# ABOUT $x^4 = \omega(x)^3 x$ TRAIN ALGEBRAS

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**Abstract**

*The purpose of this paper is to study in particular a class of train algebras of rank 4 that satisfy the t-equation $x^4 = \omega(x)^3 x$. Some results concerning to structure theorems of these algebras are given.*

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# 1. Introduction

Let $\mathbf{F}$ be an infinite field of characteristic not 2 and $\mathbf{A}$ a finite dimensional, commutative, non-associative algebra over $\mathbf{F}$. If $\omega : \mathbf{A} \to \mathbf{F}$ is a non-zero homomorphism, the ordered pair $(\mathbf{A}, \omega)$ is called a baric algebra and $\omega$ the weight function of $(\mathbf{A}, \omega)$. The ideal $\mathbf{B} = ker\omega$ has codimension 1 in $\mathbf{A}$ and if $c \in \mathbf{A}$, $\omega(c) = 1$ then $\mathbf{A}$ can be descomposed as $\mathbf{A} = \mathbf{F}c \oplus \mathbf{B}$. If there exist elements $\gamma_1, \cdots, \gamma_{n-1} \in \mathbf{F}$ such that the equation

$$x^r + \gamma_1\omega(x)x^{r-1} + \cdots + \gamma_{r-1}\omega(x)^{r-1}x = 0 \tag{1}$$

holds identically in $\mathbf{A}$, then $\mathbf{A}$ is called a *train algebra*. The equation like (1) with minimum degree is the rank equation of $\mathbf{A}$, $r$ is the rank of $\mathbf{A}$ and the roots of the algebraic equation $x^r + \gamma_1 x^{r-1} + \cdots + \gamma_{r-1}x = 0$ in some extension field of $\mathbf{F}$ are the train roots of $\mathbf{A}$. We will assume that all these train roots are in $\mathbf{F}$ itself. In these conditions, $1 + \gamma_1 + \cdots + \gamma_{n-1} = 0$ and $\mathbf{B}$ satisfies the monomial equation $x^r = 0$.

We will be mainly concerned in this note with train algebras of rank 4, for which equation (1) becomes $x^4 + \gamma_1\omega(x)x^3 + \gamma_2\omega(x)^2x^2 + \gamma_3\omega(x)^3x = 0$.

Every baric algebra with an idempotent of weight 1 can be obtained in the following way. Suppose $\mathbf{B}$ is an arbitrary commutative finite dimensional algebra over $\mathbf{F}$. Take the direct sum $\mathbf{A} = \mathbf{F} \oplus \mathbf{B}$ and define a multiplication in $\mathbf{A}$ by

$$(\alpha, a)(\beta, b) = (\alpha\beta, ab + \tau_{\mathbf{B}}(\alpha b + \beta a)), \quad \alpha, \beta \in \mathbf{F};\ a, b \in \mathbf{B} \tag{2}$$

where $\tau_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$ is an arbitrary linear mapping. Then $\omega : \mathbf{A} \to \mathbf{F}$ given by $\omega(\alpha, a) = \alpha$ is non-zero homomorphism, $(\mathbf{A}, \omega)$ is a baric algebra and $(1, 0)$ is an idempotent of weight 1. As $(1, 0)(0, a) = (0, \tau(a))$, multiplication by $(1, 0)$ is the same as the operator $\tau_{\mathbf{B}}$ acting on $\mathbf{B}$. In general two different $\tau's$ may give rise to $\mathbf{B}$-isomorphic algebras.

The theory of train algebras of rank 2 is trivial and for $r = 3$, the papers [1] and [2] contain some basic material. In this paper we try to develop the theory for $r = 4$, where a small number of results are known. For instance, it is known that every nuclear Bernstein algebra satisfies the train equation $x^4 - \frac{3}{2}\omega(x)x^3 + \frac{1}{2}\omega(x)^2x^2 = 0$. We shall study mainly the case particular of algebras that satisfy the t-equation $x^4 = \omega(x)^3x$.

