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Biclique immersions in graphs with independence number 2 ☆

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ABSTRACT

The analogue of Hadwiger's conjecture for the immersion relation states that every graph G contains an immersion of $K_{\chi(G)}$. For graphs with independence number 2, this is equivalent to stating that every such n-vertex graph contains an immersion of $K_{\lceil n/2\rceil}$. We show that every n-vertex graph with independence number 2 contains every complete bipartite graph on $\lceil n/2\rceil$ vertices as an immersion.

1. Introduction

A central problem in graph theory is guaranteeing dense substructures in graphs with a given chromatic number. Hadwiger's Conjecture [14] is one of the most important examples of this

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pursuit, stating that every loopless graph G contains the complete graph $K_{\chi(G)}$ as a minor (where $\chi(G)$ is the chromatic number of G), thus aiming to generalize the Four Color Theorem. This difficult conjecture is known to hold whenever $\chi(G) \leq 6$ [29], and it is open for the remaining values. Thus, a natural approach is to study whether it holds whenever G is restricted to a particular class of graphs. A class of graphs that has received particular attention (and yet remains open) is that of graphs with independence number 2. A recent survey of Seymour [30] emphasizes the importance of this case, which was first remarked by Mader (see [26]). Plummer, Stiebitz, and Toft [26] gave an equivalent formulation of Hadwiger's Conjecture for such graphs: every n-vertex graph with independence number 2 contains a minor of $K_{\lceil n/2 \rceil}$. Before that, in 1982, Duchet and Meyniel [9] had shown a result that implies that every such graph contains a minor of $K_{\lceil n/3 \rceil}$. Despite much work, see e.g. [12,15,16,34], it is still open whether there is a constant $c > 1/3$ such that every graph with independence number 2 contains a minor of $K_{\lceil cn \rceil}$. Given the difficulty to obtain a complete minor on $\lceil n/2 \rceil$ vertices, Norin and Seymour [25] recently turned into finding dense minors on this amount of vertices. They proved that every n-vertex graph with independence number 2 contains a (simple) minor of a graph H on $\lceil n/2 \rceil$ vertices and $0.98688 \cdot \binom{|V(H)|}{2} - o(n^2)$ edges.

The focus of this paper is a conjecture related to Hadwiger's, concerned with finding graph immersions in graphs with a given chromatic number; this type of substructure is defined as follows. To *split off* a pair of adjacent edges uv, vw (with u, v, w distinct) amounts to deleting those two edges and adding an edge joining u to v (keeping parallel edges if needed). A graph G is said to contain an *immersion* of another graph H if H can be obtained from a subgraph of G by splitting off pairs of edges and deleting isolated vertices. Notice then that if G contains H as a subdivision, it contains H as an immersion (and as a minor). Immersions have received increased attention in recent years, see e.g. [7,10,21,22,24,33], particularly since Robertson and Seymour [28] proved that graphs are well-quasi-ordered by the immersion relation. Much of this attention has been centered around the following conjecture of Lescure and Meyniel [20] (see also [1]), which is the immersion-analogue of Hadwiger's Conjecture.

Conjecture 1 (*Lescure and Meyniel [20]*). *Every loopless graph G contains an immersion of $K_{\chi(G)}$.*

The above conjecture holds whenever $\chi(G) \leq 4$ because Hajós' subdivision conjecture holds in this case, actually giving a subdivision of $K_{\chi(G)}$ [8]. The cases where $\chi(G) \in \{5, 6, 7\}$ were verified independently by Lescure and Meyniel [20] and by DeVos, Kawarabayashi, Mohar, and Okamura [6]. In general, a result of Gauthier, Le, and Wollan [13] guarantees that every graph G contains an immersion of a clique on $\lceil \frac{\chi(G)-4}{3.54} \rceil$ vertices. This result improves on theorems due to Dvořák and Yepremyan [11] and DeVos, Dvořák, Fox, McDonald, Mohar, and Scheide [5].

The case of graphs with independence number 2 has also received attention in regard to Conjecture 1. In particular, Vergara [32] showed that, for such graphs, Conjecture 1 is equivalent to the following conjecture.

Conjecture 2 (*Vergara [32]*). *Every n-vertex graph with independence number 2 contains an immersion of $K_{\lceil n/2 \rceil}$.*

As evidence for her conjecture, Vergara proved that every n-vertex graph with independence number 2 contains an immersion of $K_{\lceil n/3 \rceil}$. This was later improved by Gauthier et al. [13], who showed that every such graph contains an immersion of $K_{2\lfloor n/5 \rfloor}$. This last result was extended to graphs with arbitrary independence number [3]. Additionally, Vergara's Conjecture has been verified for graphs with small forbidden subgraphs [27]. The main contribution of this paper is the following result, which states that graphs with independence number 2 contain an immersion of every complete bipartite graph on $\lceil n/2 \rceil$ vertices.

