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Abstract

We present some conditions which permit to decide when a tilted algebra is a Koszul algebra, identifying maps between the direct summands of the tilting module. We also obtain some applications of our result. We show that a BB tilted algebra is simply connected if and only if the original algebra is simply connected

1 Introduction

Koszul algebras have played important role in several areas of Mathematics. The concept of tilting is also becoming more and more an important one. Both concepts are fundamental in the theory of representations of Artin algebras.

In this work all the algebras are quotients of finite quiver algebras by ideals I contained in the square of the ideal generated by the arrows. The radical of an algebra Λ is denoted by $r(\Lambda)$ or simply by r. We say that an algebra is graded when the ideal I is homogeneous with respect to the length grading on paths. All modules are left modules and finitely generated. Each indecomposable modules is identified with its isomorphism class. We also are denoting the group $\mathrm{Hom}_\Lambda(A, B)$ by (A, B). Given Λ a quotient of a quiver algebra we use the same notation for an arrow in the quiver and its class on the algebra.

Our main theorem is the following:

Main Theorem. *Let Λ be a finite-dimensional algebra over an algebraically closed field k, and T a Λ-module. Assume that $\Gamma = \mathrm{End}_\Lambda^{op}(T) = kQ/I$ has global dimension 2. Let :*

$$0 \longrightarrow P_{(2)} \xrightarrow{\rho_*} P_{(1)} \xrightarrow{f_*} \Gamma \to \Gamma/r \to 0$$

be the minimal projective resolution of Γ/r. Then, Γ is a Koszul algebra if and only if each component of ρ_* is defined by a component of T-sink maps.

We describe the ordinary quiver of Γ in the proof of the main theorem. In the final paragraph we prove that the Brenner-Butler tilted algebras are Koszul,

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that BB-algebras of simply connected hereditary algebras are simply connected and give a class of algebras whose iterated tilted algebras are Koszul.

We recollect now some definitions and basic facts on the theory of Koszul algebras. The reader can see these definitions and more details in [GM].

A graded algebra Γ is called a Koszul algebra when the Yoneda algebra $E(\Gamma) = \coprod_{n\geq 0} \operatorname{Ext}^n_\Gamma(\Gamma/r, \Gamma/r)$ is 1-generated, that is, the elements in $\operatorname{Ext}^1_\Gamma(\Gamma/r, \Gamma/r)$ generate all higher extension groups under the Yoneda's product. In the same work, the authors present a result which gives us a suitable condition to identify Koszul algebras, which we describe now.

A graded Γ-module M, generated in degree zero is called Koszul module when it has a linear resolution, that is, there exist a graded projective resolution

$$...P_{(n)} \to P_{(n-1)} \to ... \to P_{(2)} \to P_{(1)} \to P_{(0)} \to M \to 0$$

such that $P_{(j)}$ is generated in degree j, $\forall j \geq 0$.

In the mentioned work, [GM] it is shown that a graded k-algebra, is a Koszul algebra if and only if every simple module is a Koszul module.

As examples of Koszul algebras we have hereditary algebras, quadratic algebras with global dimension 2 , monomials quadratics algebras (cf. [GZ]) and the Brenner-Butler tilted algebras (see the last paragraph).

Graded algebras of global dimension two are Koszul if and only if they are quadratic. Since tilted algebras have global dimension two, they are Koszul if and only if they are quadratic and graded.

We define now the fundamental notion of tilting module.

Definition 1.1 *Let Λ be an algebra, a Λ-module T is called a tilting module when the following conditions are satisfied:*
(i) $pd_\Lambda T \leq 1$
(ii) $Ext^1_\Lambda(T,T) = 0$
(iii) There is a short exact sequence $0 \to \Lambda \to T' \to T'' \to 0$, *with* T' *and* $T'' \in add(T)$.

We continue this introductory section by recalling some definitions and fixing some notations.

1. Given a tilting Λ-module T we may consider two full subcategories of the category of finitely generated modules, namely the category $\mathcal{T}(T)$ of all modules generated by T and the category $\mathcal{F}(T)$ of modules M satisfying $Hom_\Lambda(T, M) = 0$. The modules on $\mathcal{T}(T)$ are called torsion modules and the ones in $\mathcal{F}(T)$ torsion free modules.

2. The endomorphism ring $\Gamma = \operatorname{End}^{op}_\Lambda(T)$ of a tilting module T is called a tilted algebra from Λ.

   If Λ is hereditary, we just say that Γ is a tilted algebra.

3. Given a Λ module homomorphism $f: M \to N$ we denote by f_* the induced Γ-module homomorphism $Hom(T,f): Hom_\Lambda(T,M) \to Hom_\Lambda(T,N)$.

4. Given T_l an indecomposable direct summand of T, we denote by P_l the indecomposable projective Γ-module $Hom_\Lambda(T,T_l)$.

There is a close connection between the representation theory of the algebra Λ and the endomorphism ring Γ, as it is shown in [BB] and [HR]. In particular when Λ is hereditary, the torsion theory defined by T over the category of finite generated modules Γ-mod, splits.

We refer to [BB], [HR] and [AS] for the most results on tilting theory. We will use freely the results and the nomenclature which are defined in the mentioned works.

2 Tilted Algebras

In this section Λ denotes a hereditary algebra, T a tilting Λ-module and Γ the tilted algebra $End_\Lambda^{op}(T)$.

Our first proposition is the following:

Proposition 2.1 *Let $f: T' \to T_l$ be a Λ-module homomorphism, between modules in Add(T), such that $Hom_\Lambda(T,T') \xrightarrow{f_*} P_l \to S_l \to 0$ is a minimal projective presentation of the simple non-projective Γ-module S_l. Then f is either a monomorphism or an epimorphism.*

Proof: Assume that f is not an epimorphism. Consider the short exact sequence of Λ-modules given by $0 \to \mathrm{Im}\ f \xrightarrow{i} T_l \to \mathrm{coker}\ f \to 0$, where i is the canonical inclusion. Since $\mathrm{Im}\ f \in \mathcal{T}(T)$ we have that $\mathrm{Ext}^1_\Lambda(T, \mathrm{Im}\ f) = 0$ therefore applying the functor $\mathrm{Hom}_\Lambda(T,-)$, we obtain the following short exact sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(T, \mathrm{Im}\ f) \xrightarrow{i_*} P_l \longrightarrow \mathrm{Hom}_\Lambda(T, \mathrm{coker}\ f) \longrightarrow 0$$

Considering the fact that $f = i \circ f'$, where f' is given by $T' \xrightarrow{f} \mathrm{Im}\ f$, it follows that $f_* = i_* \circ f'_*$ and we also have that $r_\Gamma P_l = \mathrm{Im}\ f_* = \mathrm{Im}(i_* \circ f'_*)$. Since i_* is a monomorphism, we conclude that $\mathrm{Im}\ f'* \cong \mathrm{Im}\ f_* = r_\Gamma P_l$.