# 2. General results

With the previous notation, if $\mathbf{B}$ satisfies the identity $a^4 = 0$ for all $a \in \mathbf{B}$, it is easy to see that $\mathbf{A}$ satisfies the rank equation

$$x^4 + \gamma_1\omega(x)x^3 + \gamma_2\omega(x)^2x^2 + \gamma_3\omega(x)^3x = 0 \tag{3}$$

if and only if the following identities hold:

$$2[\tau(a)a]a + \tau(a^2)a + \tau(a^3) + \gamma_1 a^3 = 0 \tag{4}$$

$$(\tau^2 + \gamma_1\tau + \gamma_2\mathbf{I_B})(a^2) + [2\tau + (1+2\gamma_1)\mathbf{I_B}](a\tau(a)) + 2\tau^2(a)a = 0 \tag{5}$$

$$2\tau^3 + (1+2\gamma_1)\tau^2 + (1+\gamma_1+2\gamma_2)\tau + \gamma_3\mathbf{I_B} = 0 \tag{6}$$

where $a$ is an arbitrary element of $\mathbf{B}$ and $I_{\mathbf{B}}$ is the identity operator in $\mathbf{B}$.

These equations are obtained by substituting $x = (\alpha, a) \in \mathbf{A} = \mathbf{F} \oplus \mathbf{B}$ in the equation (3), where

$x^2 = (\alpha^2, a^2 + 2\alpha\tau(a))$

$x^3 = (\alpha^3, a^3 + \alpha[2\tau(a)a + \tau(a^2)] + \alpha^2[\tau(a) + 2\tau^2(a)]$

$x^4 = (\alpha^4, a^4 + \alpha[2(\tau(a)a)a + \tau(a^2)a + \tau(a^3)] + \alpha^2[\tau(a)a + 2\tau^2(a)a + 2\tau(\tau(a)a) + \tau^2(a^2)] + \alpha^3[\tau(a) + \tau^2(a)2\tau^3(a)]$

and comparing like powers of $\alpha, \cdots, \alpha^4$.

Conversely, the proof is obvious, up to the calculations which are long. The equation (6) can be written as

$$(2\tau - \mathbf{I_B})[\tau^2 + (1+\gamma_1)\tau + (1+\gamma_1+\gamma_2)\mathbf{I_B}] = 0 \tag{7}$$

We shall study now the case $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = -1$, that is, algebras that satisfy the t-equation

$$x^4 = \omega(x)^3x \tag{3'}$$

Furthermore the relation (7) in this case can be written as

$$(2\tau - I_{\mathbf{B}})(\tau^2 + \tau + I_{\mathbf{B}}) = 0 \tag{7'}$$

thus the proper values of $\tau$ are $\frac{1}{2}$, $\lambda$, $\delta$, where $\lambda$ and $\delta$ are the roots of the polynomial

$$x^2 + x + 1 = 0 \tag{8}$$

We note that they satisfy the following relations: $\lambda^3 = \delta^3 = 1$, $\lambda + \delta = -1$ and $\lambda\delta = 1$.

If we have a train algebra $\mathbf{A}$ of rank 4, constructed as it is explained before, we can decompose $\mathbf{B} = \mathbf{U} \oplus \mathbf{V} \oplus \mathbf{W}$ where $\mathbf{U} = ker(\tau - \frac{1}{2}\mathbf{I_B})$, $\mathbf{V} = ker(\tau - \lambda\mathbf{I_B})$ and $\mathbf{W} = ker(\tau - \delta\mathbf{I_B})$.

On the other hand, the linearization of the equation (3') yields

$$2x[x(xy)] + x(x^2y) + x^3y = \omega(x)^3y + 3\omega(x)^2\omega(y)x \tag{9}$$

or

$$2z[x(xy)] + 2x[z(xy)] + 2x[x(yz)] + z(x^2y) + 2x[y(xz)] + y(zx^2)$$
$$+2y[x(xz)] = 6\omega(x)\omega(z)\omega(y)x + 3\omega(x)^2\omega(y)z + 3\omega(x)^2\omega(z)y \tag{10}$$

or

$$t[x(yz)+y(xz)+z(xy)]+x[y(zt)+z(yt)+t(yz)]+y[x(zt)+z(xt)+t(xz)]$$
$$+z[t(xy) + x(ty) + y(xt)] = 3[\omega(xyz)t + \omega(xyt)z + \omega(xtz)y + \omega(ytz)x] \tag{11}$$

