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# Rainbow Path Covers of Sparse Random Graphs

(Extended abstract)

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**Abstract**

We investigate the problem of covering the edges of the random graph $G(n,p)$ with rainbow paths, where the coloring is any proper edge-coloring. We show that for $pn = o((\log n/\log\log n)^2)$, this can be achieved with $O(n)$ rainbow paths with high probability, which is essentially best possible. Our techniques may offer insights for determining minimum rainbow path covers in arbitrary properly-colored graphs.

## 1 Introduction

A *path decomposition* of a graph $G$ is a family of edge-disjoint paths of $G$ that covers the edge set of $G$. In 1966, Erdős asked about the minimum size of a path decomposition of a graph $G$, for which Gallai conjectured that every graph on $n$ vertices admits a path decomposition of size at most $\lceil n/2\rceil$ (see [12]). Several results followed this conjecture: Chung [5] showed a tree decomposition with $\lceil n/2\rceil$ trees; Lovász [12] showed that every graph admits a path decomposition of size at most $n-1$; Donald [8] improved the upper bound to $\lfloor 3n/4\rfloor$; and Dean and Kouider [7] further improved it to $\lfloor 2n/3\rfloor$, which is the best known bound. Other results consider specific graph classes, including planar graphs [1, 3] and graphs with sparse subgraphs induced by even-degree vertices [4, 10, 13].

In order to approach Gallai's Conjecture, which remains open, one may consider path covers instead of path decompositions, where a *path cover* of a graph $G$ is a family of (not necessarily edge-disjoint) paths of $G$ that covers the edge set of $G$. Such a weakening of Gallai's Conjecture was posed by Chung [6], who conjectured that every graph on $n$ vertices admits a path cover of size at most $\lceil n/2\rceil$. In 1996, Pyber [13] verified this conjecture asymptotically

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by proving that every graph can be covered by $n/2 + O(n^{3/4})$ paths; and in 2002, Fan [9] verified it tightly.

We now present an analog of the path cover conjecture concerning the rainbow paths in graphs equipped with a proper edge-coloring. Let $G$ be graph and let $c\colon E(G) \to \mathbb{N}$ be a *proper* edge-coloring of $G$, i.e., a coloring of $E(G)$ in which $c(e) \neq c(f)$ for every pair of edges incident to the same vertex. We say that a subgraph $H$ of $G$ is *rainbow* if $c(e) \neq c(f)$ for every pair $e, f$ of distinct edges of $H$. In what follows, to avoid excessive wording, we simply say *colored* instead of properly edge-colored. We say that path cover $\mathcal{P} = \{P_1, \ldots, P_k\}$ of a colored graph $G$ is a *rainbow path cover* of $G$ if $P_i$ is a rainbow path (with respect to $c$) for every $i \in [k] = \{1, \ldots, k\}$. We are interested in the following problem posed by Bonamy, Botler, Dross, Naia, Skokan [2], which we present here as a conjecture.

**Conjecture 1** (Bonamy–Botler–Dross–Naia–Skokan, 2023)**.** *Every colored graph $G$ admits a rainbow path cover of size $O\big(|V(G)|\big)$.*

Although Conjecture 1 first arose as a possible strategy to handle a problem answered in [2], we believe it is interesting in its own right and it has not been thoroughly explored in the literature yet. A natural first step is thus to find evidence for Conjecture 1 among random graphs. More specifically, we would like to know for which ranges of $p$ does every edge-coloring of $G(n,p)$ asymptotically almost surely (a.a.s.) admit a rainbow path cover of size $O(n)$. In a very sparse regime, Conjecture 1 is naturally true due to the number of edges being linear. On the other hand, Kaique, Hoppen, Mendonça, Mota and Naia [11] verified Conjecture 1 for dense random graphs as follows.

**Theorem 2** (Kaique–Hoppen–Mendonça–Mota–Naia, 2025)**.** *Let $\varepsilon > 0$ and $p \geq n^{\varepsilon-1}$. Then there exists $D = D(\varepsilon) > 0$ such that the following holds with high probability in $G = G(n,p)$. For any proper edge-coloring of $G$, it is possible to cover $E(G)$ with $Dn$ rainbow paths.*

In this paper, we extend Theorem 2 to a sparse range of $p$. This range is significant as it is dense enough to require a non-trivial cover while still lacking the high connectivity present in the dense range that aided path finding.

