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ON PAIRS OF DIFFERENTIAL 1-FORMS IN THE PLANE

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RESUMO

Neste artigo apresentamos uma classificação topológica de pares de germes de 1-formas diferenciais (α, β) no plano, onde α (resp. β) é regular ou possui uma singularidade do tipo sela, no ou fóco. Utilizamos principalmente a teoria de singularidades e o método do blowing up polar.

Também apresentamos um teorema de desingularização de pares de 1-formas diferenciais no plano, análogo ao teorema da redução de Seidenberg para 1-formas.

ABSTRACT

In this paper we give a topological classification of pairs of germs of differential 1-forms (α, β) in the plane, where α (resp. β) is regular or has a singularity of type saddle/node/focus. The main tools used here are singularity theory and the method of polar blowing up.

We also present a desingularization theorem for pairs of germs of 1-forms in the plane. This result is analogous to Seidenberg's reduction theorem for 1-forms.

On pairs of differential 1-forms in the plane

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1 Introduction

In this paper we classify, up to homeomorphisms, pairs of foliations determined by differential 1-forms in the plane, equivalently, pairs of integral curves of direction fields in the plane. Some models of pairs of direction fields have already been established. In [15] Teixeira characterised stable pairs (ξ, f) where ξ is a vector field and f a real function whose singularities are disjoint from those of ξ, and proved the existence of a structurally stable pair (ξ, f) on a sphere. These results are extended in [16] to cover codimension 1 phenomena. Davydov [11] proved that the stable pairs in [15] are in fact stable up to diffeomorphic changes of coordinates and provided normal forms for such pairs.

Bruce and Fidal in [3] classified some pairs of directional fields in order to obtain topological normal forms of binary differential equations (BDE), that is, equations of the forms $a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0$, where the coefficients a, b, c are germs of smooth functions vanishing at the origin. In [3] the case where $c = -a$ is studied. This condition is dropped in [7] to cover the case where the discriminant function has a Morse singularity. This study allows the deduction of, for instance, the configurations of the lines of curvature at an umbilic. Further examples of BDEs can be found in [8].

The way BDEs are studied in [3] and in [7] is as follows. The bivalued direction field in the plane determined by the equation is lifted to a single-valued field on a cylinder M, with the centre circle of the cylinder corresponding to the origin. However, pairs of points on the cylinder which correspond to the same point on the plane must be identified. This defines an involution on M and the problem reduces to classifying certain a pair of vector fields on the quotient space, a Mobius band M'. The centre line of the Mobius band is an integral curve of both fields, so local models of pairs of vector fields not both singular with a common integral curve are given in [3].

In [17] Michel dealt with the case where both fields are singular with a singularity of type saddle or node and have common separatrices. In [4] is given a model for the case where the fields have the same eigenvalues and one separatrix in common.

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In this paper we give a list of topological models of pairs of germs of differential 1-forms (α, β) where α (resp. β) is regular or has a singularity of type saddle/node/focus. This list covers for instance all the codimension ≤ 1 phenomena except those arising from the instability of one of the 1-forms. It also includes some codimension 2 cases. The main method used here is that of the the polar blowing up and the local models obtained in [3]. We also prove in §6 a desingularization theorem for pairs of 1-forms, analogous to Seidenberg's reduction theorem for 1-forms.

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2 Preliminaries

Let α denote a germ at the origin of a smooth 1-form in the plane. If the 1-form is regular, then there exists a diffeomorphism taking its foliation to that defined by dy. Suppose α is singular and let (λ_1, λ_2) denote the eigenvalues of its singularity at the origin. These are said to satisfy a resonant condition if

$$\lambda_i = k_1\lambda_1 + k_2\lambda_2, \; i = 1 \text{ or } 2$$

for some non-negative integers k_i with $k_1 + k_2 > 1$. Following [9] a singular 1-form whose eigenvalues are non-resonant can be reduced by a diffeomorphism to its linear part. Topologically, one has only two orbits $xdy \pm ydx$ of 1-forms with non-zero eigenvalues (see for example [1]).

As we can multiply the 1-forms by non-zero functions, we can assume that the eigenvalues of saddle/node singularities are $(1, \lambda)$. We say that two 1-forms have distinct eigenvalues if the corresponding λ's are distinct.

Let (α_1, β_1) and (α_2, β_2) be two pairs of germs of 1-forms at the origin. These pairs are said to be equivalent if there exists a germ of a homeomorphism taking the pair of foliations determined by (α_1, β_1) to the pair of foliations determined by (α_2, β_2). By a topological model or normal form, we mean a representative of this equivalent relation. In some cases we deal with a stronger equivalence by considering diffeomorphisms instead of homeomorphisms.

The discriminant of (α, β) consists of points p where $\alpha(p)$ and $\beta(p)$ are parallel, that is points where $\alpha \wedge \beta$ vanishes. We shall denote this set by Δ and subdivide it into two types. A point $p \in \Delta$ is said to be of *type* 1 if it does not belong to a common integral curve of the two fields, otherwise it is of *type* 2, see Figure 1. For example the pairs studied in [3] have discriminants of *type* 2.

Figure 1 here

Let $\mathcal{W}$ denote the set of germs at the origin of pairs of smooth 1-forms in the plane. The set $\mathcal{W}$ can be identified with the module $\mathcal{O}_2^4$ of germs of applications $\mathbb{R}^2, 0 \to \mathbb{R}^4, 0$, and thus be given a Whitney topology (see for example [13]).

A pair (α, β) is said to be stable if there exist a neighbourhood V of (α, β) in $\mathcal{W}$ such that all elements V are equivalent to (α, β). It is said to be simple if it has a neighbourhood that contains only finitely many equivalent classe.

We recall the normal forms already established which are used in the rest of the paper.

Proposition 2.1 *([3]). Let α, β be two germs of regular 1-forms with a common leaf. Suppose that $\alpha - \beta$ has a non-degenerate zero, then there exists a germ of a diffeomorphism taking the pair to the model $(dy, (1+x)dy - ydx)$.*

Michel proved a general version of the above proposition, namely

Proposition 2.2 *([17]). Let α and β be two germs regular 1-forms with order of contact $k+1$ along a common leaf. Then there exists a germ of a diffeomorphism conjugating the pair to $(dy, d(ye^{y^k x}))$.*

Note that the condition $\alpha - \beta$ having a non-degenerate zero in Proposition 2.1 is equivalent to the order of contact being 1. We also have the following established results.

Proposition 2.3 *([3]). Let α, β be two germs of 1-forms with α regular and β with a singularity of type saddle or node at the origin. Suppose that the pair has a common leaf through the origin. Then there exists a germ of a homeomorphism taking the pair to the model $(dy, xdy \pm ydx)$.*

It is important to observe that in the statement of the proposition above the eigenvalues of the singular 1-form are not required to satisfy the non-resonant condition. The proof uses a result of Hartman [14] and constructs the required homeomorphism.

The normal forms in this paper are obtained using three different techniques. In the case when both forms are regular, singularity theory is used to obtain smooth normal forms. When one or both forms are singular we use polar blowing up and analyse the pair in the neighbourhood of the exceptional fibre. We also give in some cases the formal normal forms.

Let α be the germ of a 1-form at the origin and consider the map ϕ : $S^1 \times \mathbb{R} \to \mathbb{R}^2$ given by $\phi(\theta, r) = (r\cos\theta, r\sin\theta)$. The pull-back $\phi_*(\alpha)$ (modulo a division by r^k for some integer k) is a 1-form on $S^1 \times \mathbb{R}$ called the polar blowing up of α. Note that the restriction of ϕ to $S^1 \times \mathbb{R}^*_+$, where $\mathbb{R}^*_+$ denotes the set of positive real numbers, is a diffeomorphism to the image which is the plane with the origin removed.