We use the Brenner-Butler equivalence and get that $(T, \mathrm{Im}\ f) \cong r_\Gamma P_l$. Hence, $S_l \cong (T, \mathrm{coker}\ f)$, with $\mathrm{coker}\ f \in \mathcal{T}(T)$, therefore $\mathrm{pd}_\Gamma S_l = 1$. So, $r_\Gamma P_l$ is a projective Γ-module and $\mathrm{Im}\ f \cong \mathrm{Im}\ f' = T'$. It follows that $f = i \circ f'$ is a monomorphism. ∎

Corollary 2.2 *Using the same notations and conditions, as in the former proposition, we have the following:*

(a) $S_l \in \mathcal{Y}(T)$ *if and only if* f *is a monomorphism. In this case,* $S_l \cong \mathrm{Hom}_\Lambda(T, coker f)$.
(b) $S_l \in \mathcal{X}(T)$ *if and only if* f *is an epimorphism. In this case,* $S_l \cong \mathrm{Ext}^1_\Lambda(T, \ker f)$.
(c) $\mathrm{pd}_\Gamma S_l = 1$ *if and only if* $ker f \in \mathcal{F}(T)$.

Proof: (a) If $S_l \in \mathcal{Y}(T)$ then pd $_\Gamma S_l = 1$ and hence f_* is a monomorphism. Applying the functor $T \otimes_\Gamma -$ to the exact short sequence $0 \to (T, T') \xrightarrow{f_*} (T, T_l) \to (T, N) \longrightarrow 0$, with $(T, N) \cong S_l$ for some $N \in \mathcal{T}(T)$, we conclude that the sequence $0 \to T' \xrightarrow{f} T_l \to N \to 0$ is exact, since $\mathrm{Tor}^\Gamma_1(T, N) = 0$. Hence, f is a monomorphism. Conversely, if f is a monomorphism, so is f_*. It is clear that $(T, \mathrm{coker} f) \cong S_l \in \mathcal{Y}(T)$.

By the former proposition and by (a) the statement (b) follows.

Applying the functor $\mathrm{Hom}_\Lambda(T, -)$ to the exact short sequence $0 \to ker f \to T' \to T_l \to 0$, we see that the statement (c) is also valid. ■

3 Our Main Result

In this section we drop the requirement of Λ being hereditary. Here Λ will be any finite dimensional algebra over the algebraically closed field k. We fix, as before, a tilting Λ-module T and a decomposition of T in indecomposable direct summands given by $T = \bigoplus_{j=1}^{n} T_j$ and assume, without lost of generality, that T is multiplicity free, that is: the indecomposable directs summands of T are pairwise non-isomorphic. Some authors call it a basic module.

Definition 3.1 *Let* M *and* $N \in \mathcal{T}(T)$ *with* M *indecomposable. We say that the non-zero* Λ*-morphism* $\alpha : N \to M$ *is a sink-torsion map if it is a sink map on the category* $\mathcal{T}(T)$*, in other words:* α *is a minimal non-split homomorphism and every non-zero non-split homomorphism* $\beta : L \to M$ *with* $L \in \mathcal{T}(T)$ *factors through* α.

We observe that if $f : \mathrm{E} \to M$ is sink map then the restriction of f in $\mathrm{tr}_T(\mathrm{E})$ given by $f' : \mathrm{tr}_T(\mathrm{E}) \to M$ is a sink-torsion map, where $\mathrm{tr}_T(\mathrm{E})$ is the trace of T in E.

Proposition 3.2 *Let* Λ *be a finite dimensional* k*-algebra and* T *a tilting* Λ*-module. Let* T_l *be an indecomposable direct summand of* T *and* $P_l = Hom_\Lambda(T, T_l)$*. Then, there exist only one sink-torsion map* $\alpha : E \to T_l$*, up to isomorphism. Moreover* $E \cong T \bigotimes_\Gamma rP_l$ *and* $\alpha = T \otimes \alpha_*$ *with* $\alpha_* : rP_l \to P_l$ *the natural inclusion.*

Proof: Let $E \in \mathcal{T}(T)$ and $\alpha : E \to T_l$ be a Λ-module homomorphism which induces the inclusion $\alpha_* : rP_l \to P_l$. Then using the BB-equivalence we get that

E and α have the form we want. We claim that α is a sink torsion map. It is clear that α is not a split epimorphism since α_* does not splits. Considering a non-zero non-split epimorphism $\beta : N \longrightarrow T_l$, with $N \in \mathcal{T}(T)$, we have that $\beta_* : (T, N) \rightarrow (T, T_l)$ is a non-split epimorphism, therefore it factors through α_* and so do β through α. The minimality of α_* implies the minimality of α and vice-versa, by the BB-equivalence.

To show the uniqueness, we observe that the following general result is valid. If a subcategory of an abelian category has sink maps then the sink maps are unique, up to isomorphism. ∎

Definition 3.3

1. *We say that a Λ-module $E \in \mathcal{T}(T)$ is the torsion-predecessor of an indecomposable direct summand T_l of T if $rP_l \cong \mathrm{Hom}_\Lambda(T, E)$ with $P_l = \mathrm{Hom}_\Lambda(T, T_l)$. We denote E by E_l.*

2. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a module in add(T), with M_i indecomposable. Then a module $E \in \mathcal{T}(T)$ will be called the torsion-predecessor of M if E is the direct sum of the torsion-predecessors of all the $M_i^!s$.*

Corollary 3.4 *A sink-torsion map is either a monomorphism or an epimorphism.*

Proof: Assume that a sink torsion map α is not an epimorphism. Hence, $\mathrm{Im}\ \alpha \subsetneq T_l$. By hypothesis, the inclusion $j : \mathrm{Im}\alpha \rightarrow T_l$ must factors through α, it follows that $E \cong \ker \alpha \oplus \mathrm{Im}\ \alpha$, hence $\ker \alpha \in \mathcal{T}(T)$. If $\ker \alpha \neq 0$ we conclude that $rP_l \cong (T, \ker \alpha) \oplus (T, \mathrm{Im}\ \alpha)$, with $(T, \ker \alpha) \neq 0$. Since α is minimal, so is α_*, but this contradicts the fact that $\alpha_*(T, \ker \alpha) = 0$. ∎

Corollary 3.5 *Using the notation above, let S_l be the top of the Γ-module P_l.*

1. *If α is a monomorphism then $S_l \cong \mathrm{Hom}_\Lambda(T, \frac{T_l}{E})$.*

2. *If α is an epimorphism then $S_l \cong Ext_\Lambda^1(T, ker\ \alpha)$.*

Proof: If α is a monomorphism then the short exact sequence of Λ-modules, $0 \rightarrow E \xrightarrow{\alpha} T_l \rightarrow \mathrm{coker}\ \alpha \rightarrow 0$, induces the following short exact sequence of Γ-modules, $0 \rightarrow (T, E) \xrightarrow{\alpha_*} P_l \rightarrow (T, \mathrm{coker}\ \alpha) \rightarrow 0$. It follows from the proposition 3.2 that $rP_l \cong (T, E)$, therefore $S_l \cong \mathrm{Hom}_\Lambda(T, T_l/_E)$.