**Theorem 3.** *There is a constant $D$ so that if $pn = o((\log n/\log\log n)^2)$ and $G = G(n,p)$, then the following holds with high probability: for any proper edge-coloring of $G$, it is possible to cover $E(G)$ with $Dn$ rainbow paths.*

In section 2, we present the main tools and techniques used in the proof of Theorem 3 and in section 3, we present an outline of the proof of Theorem 3.

# 2 Main tools

## 2.1 Building random rainbow trails

One of the main tools we use to build rainbow paths is a randomized procedure that builds $n$ rainbow trails in parallel while traversing every edge twice. Let $G = (V, E)$ be a graph and $c$ be a proper edge-coloring of $G$. Let $T$ denote the number of colors used in $c$ and let $t \leq T$ be an integer. The procedure starts by creating trails $W(u) = (u)$ for every $u \in V$. We will also keep track of auxiliary walks that will be useful in the analysis of the procedure: let $W'(u) = (u)$.

**Procedure 1:** Repeat the following for $t$ steps:

- Choose a color uniformly at random among the colors not chosen so far;
- For each vertex $u \in V$, if there is an edge $vv'$ with the chosen color, where $v$ is the last vertex in $W(u)$, append $v'$ to $W(u)$ and to $W'(u)$; otherwise, append $v$ to $W'(u)$.

For each vertex $u$, the auxiliary walk $W'(u)$ keeps track of all steps, including the ones where the trail $W(u)$ was not extended. This simple procedure has many nice features: the trails $(W(u))_{u\in V}$ are rainbow; at each step, each of the trails is at a different vertex; and, for $t = T$, each edge is traversed exactly twice. Note that, if the goal was to cover edges with rainbow trails instead of paths, the linear upper bound would follow directly from this procedure. In order to obtain paths, the main obstacle is that the trails may contain cycles. In the next section, we present the properties of short cycles in $G(n, p)$ that we will use in our proof.

### 2.2 Short cycles in $G(n, p)$

In order to avoid too many cycles in the rainbow trails, which disrupts our construction, our strategy is to bound the number of short cycles in $G(n, p)$, which allows us to remove a small number of edges and increase the girth of the graph. Let $Y_\ell$ denote the number of cycles of length $\ell$ in $G(n, p)$. Let $Y_{\leq \ell}$ denote the number of cycles of length at most $\ell$ in $G(n, p)$. Let $d = pn$. It is well known that $\mathbb{E}(Y_\ell) \sim d^\ell/(2\ell)$ for $d \gg 1$. Using this fact, it is easy to show the following results on short cycles in $G(n, p)$:

**Proposition 4.** *Let $p = d/n$ with $1 \ll d \ll (\log n/ \log\log n)^2$. We have that $Y_{\leq \ell} \ll n$ a.a.s., for $\ell = \frac{\log n}{\log d}$.*

**Proposition 5.** *Let $\alpha > 1$ be a constant. Let $p = d/n$ with $1 \ll d \leq \alpha \log n/ \log\log n$. Then $Y_{\leq \ell} \ll n$ a.a.s., for $\ell = d/\alpha$.*

### 2.3 Large girth graphs

As mentioned before, the rainbow trails built by Procedure 1 can have many cycles, which is an obstacle for building our path cover. To address this, we will slice each trail into shorter segments of length at most $\ell \sim \varepsilon d$, where $\varepsilon$ is a small positive constant. This approach means we only need to worry about a trail coming back to itself within a window of at most $\ell$ steps. For each vertex $u$, we define the following:

- Let $W(u)$ be the trail built by Procedure 1 starting at $u$ and let $T(u)$ be the number of edges in $W(u)$. Let $(w_j(u))_{0\leq j\leq T(u)} := W(u)$
- Let $W'(u) = (w'_i(u))_{0\leq i\leq T}$ be the auxiliary walk for $u$.
- For each step $i$ of Procedure 1, let $J(u, i)$ denote the number of edges in the trail $W(u)$ built at that step. This implies that $w'_i(u) = w_{J(u,i)}(u)$.
- For every $i \in \{1, \dots, T - \ell\}$, let $P_i(u)$ be the trail in $W(u)$ with $\ell$ edges ending at $w'_i(u)$. More precisely, $P_i(u) := (w_j(u))_{j_0\leq j\leq J(u,i)}$ where $j_0 = \max(0, J(u, i) - \ell)$.
- For every $i \in \{1, \dots, T - \ell\}$, let $\mathrm{Split}_i(u)$ be the event that $w'_{i+1}(u)$ is different from $w'_i(u)$ and is in $P_i(u)$. We say that $\mathrm{Split}_i(u)$ is a *splitting event*.

One of our main lemmas will be used to show that, for high girth graphs, Procedure 1 causes $o(1)$ splitting events in expectation for each vertex (as long as some conditions are met). This means that we have $o(n)$ splitting events overall, with high probability.

**Lemma 6.** *Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. Let $\bar{d} := 2m/n$ and fix $\ell \sim \varepsilon\bar{d}$. Let $t = t(n) < T$ and $k = k(n)$ be such that $T - t = \omega(k)$. Suppose that the girth of $G$ is greater than $k$. For every vertex $u \in V$, the expected number of occurrences of Split$_i(u)$ for $i \leq t$ is $O\left((\bar{d}/k^2)\log(T/(T-t))\right)$.*

*Proof sketch.* Fix $u \in V$. We will omit $u$ in the notation for simplicity (e.g., $\mathrm{Split}_i(u)$ will be referred to simply as $\mathrm{Split}_i$). We also omit ceiling and floor operators as long as they do not affect the asymptotic bounds. Our goal is to bound $\sum_{i \leq t} \Pr[\mathrm{Split}_i]$. We start by partitioning $\{1, \ldots, t\}$ into intervals of size $k' := k/2$: for every integer $0 \leq j \leq t/k'$, define the interval $I_j := [jk', \min((j+1)k' - 1, t)]$. Given $0 \leq j \leq t/k'$, we show that

$$\textstyle\sum_{i \in I_j} \Pr\left[\mathrm{Split}_i\right] \leq \ell/(k'(T - jk' - k')). \tag{1}$$

Equation (1) holds for the following reasons. First, the probability that a splitting step occurs at step $i$ is bounded above by $d(w'_i, P_i)/(T-i)$, where $d(w'_i, P_i)$ is the number of edges from $w'_i$ to $P_i$. Now we explain the role of the lower bound $k$ on the girth. Let $i_0 = jk'$ be the smallest $i$ in $I_j$. Let $V_j := \{w'_i\}_{i \in I_j}$ and let $d(V_j, P_{i_0})$ denote the number of edges from $V_j$ to $P_{i_0}$. Since $V_j$ is connected by a trail of length $k'$, it induces no other edges because that would create a cycle of length less than $k$. Thus, $\sum_{i \in I_j} d(w'_i, P_i) \leq \sum_{i \in I_j} d(w'_i, P_{i_0}) = d(V_j, P_{i_0})$. Since the girth is greater than $k$, the number of edges from $V_j$ to any $k'$ consecutive vertices in $P_{i_0}$ is at most 1. This implies that $d(V_j, P_{i_0}) \leq \ell/k'$ since $P_{i_0}$ has $\ell$ edges. From these facts, it is easy to derive (1), from which it is straightforward to show that

$$\begin{aligned}\sum_{i \leq t} \Pr[\mathrm{Split}_i] &= \sum_{j=0}^{t/k'} \sum_{i \in I_j} \Pr[\mathrm{Split}_i] \leq \sum_{j=0}^{t/k'} \frac{\ell}{k'(T - jk' - k')} = \frac{\ell}{(k')^2} \sum_{j=0}^{t/k'} \frac{1}{T/k' - j - 1} \\ &\leq \frac{\ell}{(k')^2}\Big(\mathrm{H}(T/k' - 1) - \mathrm{H}((T-t)/k' - 2)\Big) \leq \frac{\ell}{(k')^2} \log\Big(\frac{T}{T-t}\Big)\Theta(1),\end{aligned}$$

where $\mathrm{H}(x) = \sum_{i=1}^{x} 1/i$ is the $x$-th Harmonic number.