The main idea used here is to lift the pair of 1-forms to a pair on $S^1 \times \mathbb{R}^*_+$ and produce models via homeomorphisms that extend to the exceptional fibre, that is homeomorphisms from $S^1 \times \mathbb{R}^+ \to S^1 \times \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of non-negative real numbers. Blowing down yields the required homeomorphism in a neighbourhood of the origin in the plane.

3 The 1-forms α and β are regular

The case when the discriminant is of *type* 2 is given by Proposition 2.2. We shall study the case when Δ is of *type* 1 or contains both types, recovering (and go beyond) the results in [11] using an alternative method.

Theorem 3.1 *Let α and β be two germs of regular 1-forms.*

(i) If the two forms are transverse, then there exists a diffeomorphism taking the pair (α, β) to (dy, dx), which is stable (Figure 2(i)).

(ii) Suppose that the two forms have order of contact 2 at the origin. Then the set of critical points is a smooth curve of points of type 2, and there exists a diffeomorphism taking the pair (α, β) to the model $(dy, d(y - x^2))$, which is stable (Figure 2(ii)).

(iii) Suppose the discriminant is smooth and the 1-forms have order of contact $k \geq 3$ at the origin. Then there exists a diffeomorphism taking the pair to the model

$$(dy, d(y + xy + x^k + a_1x^{k+1} + a_2x^{k+2} + \ldots + a_{k-3}x^{2k-3})).$$

A versal deformation is given by $(dy, d(y+xy+x^k+(a_1+\bar{a}_1)x^{k+1}+\ldots+(a_{k-3}+\bar{a}_{k-3})x^{2k-3}+t_{k-2}x^{k-2}+\ldots+t_2x^2))$. In particular, the stable pair occurs when $k = 3$ (Figure 2(iii)). It is also the only simple pair.

(iv) Suppose the discriminant has a Morse singularity and the two 1-forms have order of contact 3 at the origin. Then there exists a diffeomorphism taking the pair to the model $(dy, d(y + x^3 \pm xy^2 + ax^5))$ (Figure 2(iv)). This pair is of codimension 2 (with modality 1) and a versal deformation is given by $(dy, d(y + x^3 \pm xy^2 + (a + \bar{a})x^5 + tx))$.

Figure 2 here

Proof: We set $\alpha = dy$ and assume that the foliation of β is given as the level curves of a smooth function f. To obtain the models for f we shall classify germs of functions up to diffeomorphisms in the source that preserve the horizontal lines. Such diffeomorphisms are of the form $(\phi(x, y), \psi(y))$ and form a subgroup $\mathcal{G}$ of the right group $\mathcal{R}$ (for notation see [2, 5, 6]). The $\mathcal{G}$-tangent space to the orbit of f is the module $T_f\mathcal{G} = m(x, y)f_x + m(y)f_y$, where $m(x, y)$ denotes the maximal ideal of the ring $\mathcal{O}(x, y)$ of germs at the origin of smooth functions and $m(y)$ the maximal ideal in $\mathcal{O}(y)$. We also allow changes of coordinates in the target (action of the left group $\mathcal{L}$) as these do not alter the structure of the level curves of f and those of the function y. Using the complete transversal technique in [5] and the determinacy results in [6] we classify germs of functions up to $\mathcal{G} \times \mathcal{L}$-equivalence. This is done inductively on the jet level.

Since β is regular, the 1-jet of f is not identically zero. If the 1-forms are transverse then $j^1f = ax + by$ with $a \neq 0$. Then a change of coordinates can set $j^1f(x, y) = x$. This jet is 1 determined and stable.

When the forms are not transverse we can write $j^1f = y$. The complete 2-transversal is then given by $j^2f = y + axy + bx^2$.

• If $b \neq 0$, using Mather's lemma we can set $j^2 f = y + x^2$. This is 2-determined and stable.

• If $b = 0$ but $a \neq 0$ then $j^2 f = y + xy$. (Note that these conditions are equivalent to the order of contact being higher than 2 and the discriminant being smooth.) It is now easy to show that a complete k-transversal to $y + xy$ is given by $y + xy + cx^k$ which is equivalent to $y + xy + x^k$ if $c \neq 0$. This germ is $2k-3$-determined and is part of the family of unequivalent germs $y + xy + x^k + a_1 x^{k+1} + a_2 x^{k+2} + \ldots + a_{k-3} x^{2k-3}$. This has modality $k-3$ and the codimension of the stratum is $k-3$. A versal unfolding has the form given in the proposition. The only stable germ occurs when $k = 3$. It is also the only simple germ.

• If $b = a = 0$ then $j^2 f = y$ and the discriminant is singular. The 3-jet can be written in the form $j^3 f = y + ux^3 + vx^2 y + wxy^2$. If $u \neq 0$ (order of contact of the 1-forms is 3) we can reduce to $j^3 f = y + x^3 + w' xy^2$. If further $w' \neq 0$ (Morse condition on the discriminant), using Mather's Lemma we can set $j^3 f = y + x^3 \pm xy^2$. A 4-transversal of $y + x^3 \pm xy^2$ is the germ itself but a 5-transversal is given by $y + x^3 \pm xy^2 + ax^5$. The parameter a is a smooth modulus. The germ $y + x^3 \pm xy^2 + ax^5$ is 5-determined and the codimension of the stratum is 1. A versal deformation is given by $y + x^3 \pm xy^2 + (a + \bar{a})x^5 + tx$.

We observe that if the order of contact between the forms is greater than 3 or the discriminant has a singularity worse than Morse then the phenomenon will be of higher codimension than in the Morse case.

We shall need a local model for the case when the discriminant has an A_1^- singularity (*i.e.*, is $\mathcal{R}\times\mathcal{C}$-equivalent to $x^2 - y^2$) with one branch being a common leaf for the two 1-forms and the other consisting of critical points of *type* 1. This case is of infinite codimension in the setting of the proof of the above theorem but the model is needed to establish the results in the following sections.

Proposition 3.2 *Let α and β be two germs of regular 1-forms. Suppose that the discriminant has an A_1^- singularity with one branch being a common leaf to the two forms. Then there exists a diffeomorphism taking the pair (α, β) to the model $(dy, dy(1 \pm x^2))$. See Figure 3.*

Proof: We can assume that $\alpha = dy$ and the foliation of β is given as the level sets of $f(x, y) = y(1 + ay + g(x, y))$ for some smooth 1-flat function g and a scalar a. The discriminant is given by $f_x(x, y) = yg_x(x, y) = 0$, so the hypothesis implies that $g_x(0, 0) = 0$ and $g_{xx}(0, 0) \neq 0$. Using the Splitting Lemma [2] we can write $g(x, y) = \pm x^2 + k(y)$ by a diffeomorphism of the form $(\phi(x, y), \psi(y))$. A further explicit diffeomorphism preserving dy reduces f to $y(1 \pm x^2)$.

Figure 3 here

4 The 1-form α is regular and β is singular

The case when the discriminant is of *type* 2 is given in Proposition 2.3. We shall assume here that the critical points are of *type* 1 and that the singularity of β is a saddle/node/focus. It is well known that a node is topologically equivalent to a focus but one can use a topological argument to show that a pair (regular, node) is not topologically equivalent to (regular, focus).

In this section we deal principally with two cases. The first is when the leaf of the regular 1-form is transverse to the separatrices of the singular 1-form. We also show in this case the models coincide with those obtained using a reduction by formal power series. The second case deals with the situation when the leaf of the regular 1-form at the origin has an ordinary tangency with one of the separatrices of the singular 1-form. In both cases, relying on Proposition 2.3, the eigenvalues of the singular form are not required to satisfy the non-resonant condition.