If α is an epimorphism, we have the short exact sequence of Λ-modules, $0 \rightarrow \ker \alpha \rightarrow E \xrightarrow{\alpha} T_l \rightarrow 0$, which give us the following long exact sequence of Γ-modules:

$$0 \rightarrow (T, \ker \alpha) \rightarrow (T, E) \xrightarrow{\alpha_*} P_l \rightarrow \mathrm{Ext}_\Lambda^1(T, \ker \alpha) \rightarrow 0,$$

where $(T, E) \cong rP_l$. But, α_* is a monomorphism hence $(T, \ker \alpha) = 0$. It follows that $\ker\alpha \in \mathcal{F}(T)$ and $S_l \cong \mathrm{Ext}_\Lambda^1(T, \ker \alpha)$. ∎

Definition 3.6 *Let M be a non-zero Λ-module in $\mathcal{T}(T)$, and $\pi : T_M \longrightarrow M$ an homomorphism. The pair (T_M, π) will be called a T-generator if it is a minimal left add(T)-approximation of M. We recall here that this means that it satisfies the following conditions:*
(i) T_M is in addT, π is minimal.
(ii) Every morphism $\psi : T' \longrightarrow M$, with $T' \in addT$, factors through π.

Since add(T) is functorially finite we have every module has a minimal add(T) approximation, which is unique, up to isomorphism, and for modules in $\mathcal{T}(T)$ it is clear that the map π on the definition is an epimorphism. Moreover, we have that $\mathrm{Hom}_\Lambda(T, T_M)$ is the projective cover of $\mathrm{Hom}_\Lambda(T, M)$.

We show now a result which is valid for a graded algebra Γ which is tilted from a Λ-algebra .

We would like to observe that there are examples of tilted algebras which are not graded, that is, the ideal of presentation of a tilted algebra is not always a homogeneous ideal, the reader can see examples in [R].

Proposition 3.7 *Let T be a tilting Λ-module. Suppose that $\Gamma = kQ/I = End(T)^{op}$ and that I an homogeneous ideal. Let*

$$... \rightarrow P_{(3)} \rightarrow P_{(2)} \rightarrow P_{(1)} \rightarrow \Gamma \rightarrow \Gamma/r \rightarrow 0$$

be the minimal projective resolution of the top(Γ) where $P_{(j)} = \mathrm{Hom}_\Lambda(T, T_j')$ with $T_j' \in addT$. Let E_j be the torsion-predecessor T_j'. Then, Γ is a Koszul algebra if and only if, for each j, the canonical morphism from T_{j+1}' to the T-generator of E_j is a split monomorphism.

Proof: If Γ is a Koszul algebra then we have that $P_{(j)}$ is generated in degree j, for each $j \geq 0$. Since I is a homogeneous ideal it follows that $P_{(j)}$ is a direct summand of the projective cover of $rP_{(j-1)}$, for $j \geq 0$. Let's consider $E_{j-1} \in \mathcal{T}(T)$ such that $\mathrm{Hom}_\Lambda(T,\ E_{j-1}) \cong rP_{(j-1)}$ and $T_{E_{j-1}}$ the T-generator of E_{j-1}. Hence, $\mathrm{Hom}_\Lambda(T, T_{E_{j-1}})$ is the projective cover of $rP_{(j-1)}$. Therefore, T_j' is a direct summand of $T_{E_{j-1}}$.

Reciprocally, if the canonical map is a split monomorphism then $P_{(j+1)}$ is graded direct summand of the projective cover of $rP_{(j)}$ for $j \geq 0$. Since I is homogeneous, it follows that Γ is Koszul. ■

We need now to fix some more notations.

Let $P_l = \mathrm{Hom}_\Lambda(T, T_l)$ denote an indecomposable projective Γ-module, $\alpha_l : E_l \rightarrow T_l$ denotes its sink-torsion map and (T_{E_l}, π_l) the T-generator of E_l.

We also fix a decomposition $T_{E_l} = \oplus_{s=1}^{r} T_{l_s}^{m_{l_s}}$ where $T_{l_1}, ..., T_{l_r}$ are pairwise non-isomorphic, indecomposable. (In this case, m_{l_s} is the number of times that the simple Γ-module S_{l_s} appears in the top of $rP_l = (T, E_l)$.)

We have a decomposition of the map π_l such that $\pi_l = ((\pi_l)_1, \cdots, (\pi_l)_r)$ with $(\pi_l)_s : T_{l_s}^{m_{l_s}} \longrightarrow E_l$ for $s = 1, \cdots, r$.

We fix the notation $(\pi_l)_s = ((\pi_l)_s^1, \cdots, (\pi_l)_s^{m_{l_s}})$ with $(\pi_l)_s^{u_s} : T_{l_s} \longrightarrow E_l$ for $1 \leq u_s \leq m_{l_s}$. It follows that each component of $\alpha_l \pi_l$ is a map given by $\alpha_l(\pi_l)_s^{u_s} : T_{l_s} \longrightarrow T_l$ where $s = 1, \cdots, r$ and $1 \leq u_s \leq m_{l_s}$.

The map $\alpha_l \pi_l$ will be called *T-sink map of* T_l.

Definition 3.8 *Let $f : T_v \to T_l$ be a non-zero Λ-morphism between indecomposable modules in add(T). We say that f is T-irreducible if f is not an isomorphism and for any factorization $f = gh$ through addT implies that h is a split monomorphism or g is a split epimorphism.*

Lemma 3.9 *Let $f : T_v \to T_l$ be a non-zero Λ-morphism for $v \neq l$. If f is T-irreducible then f is a component of the map $\alpha_l \pi_l$, defined by the notation fixed above. Moreover, (T, T_v) is a direct summand of the projective cover of rP_l.*

Proof: Since $v \neq l$ we have that f is not an epimorphism, since α_l is a sink-torsion map, f must factors through α_l, but π_* is the projective cover of (T, E_l), hence f factors through $\alpha_l \pi_l$. It follows that there exists $\beta : T_v \longrightarrow T_{E_l}$ such that $f = \alpha_l \pi_l \beta$. Since $\alpha_l \pi_l$ is not a split epimorphism it follows that β is a split monomorphism. Hence T_v is a direct summand of T_{E_l}. We conclude that f is a component of the map $\alpha_l \pi_l$, as we wished to prove. ■

Now, we consider a presentation for Γ given by $\Gamma = kQ/I$. For each arrow $\alpha \in Q_1$ from l to v, one can associate a map $(f_\alpha)_* : P_v \to P_l$, called multiplication by the arrow α, which is defined by $(f_\alpha)_*(\varphi) = f_\alpha \circ \varphi$ for each $\varphi \in P_v$, where $f_\alpha : T_v \to T_l$ is the Λ-map that induces $(f_\alpha)_*$.