From (11) we obtain some relations between the above defined subspaces. We note also that they are obtained from [4]:

$$\mathbf{U}^2 \subset \mathbf{V} \oplus \mathbf{W},\ \mathbf{V}^2 \subset \mathbf{W},\ \mathbf{W}^2 \subset \mathbf{V}$$
$$\mathbf{UV} \subset \mathbf{U} \oplus \mathbf{W},\ \mathbf{UW} \subset \mathbf{U} \oplus \mathbf{V} \text{ and } \mathbf{VW} = \{0\} \tag{12}$$

**Remark.**
In relation to the subspaces $\mathbf{V}$ and $\mathbf{W}$ above defined, we can deduce immediately from (12) that $\mathbf{V}^3 = 0$ and $\mathbf{W}^3 = 0$. Besides from [4] it is known that $\mathbf{V}^2\mathbf{W} = 0$, $\mathbf{W}^2\mathbf{V} = 0$ and the subspace $\mathbf{Z} = \mathbf{V} \oplus \mathbf{W}$ satisfies $\mathbf{Z}^3 = 0$. ■

On the other hand, in view of these results we obtain easily:

**Corollary 1** $\mathbf{U}^2\mathbf{V}^2 = 0$, $\mathbf{U}^2\mathbf{W}^2 = 0$, $\mathbf{V}^2\mathbf{W}^2 = 0$, $(\mathbf{V}^2)^2 = 0$, $(\mathbf{W}^2)^2 = 0$, $(\mathbf{U}^2)^2\mathbf{V} = 0$, $(\mathbf{U}^2)^2\mathbf{W} = 0$ *and* $(\mathbf{U}^2)^2\mathbf{U}^2 = 0$.

■

Also from [4] it is known that:

**Corollary 2** *Let* $\mathbf{A}$ *satisfy (3'). Then (3') is a train equation of minimal degree for* $\mathbf{A}$ *if only if* $\mathbf{V}$ *and* $\mathbf{W}$ *are both nonzero subspaces.*

■

**Corollary 3** *The algebra* $\mathbf{C} = \mathbf{F}e \oplus \mathbf{U} \oplus \mathbf{V}$ *(where* $\mathbf{W} = 0$*) is a train algebra of rank 3.*

**Proof:** In these conditions and from (12) we have that:

$$\mathbf{UV} \subset \mathbf{U},\ \mathbf{U}^2 \subset \mathbf{V},\ \mathbf{V}^2 = 0$$

Now, similarly as was proved in the corollary 2 (see [4]), we obtain that $(\mathbf{C}, \omega)$ satisfies the t-equation: $x^3 - (1+\lambda)\omega(x)x^2 + \lambda\omega(x)^2 x = 0$. A similar result is obtained when $\mathbf{V} = 0$. Thus the assertion is valid.
■

**Proposition 1** *For all* $k \geq 1$, $\mathbf{B}^k$ *is an ideal of* $\mathbf{A}$.

**Proof:** This is obvious for $k = 1$ and we proceed by induction. We note that it is sufficient to show that $e \cdot \mathbf{B}^{k+1} \subset \mathbf{B}^{k+1}$, that is, $\mathbf{B}^{k+1}$ is $L_e$-invariant. Suppose that $\mathbf{B}^k$ is an ideal of $\mathbf{A}$. From (10) and taking $x \in \mathbf{B}$, $z \in \mathbf{B}^{k-1}$, $y = e$ we have

$$2z[x(ex)] + 2x[z(ex)] + 2x[x(ez)] + z(ex^2) + 2x[e(xz)]$$

$$+e(zx^2) + 2e[x(xz)] = 0$$

besides, $zx^2 + 2x(xz) \in \mathbf{B}^{k+1}$ whence $e[zx^2 + 2x(xz)] \in \mathbf{B}^{k+1}$ since all the other sumands of the above relation are in $\mathbf{B}^{k+1}$. Thus $\mathbf{B}^k$ is an ideal of $\mathbf{A}$. ■