Theorem 3. *Let G be an n-vertex graph with independence number 2, and $\ell \leq \lceil n/2 \rceil - 1$ be a positive integer. Then G contains an immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$.*

We remark that our proof of Theorem 3 is self-contained. Using an argument due to Plummer et al. [26] we show (see Corollary 17) that Theorem 3 implies the following.

Corollary 4. *Let G be a graph with independence number 2, and* $1 \leq \ell \leq \chi(G) - 1$. *Then G contains an immersion of* $K_{\ell,\chi(G)-\ell}$.

This result leads us to make the following conjecture, which is a weakening of Conjecture 1 and holds trivially when $\ell = 1$.

Conjecture 5. *If* $1 \leq \ell \leq \chi(G) - 1$, *then G contains an immersion of* $K_{\ell,\chi(G)-\ell}$.

We denote by $K_{a,b,c}$ the graph that admits a partition into parts of sizes a, b, and c such that any pair of these parts induces a complete bipartite graph. In addition to Corollary 4, as evidence for Conjecture 5, we give a short proof of the following strengthening of the case $\ell = 2$, which is also implied by a special case of a result of Mader for subdivisions [23].

Proposition 6. *If* $\chi(G) \geq 3$, *then G contains* $K_{1,1,\chi(G)-2}$ *as an immersion.*

We note that Conjecture 5 has its parallel in the minor order. Woodall [35] and, independently, Seymour (see [19]), proposed the following conjecture: every graph G with $\ell \leq \chi(G) - 1$ contains a minor of $K_{\ell,\chi(G)-\ell}$. In [35], Woodall showed that (the list-coloring strengthening of) his conjecture holds whenever $\ell \leq 2$. Kostochka and Prince [19] showed that the case $\ell = 3$ holds as long as $\chi(G) \geq 6503$. Kostochka [17] proved it for every ℓ as long as $\chi(G)$ is very large in comparison to ℓ, and later [18] improved this so that $\chi(G)$ could be polynomial in ℓ, namely, whenever $\chi(G) > 5(200\ell \log_2(200\ell))^3 + \ell$. In fact, the results in [17–19] obtain the full join $K^*_{\ell,\chi(G)-\ell}$, which is the graph obtained from the disjoint union of a K_ℓ and an independent set on $\chi(G) - \ell$ vertices by adding all of the possible edges between them. This and the above-cited result of Norin and Seymour leads us to make the following conjecture, which can be shown to be (using the arguments of Corollary 17) a particular case of the conjecture of Woodall and Seymour.

Conjecture 7. *Let G be an n-vertex graph with independence number 2, and* $1 \leq \ell \leq \lceil n/2 \rceil - 1$. *Then G contains a minor of* $K_{\ell,\lceil n/2 \rceil-\ell}$.

Note that the result of Kostochka leaves open the balanced case, thus not implying Conjecture 7. Yet, inspired by an earlier version of this paper Chen and Deng [4] showed that Conjecture 7 holds.

An extended abstract, containing a short sketch of the proof of Theorem 3, has appeared in [2].

The rest of the paper is organized as follows. In Section 2 we give some definitions and prove an interesting lemma that is used to prove Theorem 3, in Section 3. In Section 4 we prove Corollary 4 and Proposition 6. Finally, in Section 5 we present a related result together with an open problem.

2. Preliminaries and notation

All the graphs considered in this work are finite and loopless. For a graph G, we denote by $V(G)$ and $E(G)$ its set of vertices and edges, respectively. For every positive integer n, let $[n] = \{1, \ldots, n\}$. Let G be a graph. For $v \in V(G)$ and $S \subseteq V(G)$, we define $E(v, S) = \{vu \in E(G) : u \in S\}$. For a set of vertices $W \subseteq V(G)$, we write $G[W]$ to denote the subgraph induced by W. For a set of edges $F \subseteq E(G)$, we write $G[F]$ to denote the subgraph induced by F, i.e., the subgraph of G whose edge set is F, and whose vertex set is the set of all end vertices of the edges in F. We also denote by $N_G(v)$ and $N_G[v]$ the neighborhood and the closed neighborhood, respectively, of v in $V(G)$. The degree of a vertex v of a graph G is denoted by $d_G(v)$. We simply write $N(v)$, $N[v]$, and $d(v)$ when G is clear from the context. Moreover, when convenient, given a set $F \subseteq E(G)$ and a vertex $v \in V(G)$, we write $N_F(v)$, $N_F[v]$, and $d_F(v)$ to denote, respectively, $N_{G[F]}(v)$, $N_{G[F]}[v]$, and $d_{G[F]}(v)$.