Lemma 3.10 *Let $(f_\alpha)_* : P_v \to P_l$ be a Γ-morphism given by multiplication by an arrow of Q, as above. Then, f_α is T-irreducible.*

Proof: Suppose that $f = gh$ for some $g : T' \to T_l$ and $h : T_v \to T'$ with $T' \in addT$. It follows that $f_* = g_* \circ h_*$. Moreover, if we consider $P_v = \Gamma\mathcal{E}_v$ such that $\mathcal{E}_v$ is the idempotent of Γ defined by $\mathcal{E}_v = (0, \cdots, I_{T_v}, \cdots, 0) : T \to T_v$, where $I_{T_v} : T_v \to T_v$ is the identity, we have that $f_*(\mathcal{E}_v) = (gh) \circ \mathcal{E}_v = (0, \cdots, gh, 0, \cdots, 0)$. Since gh factors through T' we have that $(gh)_*(\mathcal{E}_v)$ determines a path from the vertice l to the vertice v passing through the vertices associated to the projective (T, T'). Since we have that $(gh)_*$ is a map defined by multiplication by arrow, and I is admissible, we get that either h is a split monomorphism or g is a split epimorphism. ■

Lemma 3.11 *Let $P_l = \mathrm{Hom}_\Lambda(T, T_l)$ be a projective Γ-module. Then P_j is a direct summand of the projective cover of rP_l if only if there is a T-irreducible morphism $f : T_j \to T_l$.*

Proof: Assume that $P_j = (T, T_j)$ is a direct summand of the projective cover of radical of P_l. Hence, there exist a non-zero morphism $f_* : P_j \longrightarrow P_l$ such that $Imf_* \subset rP_l$ and $Imf_* \not\subset r^2P_l$. Let $f : T_j \longrightarrow T_l$ be the map that induce f_*. We claim that f is T-irreducible. Suppose not, that is, there exists $g : T' \longrightarrow T_l$ and $h : T_j \longrightarrow T'$ non-zero maps, with $T' \in \text{add}T$, such that $f = gh$, where g is not a split epimorphism and h is not a split monomorphism. By BB-equivalence we have that $f_* = g_*h_*$ where g_* is not a split epimorphism and h_* is not a split monomorphism. Hence $Img_* \subset rP_l$ and $Imh_* \subset r(T, T')$. It follows that $Imf_* \subset r^2P_l$, a contradiction from the hypothesis on f_*. The reciprocal was proved in lemma 3.9. ■

Remarks:
(1) The maps $(\alpha_l\pi_l)_* : \oplus_{s=1}^r P_{l_s}^{m_{l_s}} \to P_l$, are such that:
(i) m_{l_s} is the multiplicity of the simple Γ-module S_{l_s} in the top of rP_l,
(ii) the maps $\alpha_l(\pi_l)_s^{u_s} : T_{l_s} \longrightarrow T_l$ where $s = 1, \cdots, r$ and $1 \leq u_s \leq m_{l_s}$ are the components of the map $\alpha_l\pi_l$ for $u_s = 1, ..., m_{l_s}$, $s = 1, ..., r$.
(2) We have that $((T, T_{E_l}), (\pi_l)_*)$ is the projective cover of rP_l and since $(\alpha_l)_* : rP_l \to P_l$ is a sink map, one can concludes that S_l has a minimal presentation given by

$$\text{Hom}_\Lambda(T, T_{E_l}) \overset{(\alpha_l\pi_l)_*}{\longrightarrow} P_l \longrightarrow S_l \longrightarrow 0.$$

As a consequence of the first remark and lemmas 3.9 and 3.10, we have that each non-zero map $f_* : P_{l_s} \to P_l$, with $l_s \neq l$, defined as a multiplication by an arrow is also a component of the map $\alpha_l\pi_l$. Actually, one can assume that $f = \alpha_l\pi_s^{u_s} : T_{l_s} \to T_l$ for some u_s.

Using the second remark given above, it is easy to see that
$\{f_\alpha : T_{l_s} \to T_l / \alpha : l \longrightarrow l_s \in (Q_\Gamma)_1\} \cong \{\alpha_l(\pi_l)_s^{u_s} : T_{l_s} \longrightarrow T_l / u_s = 1, ..., m_{l_s}\}$ for each $s = 1, \cdots, r$ fixed, with $l_s \neq l$.

For our main theorem we fix, as usual, Λ a finite dimensional k-algebra with k an algebraically closed field and T be a tilting Λ-module. We also take $\Gamma = kQ/_I = End^{op}(\Lambda)$ with Q finite quiver and I is an admissible ideal.

Main Theorem. *Suppose that $\Gamma = \text{End}_\Lambda^{op}(T)$ has global dimension 2. Consider the minimal projective resolution of $\Gamma/_r$ given by:*

$$0 \to \oplus_{t=1}^r P_{j_t}^{m_{j_t}} \xrightarrow{\rho_*} \oplus_{v=1}^s P_{l_v}^{m_{l_v}} \xrightarrow{f_*} \Gamma \to \Gamma/_r \to 0.$$

Then, Γ is a Koszul algebra if and only if each component of ρ, say $\rho_{j_t, l_v} : T_{j_t} \to T_{l_v}$ is given by $\alpha_{l_v}(\pi_{l_v})_t^{u_t}$ for some $u_t = 1, ..., m_{j_t}$ defined as above.

Proof: We denote l_v simply by l, and keep the same notation from our considerations and lemmas above.

Assume now that Γ is a Koszul algebra. Then I is quadratic ideal and by a result in [B], $\oplus_{t=1}^{r} P_{j_t}^{m_{j_t}} \cong I/I^2$. So we get that each component of the map ρ with domain and codomain indecomposable is defined, up to isomorphism, by a multiplication by an arrow. Since Γ has finite projective dimension, its ordinary quiver has no loops and we have that $j_t \neq l$ for each $t = 1, \cdots, r$. By lemmas 3.9 and 3.10 above we have that each component of ρ, say $\rho_{j_t,l} : T_{j_t} \to T_l$, is given by $\alpha_l(\pi_l)_t^{u_t}$ for some $u_t = 1, \cdots, m_{l_t}$.

Reciprocally, suppose that each component of ρ, say $\rho_{j_t,l} : T_{j_t} \to T_l$ is given by $\alpha_l(\pi_l)_t^{u_t}$ for some $u_t = 1, ..., m_{j_t}$. Since $((\pi_l)_t)_*$ is a component of the projective cover of $(T, E_l) = rP_l$, we have that $P_{j_t}^{m_{j_t}}$ is a direct summand of the projective cover of rP_l, for each $t = 1, \cdots, s$. By hypothesis, we have that $(\alpha_l)_*$ is a sink map and hence by remark (2) above, we have that each component of ρ_* with domain and codomain indecomposable is defined by multiplication by arrows. Therefore, Γ is quadratic. Since Γ has global dimension 2, it follows that Γ is Koszul. ■

We would like to observe that as consequence of lemma 3.11 one can draw a full subquiver Q' of Q and by lemmas 3.9 and 3.10 complete Q' to Q. The following examples will ilustrates our results.

Examples

1. **A finite type Brenner-Butler algebra.**

   Let Λ be a quiver algebra whose quiver is the following:

$$\begin{array}{ccccc} & & & & 3 \\ & & & \nearrow & \\ 1 & \longrightarrow & 2 & & \\ & & & \searrow & \\ & & & & 4 \end{array}$$

   Let $T = \tau^- S_2 \oplus P_1 \oplus P_3 \oplus P_4$ be a tilting Λ-module. The Auslander-Reiten quiver of Λ is given by

$$\begin{array}{ccccccccccccc} (P_3) & & & & \tau^- P_3 & & & & I_3 & & & & \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & \\ & & P_2 & \longrightarrow & (P_1) & \longrightarrow & \tau^- P_2 & \longrightarrow & S_2 & \longrightarrow & I_2 & \longrightarrow & (I_1) \\ & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & \\ (P_4) & & & & \tau^- P_4 & & & & I_4 & & & & \end{array}$$