Theorem 4.1 *(Compare [11]) Let α be a germ at the origin of a regular 1-form and β a germ of a 1-form with a saddle/node/focus singularity at the origin. The discriminant is then a smooth curve. If the leaf of α at the origin is transverse to the separatrices of β, then there exists a homeomorphism taking the integral curves of (α,β) to the model $(dy,(x+\lambda y)dy+xdx)$, with λ a fixed value in the intervals $]-\infty,0[$ for a saddle, $]0,1/4[$ for node and $]1/4,+\infty[$ for a focus. See Figure 4.*

Figure 4 here

Proof: We set $\alpha = dy$. Since β has an elementary singularity at the origin, it follows that the discriminant is smooth. Furthermore, using the transversality condition we can reduce the 1-jet of β to $(x+\lambda y)dy+xdx$ by a linear change of coordinates preserving the $y = const$ curves. The singularity of β is of type saddle if $\lambda<0$, node if $0<\lambda<1/4$ and focus if $\lambda>1/4$.

We consider now the polar blowing up $x=r\cos\theta,\ y=r\sin\theta$. This results in the following two 1-forms on $S^1\times\mathbb{R}^+$:

$$\begin{aligned}\tilde{\alpha} =&\ r\cos\theta d\theta+\sin\theta dr\\ \tilde{\beta} =&\ r(\cos^2\theta+(\lambda-1)\sin\theta\cos\theta+ra(r,\theta))d\theta+\\ &(\cos^2\theta+\sin\theta\cos\theta+\lambda\sin^2\theta+rb(r,\theta))dr\end{aligned}$$

where the functions a,b depend only on terms of order greater or equal to 2 in β.

The 1-form $\tilde{\alpha}$ has 2 zeros on the circle S^1 at $0,\pi$. Both of these singularities are of type saddle.

The zeros of $\tilde{\beta}$ on S^1 are given by $\lambda\tan^2\theta+\tan\theta+1=0$. There are 4 zeros if $1-4\lambda>0$ and $\lambda\neq 0$, that is when the singularity of β is a saddle or node, and none if $\lambda>1/4$ (when β is a focus). The singularities of $\tilde{\beta}$, when they exist, are of type saddle or nodes (more precisely we have 4 saddles if $\lambda<0$ and 2 saddles and 2 nodes otherwise).

The zeros of $\tilde{\alpha}$ and $\tilde{\beta}$ are distinct, so locally at each of these points the pair can be reduced by a homeomorphism to the model $(dy, xdy \pm ydx)$ in Proposition 2.3.

The singular set lifts to a curve on $S^1 \times \mathbb{R}^+$. The points of intersection of this curve with $S^1 \times 0$ are given by $\cos\theta = 0$. These are distinct from the zeros of $\tilde{\alpha}$ and $\tilde{\beta}$. Following Proposition 3.2 the pair (α, β) is locally smoothly equivalent to $(dy, dy(1 \pm x^2))$. The sign $\pm$ in the model depends only on the 2-jet of $(\tilde{\alpha}, \tilde{\beta})$ which in turn depends only on the 1-jet of (α, β).

Away from the singular points the two forms and the singularities of the discriminant the two forms are regular and have a common leaf along S^1. Their difference is not degenerate, so a model is given by Proposition 2.1.

We can glue the local models together sliding along integral curves (see [3] for details). This yields a model in $S^1 \times \mathbb{R}^+, (S^1, 0)$. Blowing down produces the required homeomorphism in $\mathbb{R}^2, 0$.

The models in Proposition 4.1 also coincide with those of the reduction by formal power series.

Proposition 4.2 *Suppose that the 1-jet of β is set to $(x + \lambda y)dy + xdx$ and that α is regular with a leaf at the origin transverse to the separatrices of α. Then for almost all values of λ there exists a formal diffeomorphism taking the pair (α, β) to $(dy, (x + \lambda y)dy + xdx)$.*

Proof: We set $\alpha = dy$ and $\beta = (x+\lambda y+A(x,y))dy+(x+B(x,y)dx$, where A, B are 1-flat functions. We are seeking a formal diffeomorphism that preserves the $y = const$ curves and reduces β to its linear part. We shall construct this formal diffeomorphism step by step on the jet level.

Suppose that the $n-1$-jet of β is reduced to $(x + \lambda y)dy + xdx$ by a diffeomorphism that preserves α and consider the n-jet

$$\beta_n = (x + \lambda y + A_n(x, y))dy + (x + B_n(x, y))dx$$

of β where A_n and B_n are homogeneous polynomials of degree n. We apply the change of coordinates (which preserves $y = const$ curves)

$$\begin{aligned} x &= X + P_n(X, Y) \\ y &= Y + q_0 Y^n \end{aligned}$$

to β_n, where P_n is a homogeneous polynomial of degree n and $q_0 \in \mathbb{R}$. We can also multiply β_n by a non-zero polynomial of the form $1 + R_{n-1}$, with R_{n-1} a homogeneous polynomial of degree $n-1$. We would like to anihilate homogeneous terms of degree n in the resulting 1-form. This yields the following equations in the space $H^n(X, Y)$ of homogeneous polynomials of degree n.

$$\begin{aligned} P_n + \alpha q_0 Y^n + XR_{n-1} + \alpha Y R_{n-1} + nq_0 Y^{n-1} X + X\tfrac{\partial P_n}{\partial y} &= -A_n \\ P_n + X\tfrac{\partial P_n}{\partial x} + xR_{n-1} &= -B_n \end{aligned}$$

where A_n and B_n denote the n-jets of $A_n(X+P_n(X,Y), Y+q_0Y^n)$ and $B_n(X+P_n(X,Y), Y+q_0Y^n)$ respectively. Note that these n-jets depend only on A_n

and B_n. Using monomials of degree n as a basis for the vector space $H^n(X,Y)$, the above two equations can be written as a linear system of $2n+2$ equations with $2n+2$ unknowns. The matrix of this system depends only on λ and n. It is not hard to show that its determinant is a none identically zero polynomial d_n of degree at most $2n+2$ in λ. Therefore, for λ distinct from the roots of this polynomial, the matrix is not singular and the system has a solution, that is, the reduction can be done at the n-jet level. The union of the roots of the polynomials d_n is a subset of measure zero, hence the result.

We turn now to the case when the leaf of α at the origin is not transverse to one of the separatrices of β.

Proposition 4.3 *Let α be a germ at the origin of a regular 1-form and β a germ of a 1-form with a saddle/node singularity at the origin. Suppose that the leaf of α at the origin has an ordinary tangency with on of the separatrices of β. Then there exists a homeomorphism taking the pair to the model $(d(y-x^2), xdy-\lambda ydx)$. See Figure 5.*

Figure 5 here

Proof: The proof proceeds as in Proposition 4.1. We can set $\alpha = d(y-x^2)$ and reduce the 1-jet of β to $(x+ay)dy - \lambda ydx$. The difference here with Proposition 4.1 is that when considering the polar blowing up the two zeros of the lift $\tilde{\alpha}$, the singular points of the lift of the discriminant, and two of the zeros of $\tilde{\beta}$ coincide. The eigenvalues of $\tilde{\alpha}$ and $\tilde{\beta}$ are distinct at such points and the two forms have only one separatrix in common, namely the exceptional fibre. Using Proposition 5.6 below, we obtain a local model at such points. We then proceed as in Proposition 4.1. The topological model does not depend on a so we can set $a=0$.

5 The 1-forms α and β are singular

5.1 The forms do not have common leaves away from the origin

We shall start by analysing the 1-jet of the pair.