Given an edge e of a graph G, the *multiplicity of (the edge) e in G*, denoted by $m_G(e)$, is the number of edges in G having the same pair of vertices as the edge e. For sets $A, B \subseteq V(G)$, we say that a path P of G is an *A, B-path* if P has an endvertex in A and the other in B. We denote by $\alpha(G)$ the maximum size of an independent set of G, which is the *independence number* of G.

The *complete graph* on n vertices is denoted by K_n and the *complete bipartite graph* with parts of size a and b is denoted by $K_{a,b}$. If A and B are disjoint sets, we let $K_{A,B}$ be the complete bipartite

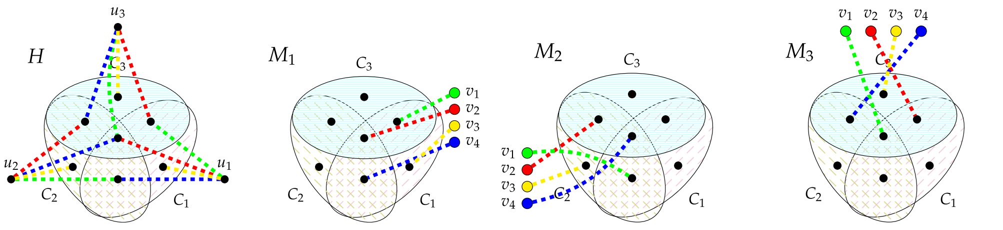

Fig. 1. An example of Lemma 8.

graph with bipartition (A, B). Complete graphs and complete bipartite graphs are also called *cliques* and *bicliques*, respectively.

In a manner that is equivalent to the definition given in the introduction, given a graph H, we say that a graph G *contains an immersion of* H if there exists an injection $f : V(H) \to V(G)$ and a collection of edge-disjoint paths in G, one for each edge of H, such that the path corresponding to an edge joining u and v has endvertices $f(u)$ and $f(v)$.

In our proof, we make use of the following lemma, which we believe to be interesting by itself, and that we could not find in the literature. The lemma below may be rephrased in terms of (disjoint) systems of distinct representatives.

Lemma 8. *Let $j \leq k$ be two positive integers, and let $C_1, \dots, C_j \subseteq [n]$ be sets of size k. Let A be a set of size k disjoint from $[n]$. Then there are disjoint matchings $M_1, \dots, M_j$ in $K_{A,[n]}$ such that M_i matches A with C_i for every $i \in [j]$.*

Proof. Let H be an auxiliary bipartite graph such that $V(H) = \{u_1, \dots, u_j\} \cup [n]$ and $E(H) = \{u_i x : x \in C_i, i \in [j]\}$. Since $|C_i| = k$, we have $d(u_i) = k$. Moreover, the degree of a vertex $x \in [n]$ is precisely the number of C_i's that contain x, which is at most $j \leq k$. Therefore, H is a bipartite graph of maximum degree precisely k. Then, by König's line coloring theorem, we can find a proper k-coloring of the edges of H, say $D_1, \dots, D_k \subseteq E(H)$.

Let $A = \{v_1, \dots, v_k\}$ be disjoint from $[n]$. Now we construct the matchings of $K_{A,[n]}$. For each $i \in \{1, \dots, k\}$, let $u_i w_1^i, \dots, u_i w_k^i$ be the edges of H incident to u_i, where $u_i w_j^i \in D_j$ (for an example, see Fig. 1). Note that, by the definition of H, we have $w_1^i, \dots, w_k^i \in C_i$. Then we create the matching $M_i = \{v_1 w_1^i, \dots, v_k w_k^i\}$, which matches A with C_i.

We claim that $M_i \cap M_j = \emptyset$ for every $i \neq j$. Suppose, without loss of generality, that $v_1 w \in M_i \cap M_j$. By the construction of M_i, we have that $u_i w \in D_1$. Analogously, $u_j w \in D_1$. Then, since D_1 is a matching, $u_i = u_j$, which means $i = j$. Thus, $M_i \cap M_j = \emptyset$ when $i \neq j$. □

3. Proof of Theorem 3

While Theorem 3 holds even when parallel edges are allowed, we will only consider simple graphs in this proof as this suffices and makes the writing simpler.