   The morphism $(P_2 \longrightarrow P_1)$, in the graph above, is a sink map; moreover, we have that $P_3 \oplus P_4$ is the T-generator of $\text{tr}_T P_2$. Also, we have that P_1 is the T-generator of I_2 and the morphism $I_2 \longrightarrow \tau^- S_2 = I_1$, given by the graph above, is the map that induces the sink map $r(T, \tau^- S_2) \hookrightarrow$

$(T, \tau^{-}S_2)$. We have that Γ is an algebra with radical squared zero and ordinary quiver $Q(\Gamma)$ given by:

$$\begin{array}{ccccc} & & & & \hat{3} \\ & & & \overset{\beta}{\nearrow} & \\ \hat{2} & \xrightarrow{\alpha} & \hat{1} & & \\ & & & \underset{\gamma}{\searrow} & \\ & & & & \hat{4} \end{array}$$

2. **An example of a concealed algebra.** Let Λ be a quiver algebra whose quiver is the following:

$$\begin{array}{ccccccc} 1 & & & & & & 5 \\ & \nwarrow & & & & \swarrow & \\ & & 3 & \longleftarrow & 4 & & \\ & \swarrow & & & & \nwarrow & \\ 2 & & & & & & 6 \end{array}$$

A local sketch of the preprojective component of the Auslander-Reiten of Λ is given by

$$\begin{array}{cccccccccccc} P_1 & & & & \tau^{-}P_1 & & & & \tau^{-2}P_1 & & & \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\ P_2 & \to & P_3 & \to & \tau^{-}P_2 & \to & \tau^{-}P_3 & \to & \tau^{-2}P_2 & \to & \tau^{-2}P_3 & \to \\ & & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & & & & P_4 & \to & P_6 & \to & \tau^{-}P_4 & \to & \tau^{-}P_6 & \to \\ & & & & & \searrow & & \nearrow & & \searrow & & \nearrow \\ & & & & & & P_5 & & & & \tau^{-}P_5 & \end{array}$$

Consider the tilting Λ-module $T = \tau^{-}P_5 \oplus \tau^{-}P_4 \oplus \tau^{-}P_3 \oplus \tau^{-}P_2 \oplus \tau^{-}P_1 \oplus P_6$. We have that the full subquiver of the Auslander-Reiten quiver of Λ, with vertex de indecomposable direct summands of T form a slice and it is the following quiver:

$$\begin{array}{ccccccc} \tau^{-}P_1 & \longrightarrow & \tau^{-}P_3 & \longrightarrow & \tau^{-}P_4 & \longrightarrow & \tau^{-}P_5 \\ & \nearrow & & \nearrow & & & \\ \tau^{-}P_2 & & P_6 & & & & \end{array}$$

where the arrows represent irreducible maps. It follows that Γ has projective radical and its quiver is the same as the quiver of Λ, moreover Γ is hereditary.

4 Some applications

4.1 *Brenner-Butler tilted algebras*

As usual, on this work, $\Lambda = kQ$ is a finite dimensional path algebra over a field k, with Q a finite connected quiver (so, it contains non-oriented cycles). Let $P_1, P_2, \cdots, P_n$ be a complete list of indecomposable projective non-isomorphic Λ-modules. Let's fix $S = S_i$ the simple Λ-module associated to the vertex i of Q. Assume that $\tau^- S_i \neq 0$. We have that $T = \tau^- S_i \oplus \bigoplus_{j\neq i} P_j$ is a tilting Λ-module. The endomorphism ring $\Gamma = \mathrm{End}_\Lambda(T)^{op}$ is called Brenner-Butler tilted algebra or, for convenience, BB-tilted, (cf. [AS], for instance). It is a known fact that the class of torsion-free modules is given by $\mathcal{F}(T) = \mathrm{Cogen}\ (S)$. We shall prove the following result:

Theorem (BB):*Every BB-tilted algebra is a Koszul algebra.*

Proof: We prove this theorem by showing that each simple Γ-module has a minimal projective resolution which satisfies the conditions of our main theorem.

We have that $\hat{S} = Ext^1_\Lambda(T,S)$ is a simple $\mathrm{End}_\Lambda(T)^{op}$-module, since $Ext^1_\Lambda(T,S) \cong \mathrm{DHom}_\Lambda(S,\tau T) = \mathrm{DHom}_\Lambda(S,S) \cong k$. Because we have that $ind\mathcal{F}(T) = \{S\}$, we conclude that $\Omega^2(\Gamma/r) = \Omega^2(\hat{S})$. If $\mathrm{pd}_\Gamma \hat{S} = 1$ then Γ is hereditary, therefore it is a Koszul algebra. So we assume $\mathrm{pd}_\Gamma \hat{S} = 2$.

Let $0 \longrightarrow S \longrightarrow I_i \longrightarrow I_1 \longrightarrow 0$ be a minimal injective coresolution of the simple Λ-module $S = S_i$, where $I_1 = I_{l_1}^{m_1} \oplus \cdots \oplus I_{l_t}^{m_t}$ is such that l_s is an immediate predecessor of i for $s = 1, \cdots, t$ and m_s is the number of arrows from l_s to i.

We have that the top of $\tau^- S = \mathrm{soc}\ I_1$, (cf. [ARS]). Since Λ is hereditary we have that $(DS)^* = 0$, hence the exact short sequence $0 \longrightarrow (DI_i)^* \longrightarrow (DI_1)^* \longrightarrow \tau^- S \longrightarrow 0$ is a Λ-projective minimal resolution of $\tau^- S$, where $(DI_j)^* = (P_j)^*$, for each vertex j of the quiver Q_Λ of Λ. Moreover, it is given by $0 \longrightarrow P_i \xrightarrow{f} \bigoplus_{s=1}^{t} P_{l_s}^{m_s} \xrightarrow{\pi} \tau^- S \longrightarrow 0$, where f is the map induced by multiplication by the arrows that join the vertices $l_1, \ldots, l_t$ to i in Q_Λ.

More precisely, let's consider a complete list of arrows joining l_s to the vertice i given by $\{\alpha_s^{u_s} : l_s \longrightarrow i\}_{s,u_s}$, where $s = 1, \cdots, t$ and $1 \leq u_s \leq m_{l_s}$, with $m_{l_s} =$ the number of arrows from l_s to i. Then $f = (f_s^{u_s})_{s,u_s}$, where $f_s^{u_s} : P_i \to P_{l_s}$ is defined by multiplication by the arrow $\alpha_s^{u_s}$, for s and u_s as above.

Applying the functor $\mathrm{Hom}_\Lambda(T,-)$ to the projective resolution of $\tau^- S$ given above, we obtain the following exact long sequence of Γ-modules

$$0 \longrightarrow (T,P_i) \xrightarrow{f_*} (T,P) = \bigoplus_{s=1}^{t} (T,P_{l_s})^{m_{l_s}} \xrightarrow{\pi_*} (T,\tau^- S) \longrightarrow Ext^1_\Lambda(T,P_i) \longrightarrow 0$$

where $\pi_* = \mathrm{Hom}_\Lambda(T,\pi)$. We observe that π_* is not an epimorphism and hence

Im $\pi_* \subseteq r_\Gamma(T,\tau^- S)$.

Moreover, we claim that coker $\pi_* \cong \hat{S}$.