I. *α is a saddle/node*: we can fix $j^1\alpha = xdy - \lambda ydx$ and reduce the 1-jet of β to the form $j^1\beta = (x+\mu y)dy - (x+\gamma y)dx$. The characteristic polynomial of the associated matrix to $j^1\beta$ is given by $t^2-(\gamma+1)t+\gamma-\mu$. So the signs of $\delta_1 = (\gamma-1)^2+4\mu$ and $\gamma-\mu$ determine the singularity of β. For example β is a saddle if $\delta_1>0$ and $\gamma-\mu<0$.

We also have to consider the singularity type of the discriminant. The 2-jet of the discriminant is given by

$$-x^2+(\lambda-\gamma)xy+\lambda\mu y^2.$$

The singularity is in general Morse. It is of type A_1^+ (Δ is an isolated point) if $(\lambda-\gamma)^2+4\lambda\mu<0$ and A_1^- (Δ is a transverse crossing) if $(\lambda-\gamma)^2+4\lambda\mu>0$.

Considering the parabolae $(\gamma-1)^2+4\mu=0$ and $(\lambda-\gamma)^2+4\lambda\mu=0$ and the set $\gamma-\mu=0$ in the (γ,μ)-plane for λ fix, we can deduce the various pairs of singularities that could occur with a particular type of Morse singularity of the discriminant. (See Figure 6 and Proposition 5.1.)

II. *α and β are foci:* we can fix $\alpha=(x-\lambda y)dy-(\lambda x+y)dx$. The 1-jet of β can then be put in the form $j^1\beta=(x+\mu y)dy-(x+\gamma y)dx$, with $(\gamma-1)^2+4\mu<0$. The two jet of the discriminant is therefore given by

$$(\lambda-1)x^2+(\lambda\mu+\lambda-\gamma+1)xy+(\lambda\gamma+\mu)y^2.$$

This is an A_1^+ singularity if $\delta_2=(\lambda\mu+\lambda-\gamma+1)^2-4(\lambda-1)(\lambda\gamma+\mu)<0$ and an A_1^- singularity if $\delta_2>0$. (See Figure 6 and Proposition 5.1.)

Figure 6 here

Denoting S for a saddle N for a node and F for a focus we have the following.

Proposition 5.1 *Examples of all possible cases of pairs of 1-jets of 1-forms with elementary singularities at the origin are as follows.*

1). The discriminant has an A_1^+-singularity:

(S,S)	$(xdy+ydx,(x+y)dy-(x-y)dx)$
(N,N)	$(xdy-2ydx,(x-\frac{1}{8}y)dy-(x+2y)dx)$
(N,F)	$(xdy-2ydx,(x-y)dy-(x+2y)dx)$
(F,F)	$((x-2y)dy-(2x+y)dx,(x-y)dy-(x+y)dx)$

See Figure 7a.

2). The discriminant has an A_1^--singularity:

(S,S)	$(xdy+ydx,(x-\frac{1}{2}y)dy-(x-y)dx)$
(S,N)	$(xdy+ydx,(x+y)dy-(x+2y)dx)$
(S,F)	$(xdy+ydx,(x-y)dy-(x+y)dx)$
(N,N)	$(xdy-2ydx,(x+y)dy-(x+2y)dx)$
(N,F)	$(xdy-2ydx,(x-\frac{1}{10}y)dy-(x+y)dx)$
(F,F)	$((x-2y)dy-(2x+y)dx,(x-3y)dy-(x+y)dx)$

See Figure 7b.

Figure 7a and Figure 7b here

Proposition 5.2 *Let α and β be two singular 1-forms without a common integral curve away from the origin and with transverse separatrices. Suppose that the discriminant has an A_1 singularity with branches transverse to the separatrices of α and β in the A_1^- case. Then there exists a homeomorphism taking the pair (α, β) to one of the models in Proposition 5.1.*

Proof: We proceed as in Proposition 4.1. We first fix $j^1\alpha = xdy - \lambda ydx$ and $j^1\beta = (x + \mu y)dy - (x + \gamma y)dx$ when the singularities are not both foci and $j^1\alpha = (x + \lambda y)dy - (\lambda x + y)dx$ and $j^1\beta = (x + \mu y)dy - (x + \gamma y)dx$ otherwise. We then consider the polar blowing up. The singularities of $\tilde{\alpha}$ and $\tilde{\beta}$ are of type saddle or node and they do not coincide. So we can obtain local models at these points using Proposition 2.3. In the case when the discriminant of (α, β) is an isolated point, the discriminant of $(\tilde{\alpha}, \tilde{\beta})$ is the circle S^1 so it has no singular points. The local model at regular points of $(\tilde{\alpha}, \tilde{\beta})$ on S^1 is given by Proposition 2.1. We then glue these local models to obtain the required homeomorphism.

When the discriminant of (α, β) has an A_1^- singularity the discriminant of $(\tilde{\alpha}, \tilde{\beta})$ has two A_1^- singularities on the circle S^1. These singularities do not coincide with the zeros of $(\tilde{\alpha}, \tilde{\beta})$ and local models at such points are given by Proposition 3.2 and the rest is as above.

Remark 5.3 *All the cases in Proposition 5.1 are codimension 2 phenomena. Given two 1-forms $\alpha = a(x, y)dy + b(x, y)dx$ and $\beta = c(x, y)dy + d(x, y)dx$, we consider the map-germ $\theta : \mathbb{R}^2, 0 \to \mathbb{R}^4$ given by $(a(x, y), b(x, y), c(x, y), d(x, y))$. For most maps θ, 0 is a regular value, that is $0 \notin Image(\theta)$. Only in 2-parameter families of pairs of 1-forms will these singularities occur generically at isolated points.*

The formal normal forms for this case do not coincide with those of the topological equivalence.

Proposition 5.4 *Suppose that the 1-jet of (α, β) is given by $(xdy - \lambda ydx, (x + \mu y)dy - (x + \gamma y)dx)$ when one of the forms is not a focus or $((x - \gamma y)dy - (\gamma x + y)dx, (x + \mu y)dy - (x + \gamma y)dx)$ when both form are foci.*

Then for almost all values of (λ, μ, γ) there exists a formal diffeomorphism taking the pair (α, β) to

$(xdy - \lambda ydx, (x + \mu y + a(y))dy - (x + \gamma y + b(y))dx)$, *when at least one of the forms is not a focus,*

$((x - \gamma y)dy - (\gamma x + y)dx, (x + \mu y + a(y))dy - (x + \gamma y + b(y))dx)$, *when both forms are foci,*

where a and b are 1-flat formal power series.

Proof: The proof follows in the same way as that of Proposition 4.2. The difference here is that at the *nth* stage of the reduction we cannot eliminate all the homogeneous monomials of degree n in β_n. Only, for instance, those divisible by x can be eliminated, provided that a determinant of a certain matrix is not zero. As in Proposition 4.2 one can show that this determinant is a non-identically zero polynomial in (λ, μ, γ), and the result follows.

5.2 The forms have a common leaf away from the origin

As highlighted in Section 2 some aspects of the case when the two forms are singular with common separatrices are dealt with in [17]. We need the following result.

Proposition 5.5 *Let α and β be two singular forms with common separatrices and distinct eigenvalues. Then there exists a homeomorphism taking the pair (α,β) to $(xdy \pm ydx, xdy - \mu ydx)$.*

Proof: The proof follows the construction in the proof of Proposition 2.3 ([3]) for the case when one of the 1-forms is regular and the other is a saddle/node. The idea is as follows. By [14], there exists a C^1-diffeomorphism which takes the foliation of α to the level sets $x^{-\lambda}y = const$. So we can set $\alpha = xdy - \lambda ydx$.

