w}, -g\breve{w}g)$;
2. *If $s(w) \notin Sp$, then* $\operatorname{Ker} p(w) = \operatorname{Ker} p(wg) = A\breve{w} + A(g\breve{w})$;
3. *$\beta(P_w^\bullet)^\bullet = P_w^\bullet$ for any $w = w_1 w_2$ such that $w_1^{-1}$ and $w_2$ are in $\mathcal{P}_{\geq 1}(Q, J)$.*

*Proof.* 1 is obvious and, for 2, let us observe that, since $A_+$ is gentle, we have $\operatorname{Ker} p(w_1^{-1}) \cap \operatorname{Ker} p(w_2) = \operatorname{Ker} p(w_1^{-1} g) \cap \operatorname{Ker} p(w_2 g) = 0$. □

As consequence we obtain the following

**Lemma 10.** *Let $A$ be a skewed-gentle algebra. Then we have*

1. *For every $w \in GBa_a$ and $f \in \operatorname{Ind}_0 k[x]$, $\beta(P_{w,f}^\bullet)^\bullet = P_{w,f}^\bullet$;*
2. *For every $w \in GBa_s$ and $M \in \mathcal{M}$, $\beta(P_{w,M}^\bullet)^\bullet = P_{w,M}^\bullet$;*
3. *In order that $P^\bullet_{\beta(P_w^\bullet)^\bullet} \notin \mathrm{K}^b(A-\mathrm{pro})$ for some $w \in GSt$ it is necessary and sufficient that $w \in GSt_c$ or $w \in GSt^c$.*

The complexes in $\mathbf{D}^b(A)$ isomorphic to $T^i(\beta(P^\bullet_\gamma)^\bullet)$, where $\gamma \in GSt_c \cup GSt^c, i \in \mathbb{Z}$ and $P^\bullet_\gamma$ is an asymmetric (resp., a dimidiate) generalized string , will be called *asymmetric* (resp., *dimidiate*) *periodic string.*

Now we are in a good position to state and proof our main theorem which gives all the indecomposable objects of the derived category.

**Theorem 3.** *Let A be a finite-dimensional skewed-gentle algebra and let us keep our foregoing notations. Then each indecomposable object of* $\mathbf{D}^b(A)$ *is a string (asymmetric or dimidiate) or a periodic string (asymmetric or dimidiate) or a band (asymmetric or dimidiate). The complexes* $T^i(P^\bullet_\delta)$ *and* $T^i(\beta(P^\bullet_\gamma)^\bullet)$*, where*

$$\delta \in GSt_a \amalg GSt_s \times \{0,1\} \amalg GBa_a \times \mathrm{Ind}_0 k[x] \amalg GBa_s \times \mathcal{M}, \tag{3}$$

$$\gamma \in GSt_c \amalg (GSt^c \setminus GSt^c_c) \text{ and } i \in \mathbb{Z}, \tag{4}$$

*(T being the translation functor) constitute an exhaustive list of pairwise non-isomorphic indecomposable objects of* $\mathbf{D}^b(A)$.

*Proof.* A straightforward calculation, which we omit, shows that for any nontrivial $\delta$ satisfying (3) and any $i \in \mathbb{Z}$ we have $\mathbf{F}(T^i(P^\bullet_\delta)) \cong B(\mathrm{T}(\tau(\delta,i)))$. Therefore it follows from Lemma 5, Theorem 2 and Lemma 8 that, for $\delta$ satisfying (3), the complexes $T^i(P^\bullet_\delta)$ constitute an exhaustive list of pairwise non-isomorphic indecomposable objects of $\mathbf{K}^b(A-$pro).

We end up our proof with the following observation. It follows from Lemma 10 that $\{\beta(M^\bullet)^\bullet | M^\bullet \in \mathbf{Ver}\,\mathfrak{p}(A)$ and $P^\bullet_{\beta(M^\bullet)^\bullet} \notin \mathbf{K}^b(A-\mathrm{pro})\} = \{T^i(\beta(P^\bullet_\gamma)^\bullet) \mid \gamma \in GSt_c \amalg (GSt^c \setminus GSt^c_c)$ and $i \in \mathbb{Z}\}$. □

As consequence we obtain the following

**Corollary 5.** *Let A be a finite-dimensional skewed-gentle algebra. Then*

*(i) A is derived tame;*

*(ii) A is derived discrete if and only if* $GBa = \emptyset$*;*

*(iii) A is derived finite if and only if* $|GSt| < \infty$.

**Remark 4.** *If A has finite global dimension, the statement (i) of the corollary follows from [*GePe*].*

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