Indeed, for any $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in r_\Gamma(T, \tau^- S)$, one can assume, without lost of generality, that $\varphi_n \in (\tau^- S, \tau^- S)$ and hence $\varphi_n = 0$. Since each component $\varphi_j : P_j \longrightarrow \tau^- S$ of φ factors through π, we can conclude that φ itself factors through π. It follows that $\varphi \in$ Im π_*, therefore Im $\pi_* = r_\Gamma(T, \tau^- S)$, and this finish the proof of the claim.

Applying the functor $\mathrm{Hom}_\Lambda(T,-)$ to the exact sequence $0 \longrightarrow rP_i \stackrel{\beta}{\longrightarrow} P_i \longrightarrow \frac{P_i}{rP_i} \longrightarrow 0$, for β a sink map, we can conclude that $(T,P_i) \cong (T, r_\Lambda P_i)$ and it follows that β_* is an isomorphism.

From the considerations above we have that the minimal projective resolution of $\hat{S}$ is given by

$$0 \longrightarrow (T, rP_i) \stackrel{f_*}{\longrightarrow} (T,P) = \bigoplus_{s=1}^{t} (T, P_{l_s}^{m_{l_s}}) \stackrel{\pi_*}{\longrightarrow} (T, \tau^- S) \longrightarrow \hat{S} \longrightarrow 0$$

where f_* is the Γ-map defined by the composition with $f\beta$.

Finally, we claim that $f\beta$ is a component of a sink-torsion map.

Indeed, we have that P_i is a direct summand of the radical of P and the map $f : P_i \longrightarrow P$ as described above, is a component of the map $rP \longrightarrow P$, defined by the minimal sink maps $(\alpha_{l_s})_* : rP_{l_s} \to P_{l_s}$. Moreover, we have that $\mathrm{tr}_T(P_i) = rP_i$. Hence, the composition $f\beta$ is a component of a sink torsion map. By our main result, we conclude that Γ is Koszul. ■

As an application of the result above, we describe a presentation of BB-tilted algebras. First we consider the Γ-map $(T, r_\Lambda P_i) \stackrel{\beta_*}{\longrightarrow} (T, P_i)$, defined by $\beta_*(\varphi) = \beta \cdot \varphi$, for each $\varphi \in (T, r_\Lambda P_i)$. We fix a decomposition $\beta = ((\beta_1^{v_1})_{1 \leq v_1 \leq v_{j_1}}, \cdots, (\beta_r^{v_r})_{1 \leq v_r \leq v_{j_m}})$ such that $\beta_m^{v_m}$ is the map defined by multiplication by the arrow joining the vertice i to the vertice j_m, denoted by $\beta_m^{v_m} : i \longrightarrow j_m$. In this notation v_{j_m} is the number of arrows between these vertices, and $m = 1, \cdots, r$.

Of course, β is a monomorphism. Using the same arguments as in the theorem above, we have that $(T, P_i) \cong (T, r_\Lambda P_i)$, hence β_* is an isomorphism. Moreover, we have that f_* has each component with domain and codomain indecomposable defined by $(\beta_m^{v_m} \alpha_s^{u_s})_*$, for some fixed pair (m, s).

Definition 4.1 *Giving a vertice i in a quiver Q the neighborhood of i is the full subquiver whose vertices are i and its immediate predecessors and successors.*

The quiver and relations of the BB-algebra can be described as follows:

1. Description of the Quiver

   (1) The vertex associated to the simple Γ-module $\hat{S}$, denoted by $\hat{\imath}$ becomes a source.

(2) The immediate successors of $\hat{\imath}$ are the vertex associated to the simple Γ-module $\hat{S}_{\hat{l}_s}$ for $s = 1, \cdots, t$, whose projective cover is given by the Γ-module $P_{\hat{l}_s} = \mathrm{Hom}_\Lambda(T, P_{l_s})$, where $l_1, \ldots, l_t$ are the immediate successors of i in the quiver of Λ.

(3) One of the immediate successors of each $\hat{l}_s$ is the vertice denoted by $\hat{\jmath}_m$, associated to the simple Γ-module $\hat{S}_{\hat{\jmath}_m}$ whose projective cover is the Γ-module $P_{\hat{\jmath}_m} = \mathrm{Hom}_\Lambda(T, P_{j_m})$, such that P_{j_m} is an indecomposable direct summand of rP_i.

(4) If $\{u, v\} \not\subset \{i, l_1, \ldots, l_t, j_1, \ldots, j_r\}$ then the number of arrows between $\hat{u}$ and $\hat{v}$ is the same as the number of arrows between u and v.

We sketch an argument for the description above.

Since we have that $\mathrm{Hom}_\Lambda(\tau^- S, P_j) = 0$ for $j \neq i$ and the projective cover of $\hat{S}$ is $\bigoplus_{s=1}^t (T, P_{l_s}^{m_{l_s}})$ it follows that both (1) and (2) are valid. One can prove (3) by checking the projective resolution of $\hat{S}$, since we have that Γ is Koszul. Finally, easy computation gives us (4) and also give us the number of arrows between the vertices in the quiver of Γ.

2. Description of the relations

   We recall that, using argument presented to prove the theorem(BB) above, we see that the arrows with origin the vertice $\hat{\imath}$ are defined by the components of the map π_*, induced by the projective cover $\pi : \oplus_{s=1}^t P_{l_s}^{m_{l_s}} \to \tau^- S$. We observe that considering the element $\psi \in (T, r_\Lambda P_i)$, such that ψ is the map defined by a multiplication of the sum of paths in the quiver of Γ that ends in the vertices $\hat{\jmath}_m$ for $m = 1, \cdots, r$, one can conclude that $(\pi_* \circ f_*)(\tilde{\psi})$ is a relation in Γ.

   Indeed, examining the projective resolution of $\hat{S}$, we can see that the set of entries of the matrix of the map $\pi_*(f\beta)_*$, defines the ideal of relations of Γ.

   More explicitly, all relations begin in $\hat{\imath}$ and end in $\hat{\jmath}_m$ and are defined by $\sum_{s=1}^{t} \sum_{u_s=1}^{m_{l_s}} (\beta_m^{v_m} \alpha_s^{u_s}) \pi_s^{u_s} = 0$, for each $v_m = 1, \cdots, v_{j_m}$ and $m = 1, \cdots, r$.

As we said above the quivers $Q(\Lambda)$ and $Q(\Gamma)$, of the hereditary algebra Λ and the associated BB-tilted algebra Γ, have the same shape outside the neighborhood from the vertice which we denoted by i. In order to exemplify the description above, we restrict ourselves to the case where $Q(\Lambda)$ has no double arrows, and give a pictures showing the connection between both quivers.

Assume that the neighborhood of the vertice i in $Q(\Lambda)$ has the following description:

$$
\begin{array}{ccccc}
l_1 & & & & j_1 \\
& \searrow^{\alpha_1} & & \beta_1\nearrow & \\
\vdots & & i & & \vdots \\
& \alpha_t\nearrow & & \searrow^{\beta_r} & \\
l_t & & & & j_r
\end{array}
$$

As we have shown, the vertice $\hat{\imath}$ is a source in the quiver $Q(\Gamma)$ and has immediate successors given by the vertices $\hat{l}_1, \ldots, \hat{l}_t$ corresponding to the projectives $Hom_\Lambda(T, P_{l_s}), s = 1, \ldots, t$. Let $\pi = (\pi_1, \ldots, \pi_t)$ be a decomposition of π, and the arrow $\overline{\pi}_s$ corresponding to the homomorphism $(\pi_s)_* = Hom(T, \pi_s)$. Each vertice $\hat{l}_s$ is an immediate predecessor of each $\hat{\jmath}_m$. The arrows between $\hat{l}_s$ and $\hat{\jmath}_m$ correspond to the s-th component of the homomorphism induced by $f \circ \beta_m$ which is defined by the product $\beta_m \alpha_s$.