Let $j^1\beta = xdy - \mu ydx$. Again by [14], there exists a local C^1-diffeomorphism $\psi(x,y) = (\psi_1(x,y), \psi_2(x,y))$, with 1-jet the identity, which takes β to $j^1\beta$. Moreover the foliation of $j^1\beta$ is given by the level sets of $x^{-\mu}y$.

Consider a line $x = \epsilon$ for a small $\epsilon > 0$, and work in the first quadrant. A leaf C of β through (x,y) intersects the line $x = \epsilon$ at a point (ϵ, c). To determine c we slove the equation $\psi_1^{-\mu}(x,y).\psi_2(x,y) = \psi_1^{-\mu}(\epsilon,c).\psi_2(\epsilon,c)$. Consider now the function germ $h : \mathbb{R},0 \to \mathbb{R},0$ defined by $h(c) = \psi_1^{-\mu}(\epsilon,c).\psi_2(\epsilon,c)$. As shown in [3] h is C^1 so the function $g(x,y) = h^{-1}(\psi_1^{-\mu}(x,y).\psi_2(x,y))$ is C^1 away from the y axis and $g(0,y) = 0$. (The value $g(x,y)$ is precisely the scalar c.)

Figure 8 here

The idea now is to map the leaf C to the leaf of $j^1\beta$ by sliding along the level sets of $x^{-\lambda}y$, see Figure 8. This transformation sends a point (x,y) to (X,Y) whose coordinates are obtained by solving the system $X^{-\lambda}Y = x^{-\lambda}y$ and $X^{-\mu}Y = \epsilon^{-\mu}c = \epsilon^{-\mu}g(x,y)$. Assuming that $\lambda > \mu$, the solution is given by is

$$\begin{aligned} X &= (\epsilon^{-\mu}x^{\lambda}g(x,y)/y)^{\frac{1}{\lambda-\mu}} \\ Y &= \epsilon^{-\mu}g(x,y)X^{\mu}. \end{aligned}$$

Using the properties of the function g in [3] one can show that X and Y are well defined and that (X,Y) is a homeomorphism of the first quadrant. The same construction works for the other quadrants. The resulting homeomorphism is in fact a diffeomorphism away from the axis. We can now set $\lambda = \pm 1$ using the homeomorphism $(sign(x).|x|^{|\lambda|}, y)$.

The case when only one separatrix is in common but the eigenvalues are the same (this is a necessary condition for the forms to be transverse away from the common separatrix) is given in [4]. When the eigenvalues are distinct then the discriminant, as we shall see, contains a curve of *type* 1.

Let α and β be two singular 1-forms with a common separatrix and distinct eigenvalues. Without loss of generality we can set

$$\begin{aligned} j^1\alpha &= xdy - \lambda ydx \\ j^1\beta &= (x+y)dy - \mu ydx. \end{aligned}$$

The 2-jet of the discriminant is then given by $y((\lambda-\mu)x+\lambda y)$. As $\lambda-\mu \neq 0$ it is clear that the discriminant has an A_1^- singularity with one branch the common separatrix and the other the set of critical points of *type* 1 which we assume is transverse to the separatrices of α and β.

We proceed now as before by considering the polar blowing up. The lifted forms $\tilde{\alpha}$ and $\tilde{\beta}$ have 4 singularities of type saddle or node and two are common to both forms. It is not hard to see that the two forms have the same separatrices and distinct eigenvalues at such points. One can also verify that the singularities of the lift of the discriminant are distinct from the common zeros of $\tilde{\alpha}$ and $\tilde{\beta}$ (this is a consequence of the fact that the branch of discriminant of *type* 1 is transverse to the separatrices of α and β). Therefore by Proposition 5.5 the two forms can be reduced locally to a normal form. Away from the common zeros we proceed as in the previous cases. We thus have

Proposition 5.6 *Let α and β be two singular 1-forms with a common integral curve and distinct eigenvalues. Suppose furthermore that the branch of the discriminant of points of type* 1 *is transverse to the separatrices of the forms. Then there exists a homeomorphism taking the pair (α, β) to the models*

$$(xdy \pm ydx, (x+y)dy - \mu ydx).$$

See Figure 9.

Figure 9 here

Remark 5.7 *1) The structurally stable pairs of 1-forms are those in Propositions 3.1 and 4.1. The normal forms for codimension 1 pairs, excluding the cases where the instability arises from one of the 1-forms, are as follows*

$(i):$	$(d(y-x^2), xdy - \lambda ydx)$	*(Proposition 4.3)*
$(ii):$	$(dy, d(y+xy+x^4+ax^5))$	*(Proposition 3.1 (ii))*
$(iii):$	$(dy, d(y+x^3 \pm xy^2+ax^5))$	*(Proposition 3.1(iii))*

All the cases in Proposition 5.1 are of codimension 2.

2) Geometrical characterisations of codimension 1 pairs (ξ, f) where ξ is a vector field and f a real function are listed in [16]. In the notation of [16] the cases Q_1 and Q_6 have the normal form in 1.(i) above, Q_7 that of 1.(ii) and Q_8 the one in 1.(iii). The remaining cases in [16] are Q_2 (eigenvalues of ξ are equal), Q_3 (ξ is a saddle node), Q_4 (ξ is a composed focus) and Q_5 (f is a cusp). The singularity Q_5 is of higher codimension when considered as a pair of foliations given by 1-forms. Alternative approachs to those used in this paper are needed to obtain the normal forms for the singularities Q_3, Q_4, and Q_5.

6 Desingularization of pairs of 1-forms

Let α be a germ of a singular 1-form in the plane. The singularity of α is said to be *simple* if the eigenvalues λ_1, λ_2 of the associated vector field satisfy one of the conditions:

(i) $\lambda_1.\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}^+$

(ii) $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

The desingularization theorem of Seidenberg [18] asserts that any singularity of an analytic or formal 1-form becomes simple after a finite number of blowing ups of the form $x = X,\ y = XY$ and $x = XY,\ y = Y$. A generalisation of this result to the smooth case is given by Dumortier in [12], where the 1-form/vector field is required to satisfy the Lojasiewicz inequality (all analytic forms satisfy this inequality).

An extended version of Seidenberg's theorem is given in [10]. This result asserts that one can reduce further to the case where all the separatrices become smooth and disjoint, no separatrix passes through a corner (intersection of two divisors) and all separatrices are transverse to the divisor.

We introduce the following definition for pairs of 1-forms.

Definition 6.1 *A pair of analytic 1-forms (α, β) is said to be desingularized when*

(i) α and β are desingularized (in the sense of [10]),

(ii) the discriminant is desingularized, i.e. given in local coordinates in the form $x^p y^q = 0$, with at least one branch being a common leaf for the pair in the case when $pq \neq 0$ or one of the 1-forms is singular,

(iii) at a common singular point of the pair the discriminant consists of the separatrices.

Lemma 6.2 *Let (α, β) be a pair of non-identical analytic 1-forms. Then the discriminant is desingularized after a finite number of blowing ups of the pair.*

Proof: Let $\alpha = a(x,y)dy + b(x,y)dx$ and $\beta = c(x,y)dy + d(x,y)dx$ so that the discriminant is the zero of the analytic function $(ad - bc)(x,y)$. Consider the blowing up $x = XY\ y = Y$, so that $\tilde{\alpha} = (a(XY,Y) + Xb(XY,Y))dY + Yb(XY,Y)dX$ and $\tilde{\beta} = (c(XY,Y) + Xd(XY,Y))dY + Yd(XY,Y)dX$. The resulting discriminant function is given by $Y(ad - bc)(XY,Y)$. (The blowing up $x = X$, $y = XY$ results in a discriminant of the form $X(ad - bc)(XY,Y)$.) Since $ad - bc$ is analytic it is desingularized (see for example [19]), so after finitely many blowing ups of the pair, the resulting disciminant can be written in local coordinates in the form $u^p v^q = 0$.