**Proposition 2** *If* $\mathbf{A}$ *is a train algebra satisfying (3') then it is not a power-associative algebra.*

**Proof:** This is trivial from [5]. ■

## 3. Some special cases

**(A)** In the following, we shall do reference to the algebra $\mathbf{C} = \mathbf{F}e \oplus \mathbf{Z}$ $(\mathbf{U} = 0)$. We do note that if $e \in I_p(\mathbf{C})$ then $\omega(e) = 1$. In fact, if $e \in I_p(\mathbf{C})$ and $\omega(e) = 0$, then $e \in ker\omega$ and $e^4 = e = 0$, a contradiction. Hence $\omega(e) \neq 0$. Besides as $\omega(e) = \omega(e^2) = \omega(e)^2$ then $\omega(e)[\omega(e) - 1] = 0$ and so $\omega(e) = 1$.

**Proposition 3** *The algebra $\mathbf{C} = \mathbf{F}e \oplus \mathbf{Z}$ has exactly one idempotent element.*

**Proof:** Let $x = \alpha e + v + w$ be a nonzero idempotent element in $\mathbf{C}$. From $x^2 = x$ we deduce that

$$\alpha^2 e + v^2 + w^2 + 2\alpha ev + 2\alpha ew = \alpha e + v + w$$

and this shows that

$$\alpha^2 = \alpha, \ 2\alpha\lambda v + w^2 = v, \ 2\alpha\delta w + v^2 = w$$

If $\alpha = 0$ then $x = x^4 = 0$. Thus $\alpha = 1$, besides we find $(2\lambda - 1)v^2 = 0$ whence $v^2 = 0$ which implies $w = 0$ and $v = 0$. Therefore $x = e$, which proves the proposition. ■

**Proposition 4** *$\mathbf{C} = \mathbf{F}e \oplus \mathbf{Z}$ is a special train algebra and hence genetic.*

**Proof:** We know that $\mathbf{C}$ is a train algebra of rank 4 and that $\mathbf{Z}^3 = 0$ then $\mathbf{Z}$ is a nilpotent algebra. Furthermore, from the proposition 1 we can conclude that $\mathbf{C}$ is a special train algebra and so consequently a genetic algebra. ■

**(B)** Now, we shall study the algebra $\mathbf{A} = \mathbf{F}e \oplus \mathbf{U} \oplus \mathbf{Z}$ with some restrictions of the proper subspaces $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$. Also, we do note that the results obtained for $\mathbf{V}$ also are verified for $\mathbf{W}$. All this by the symmetric properties of the mentioned subspaces.

**Proposition 5** *If $\mathbf{UV} \subset \mathbf{W}$ then $\mathbf{Z}(\mathbf{UV}) = 0$ and $\mathbf{UV}^2 = 0$.*

**Proof:** (a) As $\mathbf{UV} \subset \mathbf{W}$ it follows immediately that $\mathbf{V}(\mathbf{UV}) = 0$. On the other hand, we choose $x = u \in \mathbf{U}$, $y = v \in \mathbf{V}$, $z = w \in \mathbf{W}$ and $t = e$. Therefore from (11) it follows that $e[u(vw) + v(uw) + w(uv)] + u[v(ew) + w(ev) + e(vw)] + v[u(ew) + w(eu) + e(uw)] + w[e(uv) + u(ev) + v(eu)] = 0$. In this conditions we obtain that $2e[w(uv)] = w(uv)$ and so $w(uv) \in \mathbf{U}$. But also $w(uv) \in \mathbf{V}$, then $w(uv) = 0$ whence $\mathbf{W}(\mathbf{UV}) = 0$. Therefore $\mathbf{Z}(\mathbf{UV}) = 0$.
(b) Also, from (11) with $x = u \in \mathbf{U}$, $y = z = v \in \mathbf{V}$ and $t = e$ we have $e[uv^2 + 2v(uv)] + u[2v(ev) + ev^2] + 2v[u(ev) + v(eu)e(uv)] = 0$ or else $e(uv^2) + (\delta + 2\lambda)uv^2 = 0$ whence $e(uv^2) = (1 - \lambda)uv^2$ so $uv^2 = 0$ because $1 - \lambda$ is not a proper value. Therefore $\mathbf{UV}^2 = 0$. ■