For each $m = 1, \ldots, r$ we have the following local picture.

$$
\begin{array}{ccccc}
& & \hat{l}_1 & & \\
& \overset{\delta_1}{\nearrow} & & \overset{(\beta_m\alpha_1)}{\searrow} & \\
\hat{\imath} & & \vdots & & \hat{\jmath}_m \\
& \overset{\delta_t}{\searrow} & & \overset{(\beta_m\alpha_t)}{\nearrow} & \\
& & \hat{l}_t & &
\end{array}
$$

The following examples will make clear such description.

Example 1: If Λ is the quiver algebra whose quiver is :

$$
\begin{array}{ccccc}
& & 2 & & \\
& \alpha\swarrow & & \searrow & \\
Q: \quad 1 & & \xrightarrow{\beta} & & 3
\end{array}
$$

and $T = \tau^- S_1 \oplus \bigoplus_{j \neq 1} P_j$. Then we get that Γ will have the following ordinary quiver :

$$\hat{1} \xrightarrow{\overline{\pi}} \hat{2} \overset{\beta\alpha}{\underset{\overline{\gamma}}{\rightrightarrows}} \hat{3}$$

and relation $(\beta\alpha)\, \overline{\pi} = 0$.

Example 2: Let Λ be the quiver algebra whose quiver is the following :

$$1 \xleftarrow{\beta} 2 \overset{\alpha_1}{\underset{\alpha_2}{\leftleftarrows}} 3$$

and T the tilting module associated to the vertice 2, that is, $T = \tau^- S_2 \oplus P_1 \oplus P_3$. In this case Γ has a presentation whose ordinary quiver is given by:

$$\hat{2} \underset{\overline{\pi}_2}{\overset{\overline{\pi}_1}{\rightrightarrows}} \hat{3} \underset{\beta\alpha_2}{\overset{\beta\alpha_1}{\rightrightarrows}} \hat{1}$$

and only one relation $\overline{\pi}_1(\beta\alpha_1) + \overline{\pi}_2(\beta\alpha_2) = 0$.

The next lemma is an easy consequence of the exact sequence which appear in [AP] section 2.3, and it is a handy result in our next proposition. We review some of the notations and definitions.

Let (Q, I) be a presentation of a connected algebra Γ, we denote by $\Pi_1(Q, I)$ its homotopy group, v_0 the number of vertices in Q_0 and n_a the number of arrows in Q_1.

Given any abelian group G, we denote by $Z^1(\Gamma, I, G)$ the set of all G-valued function $f : Q_1 \to G$ such that $\sum_{i=1}^{u} f(\alpha_i) = \sum_{j=1}^{p} f(\beta_j)$ whenever there exists a minimal relation $\rho = \sum_{i=1}^{q} \lambda_i w_i$ such that $w_1 = \alpha_1.\cdots.\alpha_u$ and $w_2 = \beta_1.\cdots.\beta_p$. There is an exact sequence of abelian groups

$$0 \to G \to G^{v_0} \to Z^1(\Gamma, I, G) \to Hom(\Pi_1(Q, I), G) \to 0$$

Lemma 4.2 *Let (Q, I) be a presentation of connected algebra, where I is generated by a set $\{\rho_m = \sum_{i=1}^{u_m} \lambda_i w_1, m = 1, \ldots, r\} \bigcup \{\gamma_j\}$ where the γ's are the monomial relations in I and each ρ_m is a minimal relations with u_m terms. Then $dim_{\mathbb{Q}}(Hom(\Pi_1(Q, I), \mathbb{Q})) \geq n_a - v_0 + 1 - \sum_{m=1}^{r} u_m + r$.*

Proof: It is a straightforward consequence of the sequence above with G being the additive group of the rational numbers. We only need to observe that $Z^1(\Gamma, I, \mathbb{Q})$ is a subspace of $\mathbb{Q}^{n_a}$ which is determined by $\sum_{m=1}^{r} u_m - r$ linear equations. ∎

Proposition 4.3 *Let Γ be a BB-tilted algebra from an hereditary algebra Λ. Then Λ is simply connected if and only if Γ is also simply connected.*

Proof: Assume first that Λ is simply connected. Since Λ is hereditary the quiver $Q(\Lambda)$ is a tree. Let $(Q(\Gamma), I)$ be the presentation of Γ given above. Then it easy to see that the fundamental group $\pi_1(Q(\Gamma), I)$ is trivial. Since for any arrow $\gamma \in Q(\Gamma)$ we have that dim $o(\gamma)\Gamma t(\gamma) = 1$ it follows by theorem 3.5 of [BM], that the fundamental group of any presentation is trivial. Since it also known that Γ is directed, it follows that Γ is simply connected.

We assume now that the BB-tilted algebra Γ is simply connected and we shall prove that Λ is simply connected. We know that $H^1(\Gamma) = H^1(\Lambda)$ by Theorem 4.2 in [H], hence it is enough to prove that $H^1(\Gamma) = 0$.

We recall that outside the neighborhood of the vertice $i \in Q(\Lambda)$ the quivers $Q(\Lambda)$ and $Q(\Gamma)$ have the same shape, since Γ is simply connected we conclude that $Q(\Lambda)$ does not contains simple closed walks not involving arrows on the neighborhood of i. This is clear if the closed walk does not involve vertices in the neighborhood of i. We also have that there is no path in $Q(\Lambda)$ which stars at some l_i and ends at some l_j. Otherwise, we would have the following description of $Q(\Lambda)$ in the neighborhood of i.

$$\begin{array}{ccccc}
l_1 & & & & \\
\downarrow & \searrow & & & j_1 \\
\vdots & & & \nearrow & \\
l_2 & \rightarrow & i & & \vdots \\
\vdots & \nearrow & & \searrow & \\
l_t & & & & j_r
\end{array}$$

Then $Q(\Gamma)$ has the following description in the neighborhood of $\hat{\imath}$.

$$\begin{array}{ccccc}
 & & \hat{l}_1 & & \\
 & \nearrow & \downarrow & \searrow & \\
 & & \vdots & & \\
\hat{\imath} & \rightarrow & \hat{l}_2 & \rightarrow & \hat{\jmath}_m \\
 & & \vdots & & \\
 & \searrow & & \nearrow & \\
 & & \hat{l}_t & &
\end{array}$$

for each $m = 1, \ldots, r$. In this case, Γ is not simply connected, because according to our description if we compute the homotopy group of our presentation the closed walk starting at $\hat{\imath}$ and passing trough $\hat{l}_1$ and $\hat{l}_2$ is not trivial on the homotopy group.

The same kind of argument shows that there is no path in $Q(\Lambda)$ starting and ending in the vertices $j_1, \ldots, j_r$.

