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CLOSED HYPERSURFACES OF S^4 WITH CONSTANT GAUSS-KRONECKER CURVATURE

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1. Introduction

Let M be a n-dimensional Riemannian manifold and let A be a smooth symmetric tensor field on M of type (1,1). Associated to this tensor field A there are 2(n+1) smooth functions $\sigma_r(A), s_r(A), (0 \leq r \leq n)$ given by

$$\begin{cases} P(-\lambda) = \sum_{r=0}^{n} \sigma_r \lambda^{n-r} \\ s_r = trace(A^r) \end{cases} \tag{1.1}$$

where $P(\lambda) = \det(A - \lambda I)$. Those functions are related by the following algebraic formalae:

$$\begin{cases} s_r - \sigma_1 s_{r-1} + ... + (-1)^r r \sigma_r = 0, \quad r < n \\ s_r - \sigma_1 s_{r-1} + ... + (-1)^n \sigma_n s_{r-n} = 0, \ r \geq n. \end{cases} \tag{1.2}$$

Let $x : M^n \to W^{n+1}$ be an isometric immersion between two orientable Riemannian manifolds of dimension n and n+1 respectively. Suppose h is the second fundamental form of the immersion x. Associated to h there are n functions $H_1, ..., H_n$, defined by $H_r = \sigma_r(h)$. Explicitly

$$H_r = \sum_{i_1 < ... < i_r} k_{i_1} ... k_{i_r}.$$

Here $k_1, ..., k_n$ are the principal curvatures of M. They are the eigenvalues of the second fundamental form h. We say that M is isoparametric if the principal curvatures $k_1, ..., k_n$ are constant. Obviously M is isoparametric if and only if $H_1, ..., H_n$ are constant functions. We note that $H_1 = H$ is the mean curvature of x, H_n is the Gauss-Kronecker curvature, and if W^{n+1} is a real space form,

H_2 is, up a constant, the scalar curvature of M. In general H_r is the so called r-mean curvature function of the immersion x.

We now recall the following algebraic results ([AB2, AB3]).

Theorem 1.1. ([AB2]) *Let M be a compact 3-dimensional Riemannian manifold with metric g and scalar curvature $\kappa \geq 0$. Suppose a is a smooth symmetric tensor field on M of type (0,2) and let A be the tensor field of type (1,1) corresponding to a via g. Suppose in addition that*

a) the field ∇a of type (0,3) is symmetric.

b) $d(traceA) = 0$

c) $d(traceA^2) = 0$

Then

d) $d(traceA^3) = 0$.

Theorem 1.2. ([AB3]) *Let M be a compact 3-dimensional Riemannian manifold with metric g and scalar curvature $\kappa \geq 0$. Suppose a is a smooth symmetric tensor field on M of type (0,2) and let A be the tensor field of type (1,1) corresponding to a via g. Suppose in addition that*

(a) the field ∇a of type (0,3) is symmetric

(b) $d\sigma_2 = d\sigma_3 = 0$

(c) $\sigma_3 \neq 0$.

Then

d) $d\sigma_1 = 0$.

Suppose M is a 3-dimensional immersed hypersurface in a space of constant curvature. It is well known that the second fundamental form h of M satisfies condition (a) ot Theorem 1.1. The conditions (b) and (c) may be replaced by the assumption that M has constant mean curvature and constant scalar curvature. As a consequence we have the following:

Theorem 1.3. ([AB2]) *Let M be a closed 3-dimensional immersed hypersurface in a space of constant curvature. Suppose in addition that M has constant mean curvature H and constant scalar curvature $\kappa \geq 0$. Then M is isoparametric.*

From Theorem 1.2 it follows that

Theorem 1.4. *Let M be a closed 3-dimensional immersed hypersurface in a space of constant curvature. Suppose in addition that M has constant scalar curvature $\kappa \geq 0$ and constant Gauss-Kronecker curvature $K \neq 0$. Then M is isoparametric.*

We will now assume that the hypersurfaces $M \subset S^4$ have constant mean curvature (H) and constant Gauss-Kronecker Curvature ($K \neq 0$). A large class of such hypersurfaces carry a constant non-negative scalar curvature metric. This is accomplished by taking the associated Gauss map of each hypersurface. Their images are 3-dimensional hypersurfaces of S^4 with constant scalar curvature and constant Gauss-Kronecker curvature. Taking the associated Gauss map twice will produce the original hypersurface. With this construction we are able to prove the following theorem.

Theorem 1.5. *Let M be a closed oriented 3-dimensional hypersurface immersed in the standard 4-sphere with constant mean curvature H and constant Gauss-Kronecker curvature $K \neq 0$. Suppose in addition that $\frac{H}{K} \geq -3$. Then M is isoparametric .*

Using Theorem 1.5 we retrieved the following result.

Theorem 1.6. ([AB1]) *Let $M^3 \subset S^4$ be a closed minimally immersed hypersurface of S^4 with constant Gauss-Kronecker $K \neq 0$. Then M is the minimal Clifford torus $S^2(\sqrt{2/3}) \times S^1(\sqrt{1/3}) \subset S^4$.*

2. Isoparametric Hypersurfaces

The results presented in this work in some sense characterize the isoparametric hypersurfaces of the 4-dimensional sphere S^4. From well known results we know that in S^4 there are only three families of isoparametric hypersurfaces. We will now describe explicitly those hypersurfaces.