Indeed we can consider graphs with independence number at most 2. The proof follows by induction on $n + \ell$. Let G be an n-vertex graph with $\alpha(G) \leq 2$ and let $\ell \leq \lceil n/2 \rceil - 1$ be a positive integer. It is not hard to check that the result holds when $n \leq 4$, and thus we may assume that $n \geq 5$. Note also that it suffices to prove the statement in the case G is *edge-critical*, i.e., that the removal of any edge of G increases its independence number. Now, if $n \leq 4\ell - 2$, then $\lceil n/2 \rceil - \ell \leq 2\ell - 1 - \ell < \ell$. Thus, by induction, there is an immersion of $K_{\ell', \lceil n/2 \rceil - \ell'}$ in G, where $\ell' = \lceil n/2 \rceil - \ell$. But this is the desired immersion because $K_{\lceil n/2 \rceil - \ell, \lceil n/2 \rceil - \lceil n/2 \rceil + \ell} = K_{\lceil n/2 \rceil - \ell, \ell}$. Thus, from now on, we may assume that

$$n \geq 4\ell - 1. \quad (1)$$

Our proof is divided into four parts, and the following part is the only step of the proof in which we directly use induction.

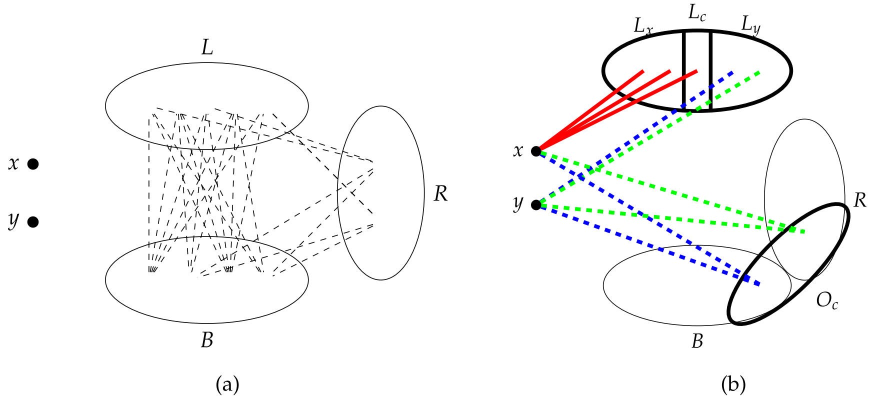

Fig. 2. Fig. 2(a) Partition of $V(G)$ with respect to H'; Fig. 2(b) Paths from x to L_y using O_c and y depicted in dashed lines.

3.1. Non-adjacent vertices with at least $\ell - 1$ common neighbors

Our first task is to show that we can assume that

$$|N(u) \cap N(v)| \leq \ell - 2 \text{ for every pair of non-adjacent vertices } u, v, \tag{2}$$

as, otherwise, we have the desired immersion. For that, suppose that G contains two non-adjacent vertices, say x and y, with at least $\ell-1$ common neighbors, and let $G' = G-x-y$. If $\ell \leq \lceil n/2 \rceil - 2 = \lceil (n-2)/2 \rceil - 1$ the induction hypothesis guarantees that G' contains an immersion of $K_{\ell, \lceil (n-2)/2 \rceil - \ell}$, which we call H'. Otherwise, if we have $\ell = \lceil n/2 \rceil - 1$ we let H' be an arbitrary set of ℓ vertices. Let L and B be the parts (i.e., the sets of vertices of G on which the parts of $K_{\ell, \lceil (n-2)/2 \rceil - \ell}$ were mapped) of H' having size ℓ and $\lceil n/2 \rceil - 1 - \ell$, respectively, and let $R = V(G') \setminus (L \cup B)$ (see Fig. 2(a)). As $\alpha(G) = 2$, every vertex in G' is either adjacent to x or to y in G. This is true in particular for the vertices in L. In what follows, we add either x or y to B, in order to obtain the desired immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$. This is immediate if x or y is adjacent to every vertex in L. Thus we may assume that $|E(y, L)|, |E(x, L)| < \ell$. Now, consider the following sets.

- ▹ L_x, the set of vertices in L adjacent to x but not adjacent to y;
- ▹ L_y, the set of vertices in L adjacent to y but not adjacent to x;
- ▹ L_c, the set of vertices in L adjacent to both x and y; and
- ▹ O_c, the set of vertices adjacent to both x and y that are not in L.

As x is adjacent to every vertex in $L_x \cup L_c$, it is enough to find (edge-disjoint) paths from x to L_y without using edges of H'. Notice that $|L_y| + |L_c| = |E(y, L)| \leq \ell - 1$ and that, by hypothesis, we have $|L_c| + |O_c| \geq \ell - 1$. Thus $|O_c| \geq |L_y|$. Let $O_c = \{o_1, o_2, \ldots, o_{|O_c|}\}$ and $L_y = \{\ell_1, \ell_2, \ldots, \ell_{|L_y|}\}$. For $1 \leq i \leq |L_y|$, we take the path $xo_iy\ell_i$ (see Fig. 2(b)). These paths are clearly pairwise edge-disjoint and do not use edges of H' because the edges xo_i, o_iy, and $y\ell_i$ are not in G'. This concludes the proof of (2).