It follows from our presentation and the fact that Γ is simply connected all simple closed walks in $Q(\Gamma)$ have vertices belongs to the set $\{\hat{\imath}, \hat{l}_1, \ldots, \hat{l}_t, \hat{\jmath}_1, \ldots, \hat{\jmath}_r\}$.

Let Γ' be full subcategory of Γ whose vertices are $\hat{\imath}, \hat{l}_1, \ldots, \hat{l}_t, \hat{\jmath}_1, \ldots, \hat{\jmath}_r$. Hence we have that Γ' is a full convex subcategory of Γ.

If we identify all the vertices $\{\hat{\imath}, \hat{l}_1, \ldots, \hat{l}_t, \hat{\jmath}_1, \ldots, \hat{\jmath}_r\}$, we get a quiver which is a tree. Using one point extensions and coextensions, and Happel's long exact sequence we get that $H^1(\Gamma) = H^1(\Gamma')$.

We claim that Γ' is the BB-tilted algebra from the hereditary algebra Λ' whose quiver is such that all the arrows start or end at i, that is it has the following description:

$$\begin{array}{ccccc} l_1 & & & & j_1 \\ & \searrow & & \nearrow & \\ \vdots & & i & & \vdots \\ & \nearrow & & \searrow & \\ l_t & & & & j_r \end{array}$$

The result follows from the claim. Indeed Λ' is hereditary and $Q(\Lambda')$ is a tree therefore $H^1(\Lambda') = 0$. We also know that $H^1(\Gamma') = H^1(\Lambda')$, hence $H^1(\Lambda) = 0$, as we wished to prove.

We prove now the claim. We have shown that, in the quiver Q_Λ, there is no path between vertices $l's$ or between $j's$ and no paths from some l to some j. Hence, in the quiver of Λ' there is no closed simple walks with origin in some l_s passing through some j_m or between themselves. Next we prove that there is no multiply arrows starting in some l_s or ending in some j_m, for every $s = 1, \ldots, t$ and $m = 1, \ldots, r$.

We recall that Γ' is a connected algebra such that the relations are given by $\rho_{v_m} = \sum\limits_{s=1}^{t} \sum\limits_{u_s=1}^{m_{l_s}} (\beta_m^{v_m} \alpha_s^{u_s}) \pi_s^{u_s}$, for each pair (m, v_m). We shall prove that if $m_{l_s} > 1$ or $v_{j_m} > 1$ for some s or some m then Γ' is not simply connected.

We denote $l = m_{l_1} + \ldots + m_{l_t}$ and $v = v_{j_1} + \ldots + v_{j_r}$. We observe that the number of vertices of Γ' is $t + r + 1$, by the description for the quiver of the BB-tilted algebras. If $l > 1$ then by the description of the minimal relations of Γ' and the lemma above we have that $dim_{\mathbb{Q}}(Hom(\Pi_1(Q, I), \mathbb{Q})) \geq (l + v) - (t + r)$. If $Q(\Lambda')$ has multiple arrows then $l > t$ or $v > r$, in this case $dim_{\mathbb{Q}}(Hom(\Pi_1(Q, I), \mathbb{Q})) > 0$ therefore Γ' is not simply connected.

Finally, if $l = 1$ then all relations are monomial. In this case if $v_{j_m} > 1$ for some m, Γ' is not simply connected. ■

4.2 A class of Koszul iterated tilted algebras

A complete classification of the iterated tilted algebra of type A_n is given in [AH]. In that work they show the following lemma on the section 2.

Lemma:[AH] *Let A be a finite dimensional algebra of finite representation type satisfying the following properties:*

1. *There are at most two irreducible maps with prescribed domain or codomain,*

2. *If P_A is projective with indecomposable radical R then there is at most one irreducible map of codomain R. Dually if $I_A i$ is injective with $I/(socI)$ indecomposable then there is at most one irreducible map of domain $I/(socI)$.*

Then, for every indecomposable M_A, the set of all (isomorphism classes) of indecomposable modules M_A such that $Hom(N, M) \neq 0$ and $Hom(N, \tau M) = 0$ is the union of two full linear subquivers of the AR-quiver intersecting at the vertex $[M]$. The dual conclusion also holds.

Using this lemma, proposition 3.2 and the same kind or arguments presented in that paper we show the following:

Proposition 4.4 *Let Λ be a finite dimension algebra of finite representation type satisfying the hypothesis of the lemma above and Γ a tilted algebra from Λ. Then all presentations of Γ are monomial quadratic, in particular it is Koszul.*

Proof: Using the lemma above, the correspondence between indecomposable summands of the tilting module T and the vertices of the quiver $Q(\Gamma)$ can be described, in the following sense.

Consider T_l an indecomposable direct summand of T corresponding under $Hom_\Lambda(T, -)$ to the projective Γ-module associated to the vertice $l \in Q(\Gamma)$. Then, in the AR quiver of Λ, there is at most two irreducible maps f and g of codomain T_l and at most two irreducible maps u, v of domain T_l and these determines at most four linear subquivers intersecting at T_l, namely $L(f), L(g), R(u)$ and $R(v)$.

Let T_v be an indecomposable direct summand of T corresponding to the vertice $v \in Q(\Gamma)$. Then if $Hom_\Lambda(T_l, T_v) \neq 0, T_v \in R(u)$ or $T_v \in R(v)$ and if $Hom_\Lambda(T_v, T_l) \neq 0, T_v \in L(f)$ or $T_v \in L(g)$.

Assume that v and j are two neighbors vertices from l. If T_v, T_l and T_j are collinear then the composition $T_v \longrightarrow T_l \longrightarrow T_j$ is non-zero since both T_v, T_l belongs to a linear subquiver determined by an irreducible map of codomain T_j. Otherwise the composition $T_v \longrightarrow T_l \longrightarrow T_j$ is zero, since T_v is not in the linear subquiver determined by an irreducible map of codomain T_j on which T_l lies.

Consider now T_l an indecomposable direct summand of T, $\alpha : E_l \to T_l$ the sink torsion map and (T_{E_l}, π_l) the T-generator of E_l. Since T_{E_l} belongs to add(T) and $Hom(T_{E_l}, T_l) \neq 0$, it follows that T_{E_l} has, at most, two indecomposable direct summands, by the considerations above. Let $T_{l,1}$ and $T_{l,2}$ be these summands. Applying the same argument to these summands we have the following picture:

$$
\begin{array}{cccccc}
T_{l,1,1} & & & & & \\
 & \searrow & & & & \\
 & & T_{l,1} & & & \\
 & \nearrow & & \searrow & & \\
T_{l,1,2} & & & & T_l & \\
T_{l,2,1} & & & & & \\
 & \searrow & & \nearrow & & \\
 & & T_{l,2} & & & \\
 & \nearrow & & & & \\
T_{l,2,2} & & & & &
\end{array}
$$

We discuss now the possible relations starting at $l \in Q(\Gamma)$. Since the composition of any maps with codomain T_l and domain any indecomposable module whose class is not on the full linear subquivers intersecting at T_l are zero, it follows that the presentation of Γ which we gave is monomial quadratic. Since we also have dim $o(\gamma)\Gamma t(\gamma) = 1$ for all arrows γ, it follows by proposition 2.5 in [BM] that all presentations are monomial quadratic. ■

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