Theorem 6.3 *Let (α, β) be a pair of non-identical analytic 1-forms. Then there exists a desingularization for (α, β).*

Proof: Using the extended desingularization theorem in [10] we can assume that both α and β are desingularized, and by Lemma 6.2 we can also suppose that the discriminant is desingularized. We need to analyse the following possibilities concerning the pair.

(i) Both forms are regular. Then by Lemma 6.2 we can assume that the discriminant is desingularized and hence the pair is as in Definition 6.1(ii).

(ii) One 1-form is regular and the other has a simple singularity. If the forms have a common leaf then the pair is desingularized. In the case where the leaf of the regular 1-form at the origin has order of contact $k \geq 0$ with one of the separatrices of the singular 1-form, then blowing up a finite number of times reduces to the case where the discriminant is as in Definition 6.1(ii). (Note that since the leaf and the separatrix are analytic, the order of contact is finite, otherwise they would coincide.)

(iii) Both forms are singular. If the forms have the same separatrices then by Lemma 6.2 we can assume that the discriminant consists of the common separatrices. In the case where the two forms have only one separatrix in common and the other two have order of contact $k \geq 0$ (this order of contact is finite as we are in the analytic case) then blowing up again reduces locally to the cases where the two forms have the same separatrices or are as in Definition 6.1(ii) . If the separatrices are distinct they have a finite number of contact, hence blowing up a finite number of times desingularizes the pair.

6.1 Remarks on the smooth case

In the smooth case, assuming that the 1-forms satisfy the Lojasiewicz inequality, then Dumortier's theorem [12] ensures that both 1-forms are desingularized. To desingularize the discriminant which is given by a smooth function, we can work for example under the hypothesis that this function is finitely determined, that is, equivalent by a change of coordinates to a polynomial. The discriminant can then be considered as an algebraic curve, and hence admits a desingularization.

To procceed as in the analytic case we still need to consider the contact between the leaves of the 1-forms. As we are in the smooth case, two curves could have infinite order of contact without coinciding. Consider for example the case when $\alpha = xdy - \lambda ydx$ and the foliation of β given by the level set of $y - f(x)$ where f is ∞-flat at the origin, so that the leaf of β at the origin has infinite order of contact with a separatrix of α. The discriminant given by $\lambda y - xf'(x) = 0$ is desingularized and the the discriminant function is 1-determined.

It appears that beside the Lojasiewicz condition and the finite determinacy of the discriminant function, one has to impose on the 1-forms to have a finite order of contact in order to desingularize the pair in a finite number of steps.

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FIGURE CAPTIONS

Figure 1. Discriminants of *type* 1 and *type* 2.

Figure 2.

Figure 3. The foliations of the pair $(dy, dy(1 \pm x^2))$, $(-)$ left and $(+)$ right.

Figure 4. α regular and β singular, the transverse case.

Figure 5. α regular and β singular, the non-transverse case.

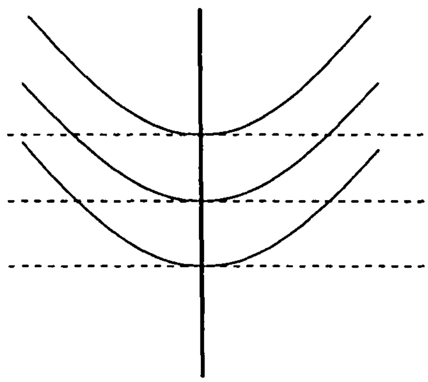
Figure 6. Partition of the (μ, γ)-plane for λ fixed.

Figure 7a. Models for Δ an isolated point.

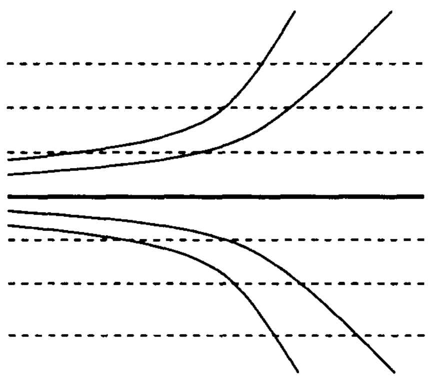
Figure 7b. Models for Δ a transverse crossing.

Figure 8. Sliding along the integral curves of α.

Figure 9. Two saddles with a common separatrix.