□

## 3 Proof overview

In this section we outline the proof of Theorem 3. In the range $pn = O(\log n / \log\log n)$, it is possible to remove $o(n)$ edges from $G(n, p)$ obtain a graph with very large girth a.a.s., preventing splitting events. This can be done by applying Proposition 5 and directly analysing the sparser range where Proposition 5 does not apply. So assume $pn = \Omega(\log n / \log\log n)$. Let $d := pn$ and $G = G(n, p)$. We have that, a.a.s., the number of edges $m$ of $G(n, p)$ satisfies $m \sim \binom{n}{2}p$ and so $\bar{d} := \frac{2m}{n} \sim d$. By Proposition 4, the number of cycles in $G$ of length at most $k$ is $o(n)$ a.a.s., for $k := \log n / \log d$. Sample $G = G(n, p)$ and assume that $G$ satisfies these conditions. Let $c : E(G) \to [T]$ be any proper edge-coloring of $G$.

Let $E_{\mathrm{cycle}} \subseteq E(G)$ be a set of edges of size $o(n)$ containing one edge from each cycle of length at most $k$. Let $\mathcal{P}_0$ denote the set of paths such that each path consists of a single edge of $E_{\mathrm{cycle}}$. Clearly, $|\mathcal{P}_0| = |E_{\mathrm{cycle}}| = o(n)$ and $\mathcal{P}_0$ is a collection of rainbow paths.

Let $G' := G - E_{\mathrm{cycle}}$, which has girth greater than $k$. Let $m' \leq m$ denote the number of edges of $G'$ and let $d' = 2m'/n$. The number of colors in $G'$ may be of the same order of $k$, so we raise the number of colors as follows. Choose $\psi(n)$ edges of $G'$ and assign each of them a new color. (Any large enough $\psi(n) = o(n)$ works here.) Let $c'$ denote the coloring obtained by restricting $c$ to the edges of $G'$ and recoloring $\psi(n)$ edges to distinct new colors, and let $T' \geq \psi(n)$ denote the number of colors used by $c'$.

Let $\phi = o(1)$. Here we also need $\phi$ to be large enough so that some properties hold. Let $t' := (1 - \phi/k)T'$. We have that $(T' - t')/k = \omega(1)$ since $T' \geq \psi(n)$. Let $W_1, \ldots, W_n$ be the random trails obtained by Procedure 1 for $(G', c', t')$. We partition them into trails of length $\ell = \varepsilon d'$, yielding a collection $\mathcal{P}'$ of $O(n)$ rainbow trails. By applying Lemma 6 to graph $G'$ and $c'$, we have that the expected total number of splitting events in the trails in $\mathcal{P}'$ is $O\left((d'n/k^2)\log(T'/(T'-t'))\right) \ll n$. Thus, by Markov's inequality, the number of splitting events is $o(n)$ a.a.s. By splitting the trails in $\mathcal{P}'$ whenever a splitting event occurs, we obtain $O(n)$ rainbow paths for $c'$. For each edge with a new color, we may have to split the paths using this edge since they may not be rainbow with respect to $c$. Since the number of new colors is $\psi(n) = o(n)$ and each edge appears in at most two paths, this causes $o(n)$ splits. Thus, we obtain a collection of $\mathcal{P}_1$ of $O(n)$ rainbow paths for $G'$.

The expected number of edges $E''$ not covered by $W_1, \ldots, W_n$ is $o(m'/k^2) = o(n)$ and so $|E''| = o(n)$ a.a.s. Let $\mathcal{P}_2$ denote the set of paths such that each path consists of a single edge of $E''$. Then $\mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2$ is a rainbow path cover of size $O(n)$ for $G$ with respect to $c$.

## 4 Future directions

There are many directions left to be explored. Naturally, covering the gap between our current bound of $pn = o((\log n/\log\log n)^2)$ and the denser regime of $pn = n^{\varepsilon}$ is a problem of interest, as is determining the exact constant in the asymptotic bound. Additionally, the techniques and properties established in our random graph analysis could provide valuable approaches to prove (or disprove) Conjecture 1, which concerns the minimum number of rainbow paths needed to cover arbitrary properly-colored graphs.

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