Example 1. *(Spheres) - Let $x : S^3(r) \to S^4$ be the isometric immersion given by $x(p) = (p,s)$, where $s^2 + r^2 = 1$. It is not difficult to see that $M = S^3(r)$ is all umbilic with principal curvatures $k_i = s/r, i = 1,2,3$. An elementary computation shows that the mean curvature (H), the scalar curvature (κ) and the Gauss-Kronecker curvature (K) satisfy the following relations:*

$$\begin{cases} H = 3K^{1/3} \\ \kappa = 6(1+K^{2/3}) = 6 + 2H^2/3 \end{cases} \tag{2.1}$$

Example 2. *(Clifford tori) - Let $x : S^2(r) \times S^1(s) \to S^4$ be the isometric immersion given by $X(p,q) = (p,q)$. It has principal curvatures given by $k_1 = -r/s$, $k_2 = k_3 = s/r$. The mean curvature (H), the scalar curvature (κ) and Gauss-Kronecker curvature (K) of the immersion x satisfy the equations:*

$$\begin{cases} H = -2K + 1/K \\ \kappa = 2(1+K^2) = 3 + (H^2 \pm H\sqrt{8+H^2})/4 \end{cases} \tag{2.2}$$

Example 3. *(Cartan's isoparametric family) - Let $N_1(\Sigma)$ denote the unit normal sphere bundle of the Veronese surface $\Sigma \subset S^4$. We can express $N_1(\Sigma)$ as*

$$N_1(\Sigma) = \{(x,\nu) \in \Sigma \times S^4 : \nu \perp T_x\Sigma \text{ and } \nu \perp x\}.$$

Finally, we define the isoparametric family $\Phi_t : N_1(\Sigma) \to S^4$ by

$$\Phi_t(x,\nu) = cost\ \nu + sint\ x$$

The principal curvatures of the immersion Φ_t are:

$$\frac{\sqrt{3}+\tan t}{1-\sqrt{3}\tan t}, \quad \frac{\tan t - \sqrt{3}}{1+\sqrt{3}\tan t}, \quad \tan t \tag{2.3}$$

An easy computation shows that the mean curvature (H_t), the scalar curvature (κ_t) and Gauss-Kronecker curvature (K_t) of the immersion Φ_t satisfy the following interesting relations:

$$\begin{cases} \kappa_t \equiv 0 \\ H_t + 3K_t \equiv 0 \end{cases} \tag{2.4}$$

3. Further Remarks

In ([CDK]), Chern-do Carmo-Kobayashi asked whether the value of the scalar curvature κ_M of a closed minimal hypersurface $M \subset S^{n+1}$ would determine the hypersurface up to a rigid motion of S^{n+1}. In their conjecture they assumed that κ_M was a constant function. They also asked if the values of the scalar curvature κ_M was a discrete set of real numbers. Recently, S. Chang ([C]), announced the following result

Theorem 3.1. ([C]) *Let M^3 be a closed hypersurface of constant scalar curvature κ in S^4. If M has constant mean curvature then $\kappa \geq 0$.*

To solve the conjecture of Chern-do Carmo-Kobayashi for 3-dimensional hypersurfaces one just have to combine Theorem 1.3 together with the non-existence result in ([C]).

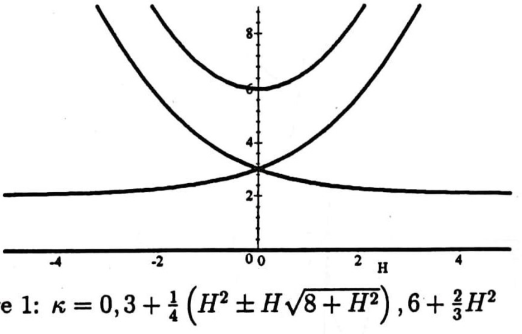

Figure 1: $\kappa = 0, 3 + \frac{1}{4}\left(H^2 \pm H\sqrt{8+H^2}\right), 6 + \frac{2}{3}H^2$

Remark 1. *Figure 1 shows the possible values for the mean curvature* (H) *and for the scalar curvature* (κ) *of a closed hypersurface* $M \subset S^4$ *with* $dH = d\kappa = 0$.

For fixed $n \geq 3$, we will denote by $\mathcal{F}_{r,s}$ the collection of all closed hypersurfaces $M \subset S^{n+1}$ having $dH_r = dH_s = 0$. The conjecture above is a particular case of the following more general question:

Question 1 *Determine* $\mathcal{F}_{r,s}$, *for all* $r \neq s$.

In this note we considered only the case $n = 3$. It is interesting to note that, even in this particular case, Question 1 is not completely solved.