3.2. Direct consequences of edge-criticality

Recall that G is edge-critical, meaning that the removal of any edge $uv \in E(G)$ creates an independent set of size 3. Hence, for such an edge there is a vertex w that is not adjacent to both u and v. We formalize this argument in the following claim.

Claim 9. *For any $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $N[u] \cup N[v] \neq V(G)$.*

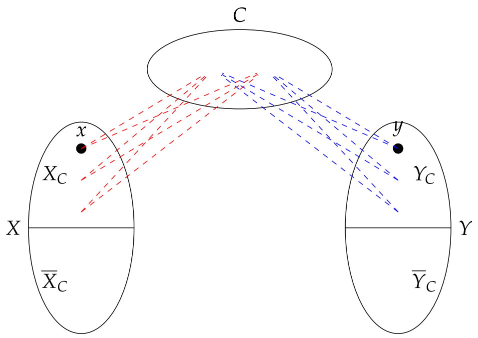

Fig. 3. Partition of G according to non-adjacent vertices x and y and their common neighborhood C.

For the rest of the proof, we fix two non-adjacent vertices x and y, and partition the vertices of G into the following three sets:

- ▷ $C = N(x) \cap N(y)$, the set of common neighbors of x and y;
- ▷ $X = \overline{N[y]}$, the set of non-neighbors of y excluding y, which contains x; and
- ▷ $Y = \overline{N[x]}$, the set of non-neighbors of x excluding x, which contains y.

Note that by (2), we have $|C| \leq \ell - 2$. Moreover, both X and Y induce complete subgraphs of G, otherwise we could find an independent set of size 3. This implies that $|X|, |Y| < \lceil n/2 \rceil$, otherwise we could find $K_{\ell, \lceil (n-2)/2 \rceil - \ell}$ as a subgraph, and hence $C \neq \emptyset$. Moreover, the edge-criticality of G yields the following claim.

Claim 10. *For every vertex $a \in C$, we have $X, Y \nsubseteq N(a)$.*

Proof. First, note that $N[x] = X \cup C$. Thus, since $a \in N(x)$, if $Y \subseteq N(a)$, then xa is an edge of G for which $N[x] \cup N[a] = V(G)$, a contradiction to Claim 9. Analogously, we have $X \nsubseteq N(a)$. □

3.3. Key vertex sets and their sizes

We partition X and Y in different ways. Let $X_C \subseteq X$ (resp. $Y_C \subseteq Y$) be the set containing vertices $v \in X$ (resp. $v \in Y$) for which $C \subset N(v)$, and put $\overline{X}_C = X \setminus X_C$ (resp. $\overline{Y}_C = Y \setminus Y_C$) (see Fig. 3). Similarly, given a vertex a in C, we denote by X_a (resp. Y_a) the set of vertices in X (resp. in Y) that are adjacent to a, and put $\overline{X}_a = X \setminus X_a$ and $\overline{Y}_a = Y \setminus Y_a$. Notice that if $v \in \overline{X}_a$ and $w \in \overline{Y}_a$, then v and w must be adjacent, as the independence number of G is 2. Thus we get $K_{\overline{X}_a, \overline{Y}_a}$ as a subgraph of G. Note that $X_C \subseteq X_a$ and $\overline{X}_C \supseteq \overline{X}_a$ (resp. $Y_C \subseteq Y_a$ and $\overline{Y}_C \supseteq \overline{Y}_a$) for every $a \in C$. Indeed, we have $X_C = \bigcap_{a \in C} X_a$ (resp. $Y_C = \bigcap_{a \in C} Y_a$) and $\overline{X}_C = \bigcup_{a \in C} \overline{X}_a$ (resp. $\overline{Y}_C = \bigcup_{a \in C} \overline{Y}_a$). The following claims give bounds on the sizes of some of these sets. This control is the key to build the desired immersion.

Claim 11. *Given $a \in C$, we have that $|X_C| \leq |X_a| \leq \ell - 2$ and $|Y_C| \leq |Y_a| \leq \ell - 2$. Furthermore, we have $|\overline{X}_a| \geq \lceil n/2 \rceil - |Y| + 3$ and $|\overline{Y}_a| \geq \lceil n/2 \rceil - |X| + 3$.*

Proof. We prove the bounds for X_a and $\overline{X}_a$, as the remaining bounds follow analogously. By Claim 10, there is a vertex $w \in \overline{X}_C$ such that $wa \notin E(G)$. By the definition of X_a we have $X_C \subseteq X_a \subseteq N(a)$, which gives us $|X_C| \leq |X_a|$. Moreover, since X induces a complete graph, we have $X_a \subseteq N(w)$, and hence $X_a \subseteq N(a) \cap N(w)$. Therefore, (2) implies $|X_a| \leq \ell - 2$.