**Proposition 6** *If* $\mathbf{UZ} \subset \mathbf{Z}$ *then* $u(u\mathbf{Z}) = 0$, $u \in \mathbf{U}$.

**Proof:** Let $x = y = u \in \mathbf{U}$, $z = v \in \mathbf{V}$ and $t = e$. Then from (11) we have $e[2u(uv) + vu^2] + 2u[u(ev) + v(eu) + e(uv)] + v[eu^2 + 2u(eu)] = 0$ or else $(2\lambda - 1)u(uv) + (\delta + 1)vu^2 + v(eu^2) = 0$. As $\mathbf{U}^2 \subset \mathbf{Z}$ hence we do $u^2 = v' + w'$ whence we get $(2\lambda - 1)u(uv) + (\delta + 1)vu^2 + \lambda vv' = 0$ and so finally we obtain $(2\lambda - 1)u(uv) + (\lambda^2 + \lambda + 1)vu^2 = 0$, remember that $\lambda^2 + \lambda + 1 = 0$. Therefore $u(uv) = 0$ for all $v \in \mathbf{V}$, that is, $u(u\mathbf{V}) = 0$. Analogously we have $u(u\mathbf{W}) = 0$ and so $u(u\mathbf{Z}) = 0$. ■

Now, we want to determine some theorems of structure of these algebras. We shall analize the case $\mathbf{U}^2 \subset Ann(ker\omega)$

(1) $\mathbf{U}^2 \neq 0$.

In this situation we have the following results:

**Proposition 7** $\mathbf{B} = \mathbf{U} \oplus \mathbf{Z}$ *is a nilalgebra.*

**Proof:** Let $x \in \mathbf{U} \oplus \mathbf{Z}$ then $x = u + v + w$ where $u \in \mathbf{U}$, $v \in \mathbf{V}$ and $w \in \mathbf{W}$. Thus $x^2 = u^2 + v^2 + w^2 + 2uv + 2uw$ and immediately it follows from above facts that $x^3 = 0$. ■

**Proposition 8** *The set of idempotent elements of* $\mathbf{A}$ *are given by*

$$I_p(\mathbf{A}) = \{e_u := e + u + \frac{1}{7}(3u^2 + 2eu^2)/u \in \mathbf{U}\}$$

*where* $e$ *is an idempotent element of* $\mathbf{A}$.

**Proof:** Let $e' = e+u+v+w$ be a nonzero idempotent element in $\mathbf{A}$. From $e'^2 = e'$ we have that $u^2+v^2+w^2+2\lambda v+2\delta w+2uv+2uw = v+w$. Thus $uv = 0$, $uw = 0$, $v^2 = 0$ and $w^2 = 0$. Let $u^2 = v' + w'$ then it is clear that $v' = (1-2\lambda)v$ and $w' = (1-2\delta)w$. Therefore $u^2 = (1-2\lambda)v + (1-2\delta)w$.
On the other hand, if $e' = e + u + x$ with $x \in \mathbf{Z}$ and as $e'^2 = e'$ then $x = 2ex + u^2$ hence $2ex = 4e(ex) + 2eu^2$ and besides $4e(ex) = 8e(e(ex)) + 4e(eu^2)$, thus we obtain $x = -\frac{1}{7}[u^2 + 2eu^2 + 4e(eu^2)]$. But from (10) we have $e(eu^2) + eu^2 + u^2 = 0$ and so finally

$$e' = e + u + \frac{1}{7}(3u^2 + 2eu^2)$$

■

(2) $\mathbf{U}^2 = 0$.