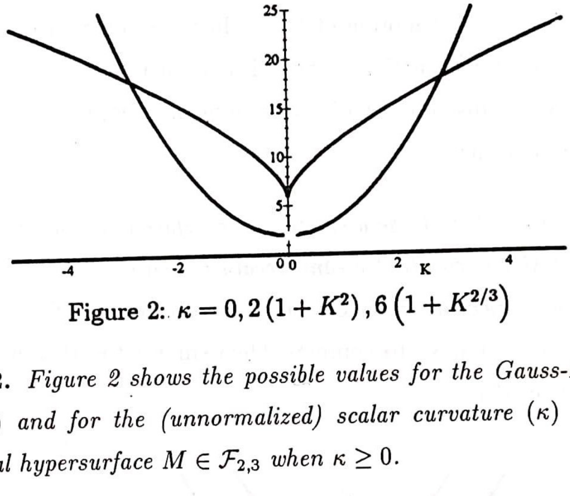

Figure 2: $\kappa = 0, 2\left(1+K^2\right), 6\left(1+K^{2/3}\right)$

Remark 2. *Figure 2 shows the possible values for the Gauss-Kronecker curvature* (K) *and for the (unnormalized) scalar curvature* (κ) *of a closed 3-dimensional hypersurface* $M \in \mathcal{F}_{2,3}$ *when* $\kappa \geq 0$.

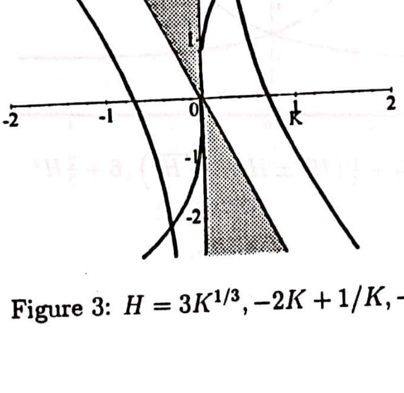

Figure 3: $H = 3K^{1/3}, -2K + 1/K, -3K$

Remark 3. *Figure 3 shows the possible values for the Gauss-Kronecker curvature (K) and for the mean curvature (H) of a closed 3-dimensional hypersurface $M \in \mathcal{F}_{1,3}$. Only the shaded region was not considered in Theorem 1.5.*

References

[AD] Alencar, H. and do Carmo, M., *Hypersurfaces with constant mean curvature in spheres*, Proc. Amer. Math. Soc. 120, (1994) 1223-1229.

[A] Almeida, S.C., *3-dimensional hypersurfaces with constant non-negative scalar curvature*, Universidade Federal do Ceará, 1994.

[AB1] Almeida, S.C. and Brito, F. G. B, *Minimal hypersurfaces of S^4 with constant Gauss-Kronecker curvature*, Math. Z. 195, (1987), 99-107.

[AB2] Almeida, S.C. and Brito, F. G. B., *Closed 3-dimensional hypersurface with constant mean curvature and constant scalar curvature*, Duke Math. J., 61, (1990), 195-206.

[AB3] Almeida S. C. and Brito, F. G. B., *Closed hypersurfaces of S^4 with two constant symmetric curvatures*, Annales de la Faculte des Sciences de Toulouse.

[BD] Barbosa , J. L. M. and Delgado, J. A., *Ruled submanifolds of space forms with mean curvature of nonzero constant length*, American Journal of Mathematics, 106, (1984), 763-780.

[Ca1] Cartan, E. *Familles de surfaces isoparamétriques dans les espaces à courbure constant*, Annali di Mat. 17, (1938), 177-191.

[C] Chang, S., *A closed hypersurface with constant scalar curvature and mean curvature in S^4 is isoparametric*, Comunications in Analysis and Geometry, Vol. 1, $N^{\underline{o}}$ 1, (1993), 71-100.

[CDK] Chern, S.S., do Carmo, M. and Kobayashi, S., *Minimal submanifolds of the sphere with second fundamental form of constant length*, Functional analysis and related fields, (1970), 59-75, ed. F. Browder, Berlim Heidelberg New York, Springer.

[H] Hsiang, W-Y., *Minimal cones and the spherical Bernstein problem*, I., Ann. of Math., 118 (1983), 61-73.

[L1] Lawson Jr., H.B., *Minimal Varieties in Real and Complex Geometry*, Séminaire de Mathématiques Supérieures, Département de Mathématiques - Université de Montréal (1974).

[L2] Lawson Jr., H. B., *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math., 89 (1969), 187-197.

[N] Nomizu, K., *Elie Cartan's work on isoparametric hypersurfaces*, Diff. Geom., Symp. Pure Math. 27 II, Stanford (1973).

[PT1] Peng, C. K. and Terng, C. L., *Minimal hypersurfaces of spheres with constant scalar curvature*, Ann. Math. Stud. 103, (1983).

[PT2] Peng, C. K. and Terng, C. L., *The scalar curvature of minimal hypersurfaces in spheres*, Math Ann. 266 (1983), 105-113.

[R] Ramanathan, J., *Minimal hypersurfaces in S^4 with vanishing Gauss-Kronecker curvature*, Math. Z., 205 (1990), 645-658.

[W] Vander Warden, B. L., *Algebra* , Vol. 1, Frederick Ungar Publishing Co., Inc. (1970).

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