Now, we use (1) in the following to obtain the claim:

$$|Y| + |\overline{X}_a| = n - |C| - |X_a|$$

$$
\begin{aligned}
&\geq n - \ell + 2 - \ell + 2 \\
&= \lceil n/2 \rceil + \lfloor n/2 \rfloor - 2\ell + 4 \\
&\geq \lceil n/2 \rceil + \lfloor (4\ell - 1)/2 \rfloor - 2\ell + 4 \\
&= \lceil n/2 \rceil + \lfloor 2\ell - (1/2) \rfloor - 2\ell + 4 \\
&= \lceil n/2 \rceil + 2\ell - 1 - 2\ell + 4 \\
&= \lceil n/2 \rceil + 3 . \quad \square
\end{aligned}
$$

The claim above is used to prove the following three claims. In fact, the "+3" term in its bounds is almost necessary for the proof of the next claim.

Claim 12. *For every $a \in C$, we have either $|X_a| \geq 2(\ell - |\overline{Y}_a|)$ or $|Y_a| \geq 2(\ell - |\overline{X}_a|)$.*

Proof. For a contradiction, assume $|X_a| < 2(\ell - |\overline{Y}_a|)$ and $|Y_a| < 2(\ell - |\overline{X}_a|)$. Then we have

$$
\begin{aligned}
|X| + |Y| &= |X| - |\overline{X}_a| + |\overline{X}_a| + |Y| - |\overline{Y}_a| + |\overline{Y}_a| \\
&< 2(\ell - |\overline{Y}_a|) + |\overline{X}_a| + 2(\ell - |\overline{X}_a|) + |\overline{Y}_a| \\
&= 4\ell - |\overline{Y}_a| - |\overline{X}_a| .
\end{aligned}
$$

Also, by Claim 11, we have $|\overline{Y}_a| \geq \lceil n/2 \rceil - |X| + 3$ and $|\overline{X}_a| \geq \lceil n/2 \rceil - |Y| + 3$, which implies

$$
\begin{aligned}
4\ell &> |X| + |Y| + |\overline{Y}_a| + |\overline{X}_a| \\
&\geq |X| + |Y| + \lceil n/2 \rceil - |X| + 3 + \lceil n/2 \rceil - |Y| + 3 \\
&\geq n + 6 .
\end{aligned}
$$

So we have $n < 4\ell - 6$, which contradicts (1). $\square$

Claim 13. *For every $v \in \overline{X}_C$ (resp. $w \in \overline{Y}_C$), we have $|N(v) \cap \overline{Y}_C| > \lceil n/2 \rceil - |X|$ (resp. $|N(w) \cap \overline{X}_C| > \lceil n/2 \rceil - |Y|$).*

Proof. We prove the bound for $|N(v) \cap \overline{Y}_C|$, as the bound on $|N(w) \cap \overline{X}_C|$ follows analogously. By the definition of $\overline{X}_C$, there is a vertex $a \in C$ such that $va \notin E(G)$. Note that $\overline{Y}_a \subseteq N(v)$, otherwise G would contain an independent set of size 3. Since $\overline{Y}_a \subseteq \overline{Y}_C$, we have $\overline{Y}_a \subseteq N(v) \cap \overline{Y}_C$, and the desired bound follows from Claim 11. $\square$

Finally, we need the following relation between the sizes of $\overline{X}_C$ and X_C (resp. $\overline{Y}_C$ and Y_C).

Claim 14. *For every $a \in C$ we have the following:*

(i) If $|\overline{X}_C| < \ell$, then $|X_C| > \lceil n/2 \rceil - \ell - |\overline{Y}_a| \geq \ell - |\overline{Y}_a|$; and
(ii) If $|\overline{Y}_C| < \ell$, then $|Y_C| > \lceil n/2 \rceil - \ell - |\overline{X}_a| \geq \ell - |\overline{X}_a|$.

Proof. We only prove (i), as (ii) follows analogously. By Claim 11, if $|\overline{X}_C| < \ell$, then

$$\lceil n/2 \rceil + 3 \leq |\overline{Y}_a| + |X| = |\overline{Y}_a| + |\overline{X}_C| + |X_C| < |\overline{Y}_a| + \ell + |X_C| .$$

By (1) we have $\lceil n/2 \rceil \geq 2\ell$, which implies

$$|X_C| > \lceil n/2 \rceil - \ell - |\overline{Y}_a| \geq 2\ell - \ell - |\overline{Y}_a| = \ell - |\overline{Y}_a| . \quad \square$$

3.4. Constructing the immersion

The rest of the proof is divided into two cases which depend on the sizes of $\overline{X}_C$ and $\overline{Y}_C$. The sets that form the bipartition of the immersion depend on which case we are dealing with. The construction requires more care in Case 2, where one of $\overline{X}_C, \overline{Y}_C$ is large. In that case, we apply Lemma 8 and obtain a set of paths that are not necessarily edge-disjoint. Nevertheless, the