Type 1



Type 2

Figure 1

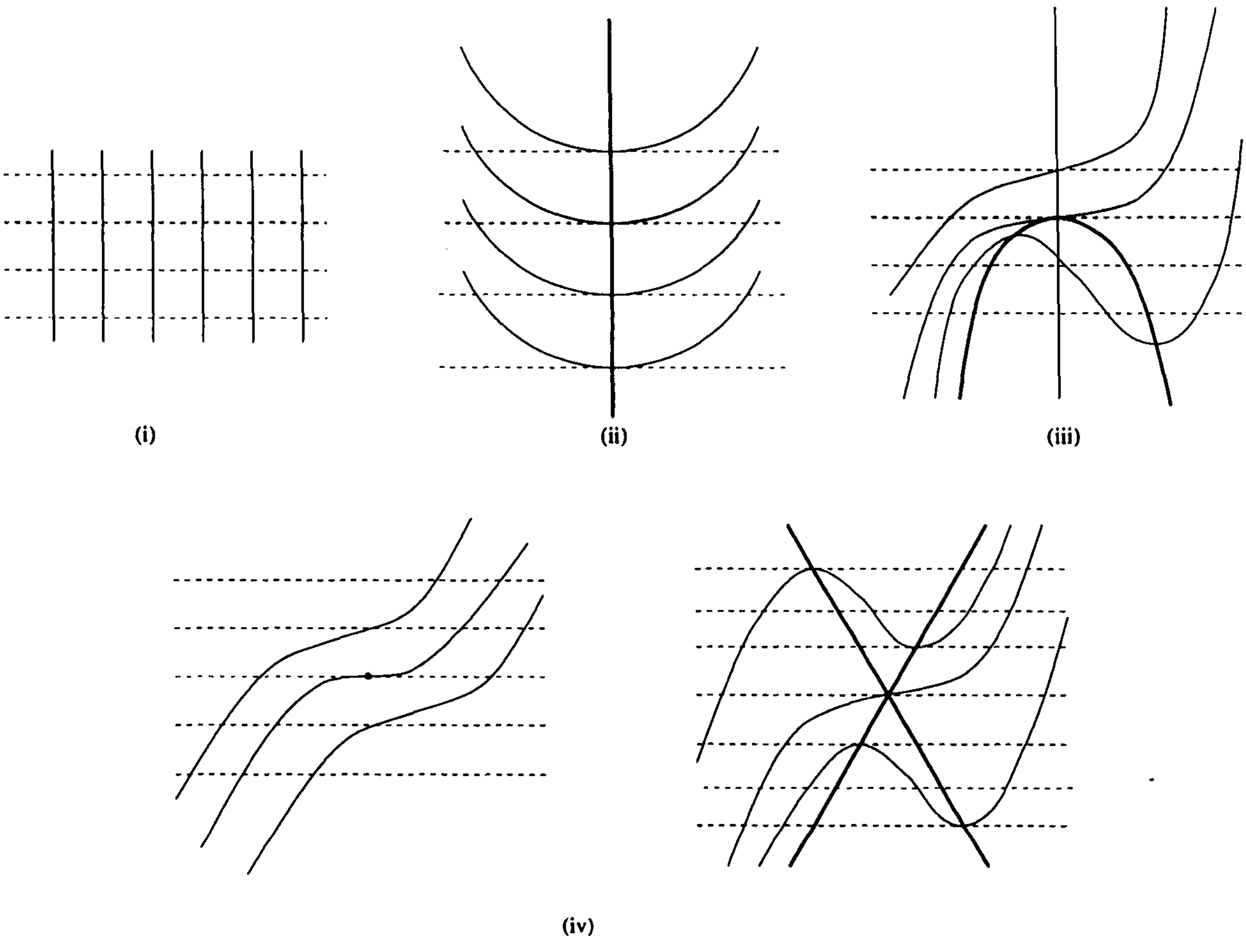


Figure 2

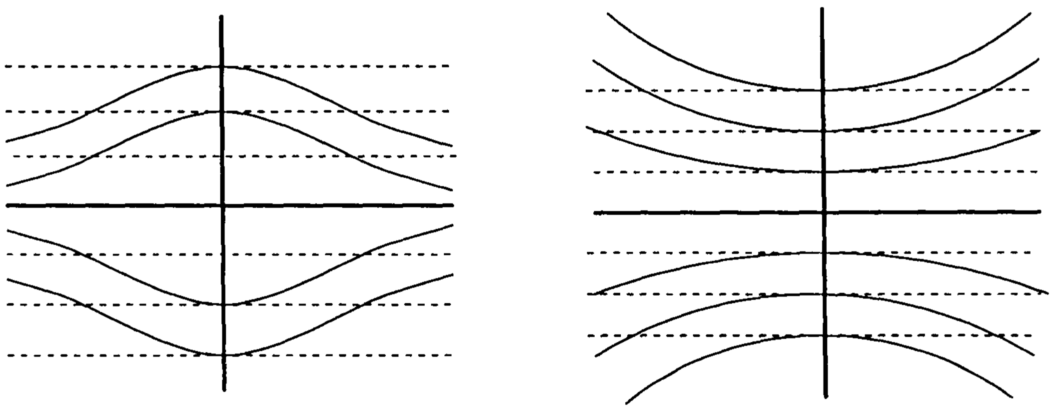
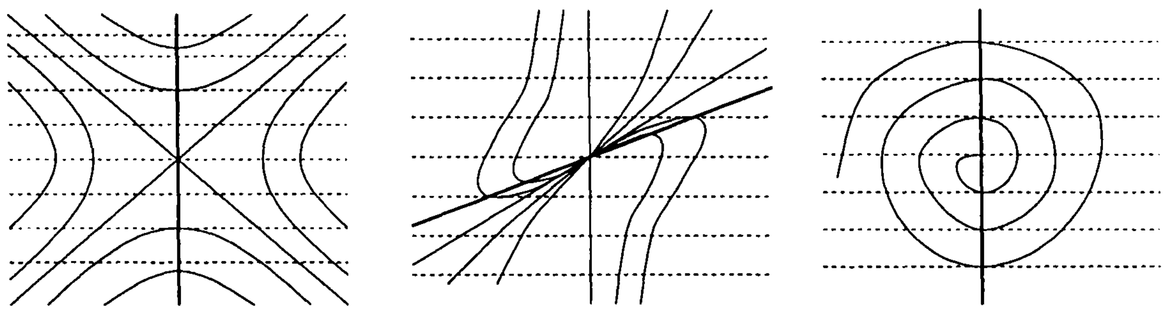
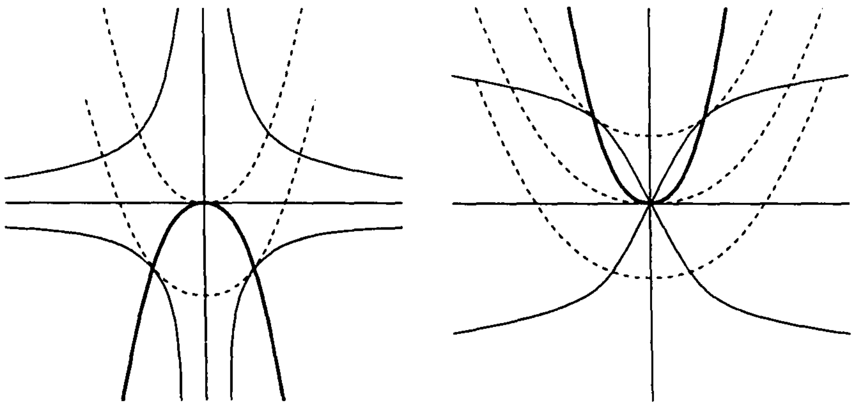


Figure 3



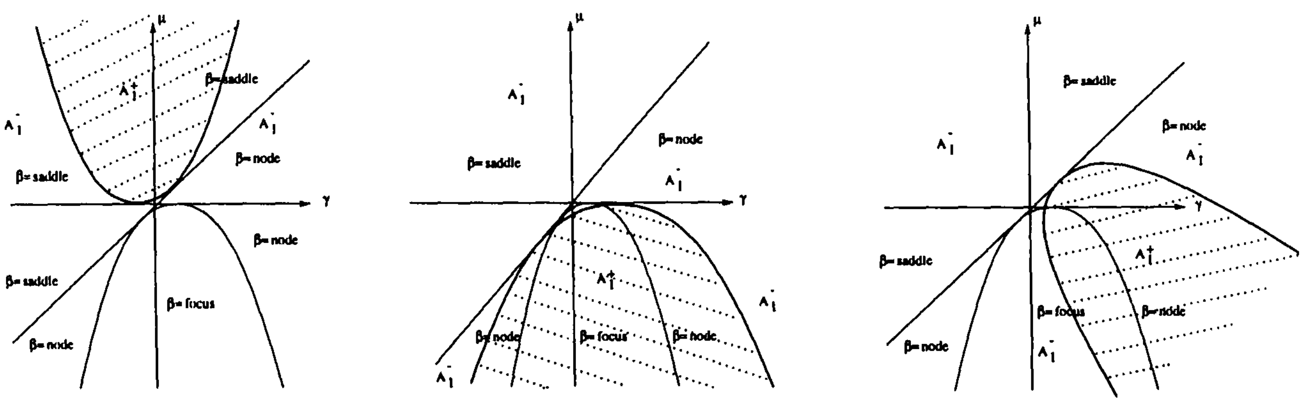
(regular, saddle)　(regular, node)　(regular, focus)

Figure 4



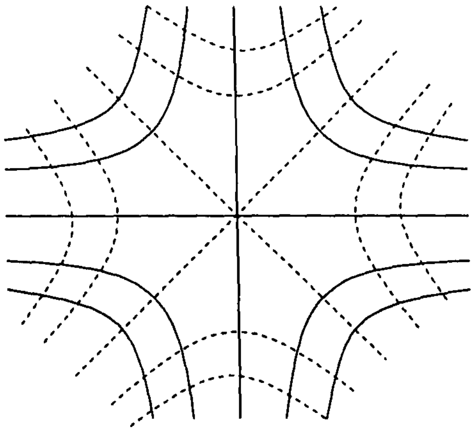
(regular, saddle) (regular, node)