In this case we have the following new property:

**Proposition 9** *The set of idempotent elements of* $\mathbf{A}$ *are given by*

$$I_p(\mathbf{A}) = \{e_u := e + u / u \in \mathbf{U}\}$$

**Proof:** Let $x = \alpha e + u + v + w$ be a nonzero idempotent element in $\mathbf{A}$. From $x^2 = x$ we have that

$$\alpha^2 e + v^2 + w^2 + 2\alpha\lambda v + 2\alpha\delta w + 2uv + 2uw = \alpha e + u + v + w$$

Therefore $\alpha^2 = \alpha$, $w^2 + 2\lambda v + 2uw = v$ and $v^2 + 2\delta w + 2uv = w$. whence

$$\alpha = 1,\ w^2 + 2uw = (1-2\lambda)v \text{ and } v^2 + 2uv = (1-2\delta)w$$

Hence $w^2 v + 2v(uw) = (1-2\lambda)v^2 \implies v^2 = 0$ because $v(uw) = 0$. Analogously $w^2 = 0$ and this show that $v = 2(1-2\lambda)^{-1}uw$ and $w = 2(1-2\delta)^{-1}uv$, but $uv = 2(1-2\lambda)^{-1}u(uw) = 0$ and $uw = 2(1-2\delta)^{-1}u(uv) = 0$. Finally we deduce that $v = w = 0$ and so $e_u = e + u$. ■

# 4. Train algebras of rank 4 in fields of characteristic 2

In the following, we shall assume that $\mathbf{F}$ has characteristic 2.

**Proposition 10** *If* **A** *verifies the t-equation (3'), then the following assertions are satisfied:*
*(a)* **A** *admit an unique idempotent element e.*
*(b) If besides,* **A** *is a power-associative algebra then it is a quasi-constant algebra of order 2.*

**Proof:** (a) Let $x \in \mathbf{A}$ such that $\omega(x) = 1$ then $x^2 - x \in ker\omega$ therefore $(x^2 - x)^4 = 0 \Longleftrightarrow (x^2 - x)^2(x^2 - x)^2 = (x^2)^2(x^2)^2 + x^2x^2 = 0$ whence $(x^2)^2(x^2)^2 = x^2x^2 = (x^2)^2$ which implies $e = (x^2)^2$.
(b) Let $x = \omega(x)e + y$, $y \in ker\omega$ then easily we conclude that $x^4 =$ $\omega(x)^4 e$ since $y^4 = 0$.

On the other hand, in these conditions we can prove that the idem-potent element $e$ is unique. Since, suppose that $e'$ is another idem-potent, that is, $e'^2 = e'$ then $\omega(e') = 1$ and $e' = e'^2 = (e'^2)^2 = e'^4 =$ ■
$\omega(e')^4 e = e$ so $e = e'$.

**Proposition 11** *Let* **A** *be a train algebra satisfying (3'). Then we have the following results:*
*(a)* **B** *is a nil Jordan algebra.*
*(b)* **A** *is a special train algebra and hence genetic.*

**Proof:** (a) Use (9) to obtain $x(x^2y) + x^3y = \omega(x)^3y + \omega(x)^2\omega(y)x$.
Now, replace $x$ by $x^2$ in this equation and $y \in \mathbf{B}$ we get $x(yx^2) =$ $x^2(xy)$. Also for $x \in \mathbf{B}$ from (3) we find $x^4 = 0$. So **B** is a nil Jordan algebra and consequently a nilpotent algebra. Besides clearly **B** is a power-associative algebra.
(b) We will prove that $\mathbf{B}^k$ is an ideal of **A** and this is easily established by induction on $k$. Suppose that $\mathbf{B}^k$ is an ideal of **A**. From (10) we have that $z(x^2y) + y(zx^2) = \omega(x)^2\omega(y)z + \omega(x)^2\omega(z)y$. Now taking $x \in \mathbf{B}$, $z \in \mathbf{B}^{k-1}$, $y = e$ we get $e(zx^2) = -z(ex^2) \in \mathbf{B}^{k+1}$. Therefore $e \cdot \mathbf{B}^{k+1} \subset \mathbf{B}^{k+1}$ and so $\mathbf{B}^k$ is an ideal of **A** as required. ■

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