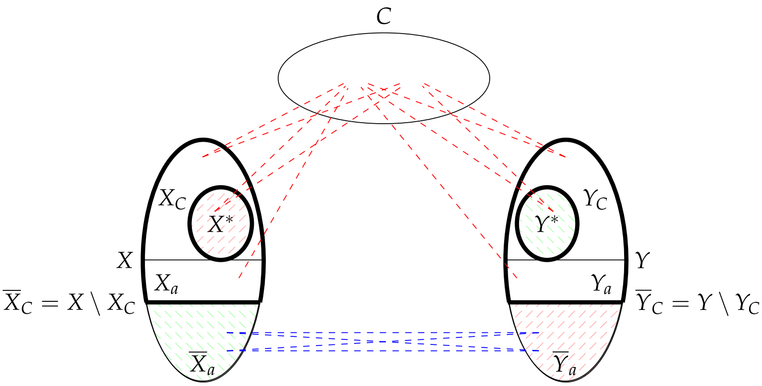

Fig. 4. Immersion for Case 1, when $|\overline{X}_C|, |\overline{Y}_C| < \ell$: between $Y^* \cup \overline{X}_a$ and $X^* \cup \overline{Y}_a$.

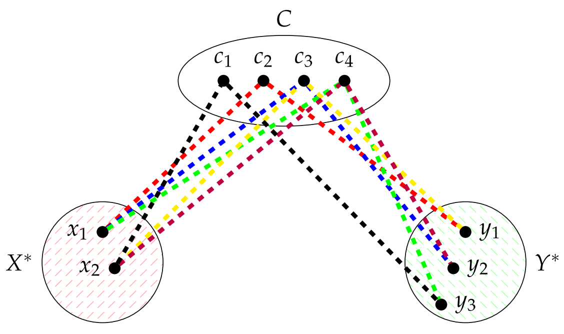

Fig. 5. Construction for Case 1, connecting X^* to Y^* through C.

intersections are relatively few and with a combination of different techniques, we are able to fix them and obtain the desired immersion. For the rest of the proof, we fix $a \in C$.

Case 1, $|\overline{X}_C|, |\overline{Y}_C| < \ell$. By Claim 14, we can choose $X^* \subset X_C$ and $Y^* \subset Y_C$ such that $|X^*| = \lceil n/2 \rceil - \ell - |\overline{Y}_a|$ and $|Y^*| = \ell - |\overline{X}_a|$. In what follows, we find an immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$ with bipartition $(Y^* \cup \overline{X}_a, X^* \cup \overline{Y}_a)$ (see Fig. 4).

Recall that X, Y, and $\overline{X}_a \cup \overline{Y}_a$ each induce a clique (each is contained in the non-neighborhood of a vertex and G has independence number 2), and thus G contains all edges joining (i) vertices in $\overline{X}_a$ to vertices in $\overline{Y}_a$; (ii) vertices in $\overline{X}_a$ to vertices in X^*; and (iii) vertices of $\overline{Y}_a$ to vertices in Y^*. It remains to find edge-disjoint paths joining vertices in X^* to vertices in Y^*. For that, we use only edges incident to vertices in C, which guarantees that our new paths do not use any previously used edge. Let $X^* = \{x_1, \ldots, x_{|X^*|}\}$, $Y^* = \{y_1, \ldots, y_{|Y^*|}\}$, and $C = \{c_1, \ldots, c_{|C|}\}$. For each pair of vertices x_i, y_j, consider the path $x_i c_{i+j} y_j$, where the additions in the subscript are taken modulo $|C|$ (see Fig. 5).

It is easy to see that these paths are edge-disjoint if and only if $|C| \geq |X^*|, |Y^*|$. Since, Claim 11 gives us $|X_C|, |Y_a| \leq \ell - 2$, and since in Case 1 we assume $|\overline{X}_C| < \ell$, we have

$$\begin{aligned}
|C| &= n - |\overline{X}_C| - |X_C| - |\overline{Y}_a| - |Y_a| \\
&> n - \ell - (\ell - 2) - |\overline{Y}_a| - (\ell - 2) \\
&= n - 3\ell - |\overline{Y}_a| + 4 \\
&= \lceil n/2 \rceil + \lfloor n/2 \rfloor - 3\ell - |\overline{Y}_a| + 4
\end{aligned}$$