Figure 5

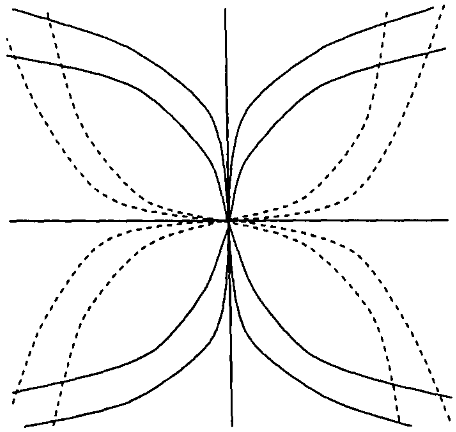


α = saddle α = node α = focus

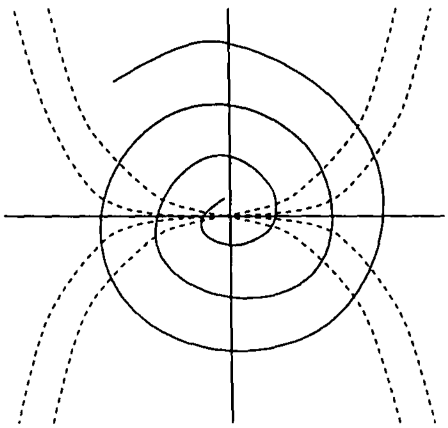
Figure 6



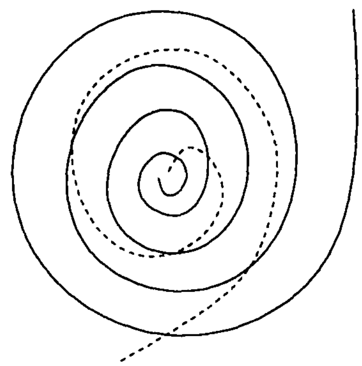
(saddle, saddle)



(node, node)



(node, focus)



(focus, focus)

Figure 7a

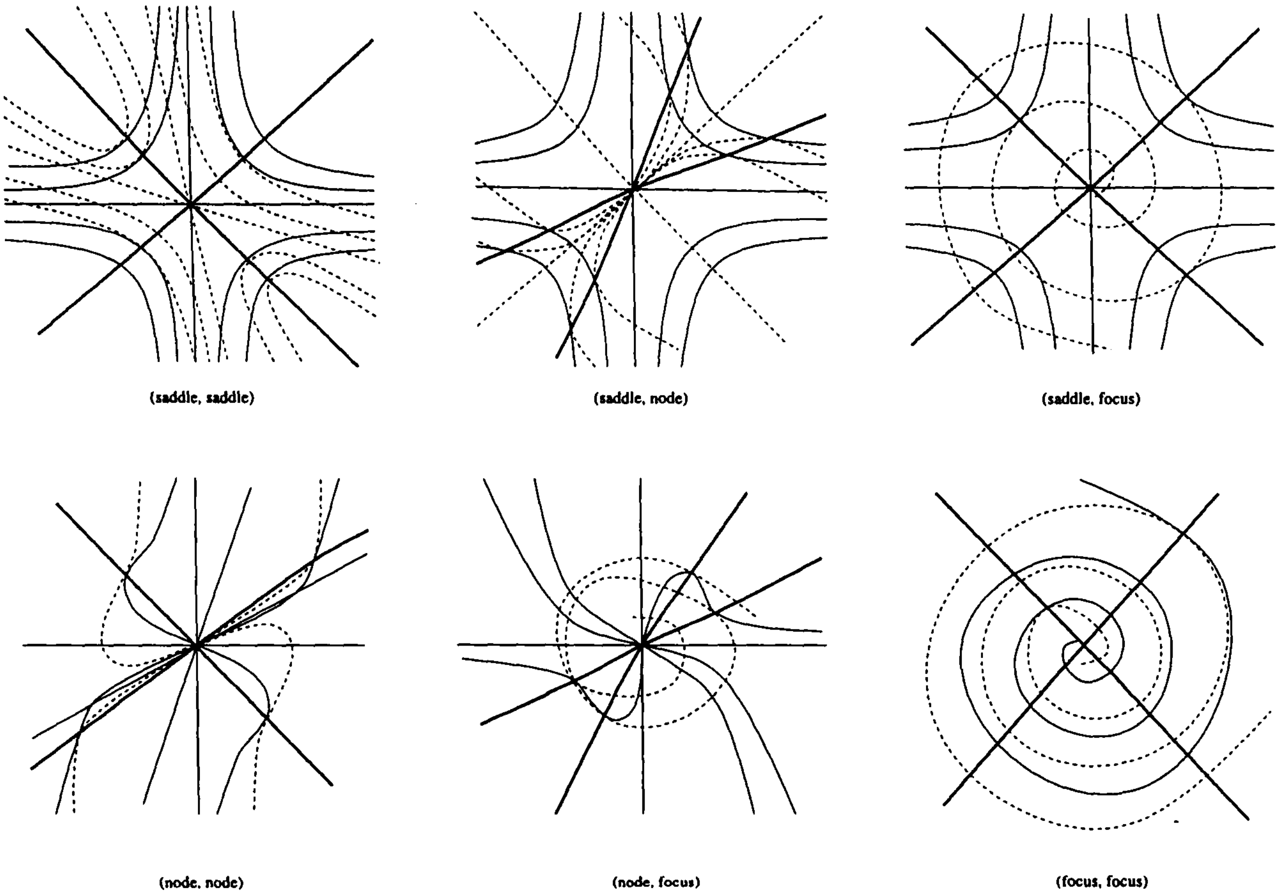

(saddle, saddle)
(saddle, node)
(saddle, focus)
(node, node)
(node, focus)
(focus, focus)

Figure 7b

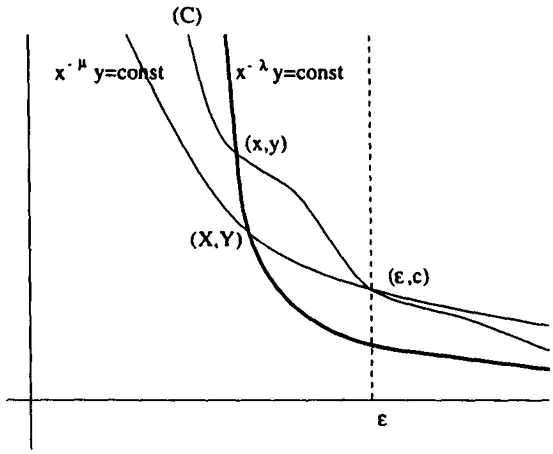


Figure 8

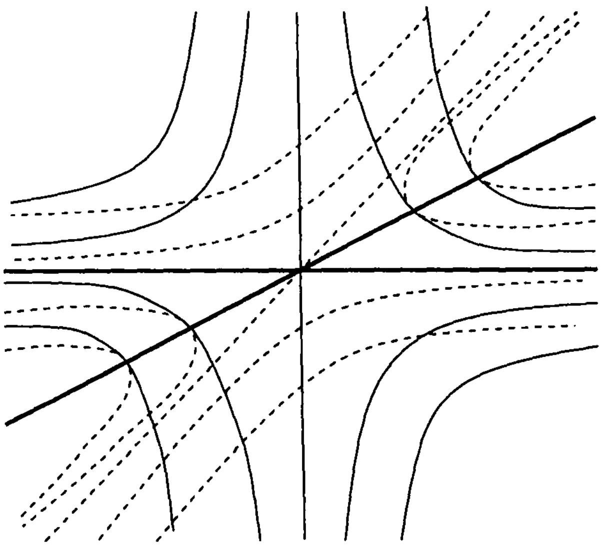


Figure 9

NOTAS DO ICMSC

SÉRIE MATEMÁTICA

059/97 ARRIETA, J.M.; CARVALHO, A.N.; RODRIGUEZ-BERNAL, A. - Parabolic problems with nonlinear boundary condition and critical nonlinearities.

058/97 BIASI, C.; GONÇALVES, D.L.; LIBARDI, A.K.M. - Metastable immersion with the normal bordism approach.

057/97 BIASI, C.; DACCACH, J.; SAEKI, O - A primary obstruction to topological embeddings and its applications.

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055/97 RUAS, M.A.S.; SEADE, J. - On real singularities which fiber as complex singularities.

054/97 CARVALHO, A.N.; CHOLEWA, J. W.; DLOTKO, T. - Global attractors for problems with monotone operators.

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051/97 ARRIETA, J. M.; CARVALHO, A.N. - Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations

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