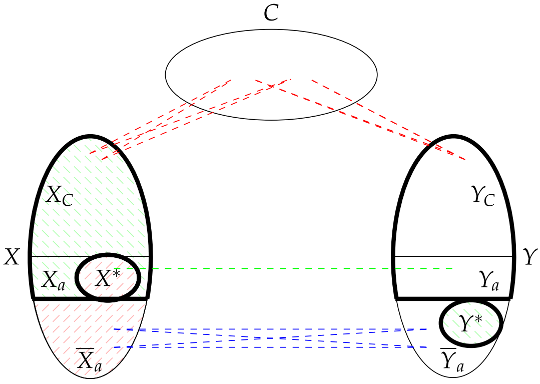

Fig. 6. Immersion for Case 2, when $|\overline{X}_C| \geq \ell$: between $X^* \cup \overline{X}_a$ and $(X_a \setminus X^*) \cup Y^*$.

$$
\begin{aligned}
&\geq \lceil n/2 \rceil + 2\ell - 1 - 3\ell - |\overline{Y}_a| + 4 \qquad \text{by (1)} \\
&= \lceil n/2 \rceil - \ell - |\overline{Y}_a| + 3\,. \qquad (3)
\end{aligned}
$$

Hence, $|C| > \lceil n/2 \rceil - \ell - |\overline{Y}_a| + 3 > |X^*|$. Analogously to (3), we can show that $|C| \geq \lceil n/2 \rceil - \ell - |\overline{X}_a| + 3$, and by (1), that $\lceil n/2 \rceil - \ell - |\overline{X}_a| + 3 > |Y^*|$, as desired. Since these X^*, Y^*-paths are mutually edge-disjoint and edge-disjoint from all previous paths, we get the desired immersion.

Case 2, $|\overline{X}_C| \geq \ell$ or $|\overline{Y}_C| \geq \ell$. Let $\gamma = \min\{|\overline{X}_C|, |\overline{Y}_C|\}$. The proof of this case is again divided into two cases, depending on whether $\gamma \geq \ell$ or $\gamma < \ell$. Since the constructions of the immersions in these two cases are similar, we unify them as follows. Recall that by Claim 12, either $|X_a| \geq 2(\ell - |\overline{Y}_a|)$ or $|Y_a| \geq 2(\ell - |\overline{X}_a|)$. Then, if $\gamma \geq \ell$, we assume without loss of generality that $|Y_a| \geq 2(\ell - |\overline{X}_a|)$; and if $\gamma < \ell$, we assume without loss of generality that $|\overline{Y}_C| = \gamma$ and that $|\overline{X}_C| \geq \ell$. Notice that in both cases we have $|\overline{X}_C| \geq \ell$.

By Claim 11, we can choose $Y^* \subset \overline{Y}_a$ with $|Y^*| = \lceil n/2 \rceil - |X|$, and since $|\overline{X}_C| \geq \ell$, we can choose $X^* \subset \overline{X}_C \setminus \overline{X}_a$ with $|X^*| = \ell - |\overline{X}_a|$. Note that $|X_a| - |X^*| + |Y^*| = |X_a| - \ell + |\overline{X}_a| + |Y^*| = |X| + |Y^*| - \ell = \lceil n/2 \rceil - \ell$. Now, we show that

$$
|X^*| \leq |Y^*|. \qquad (4)
$$

Indeed, by Claim 11, we have $|X_a| \leq \ell - 2$, and hence

$$
\begin{aligned}
|Y^*| &= \lceil n/2 \rceil - |X| \\
&= \lceil n/2 \rceil - |X_a| - |\overline{X}_a| \\
&\geq 2\ell - (\ell - 2) - |\overline{X}_a| \\
&= \ell + 2 - |\overline{X}_a| \\
&> \ell - |\overline{X}_a| \\
&= |X^*|\,.
\end{aligned}
$$

In what follows, we find an immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$ with bipartition $\left(X^* \cup \overline{X}_a, (X_a \setminus X^*) \cup Y^*\right)$ (see Fig. 6).

As before, since X, Y, and $\overline{X}_a \cup \overline{Y}_a$ induce complete graphs, G contains all edges joining (i) vertices in $\overline{X}_a$ to vertices in Y^*; (ii) vertices in $\overline{X}_a$ to vertices in $X_a \setminus X^*$; and (iii) vertices in X^* to vertices in $X_a \setminus X^*$. It remains to find edge-disjoint paths joining vertices in X^* to vertices in Y^*. For these paths, we only use edges that are incident to vertices in Y and not to vertices in $\overline{X}_a$; this assures that they are disjoint from the edges already used. Let $X^* = \{v_1, v_2, \ldots, v_{|X^*|}\}$ and $Y^* = \{y_1, y_2, \ldots, y_{|Y^*|}\}$. The first step is to use Lemma 8 to find paths joining each vertex v_i to all vertices in Y^* allowing edges between vertices of Y^* to be used at most twice. This is done in the next claim, which is a key step in the proof.

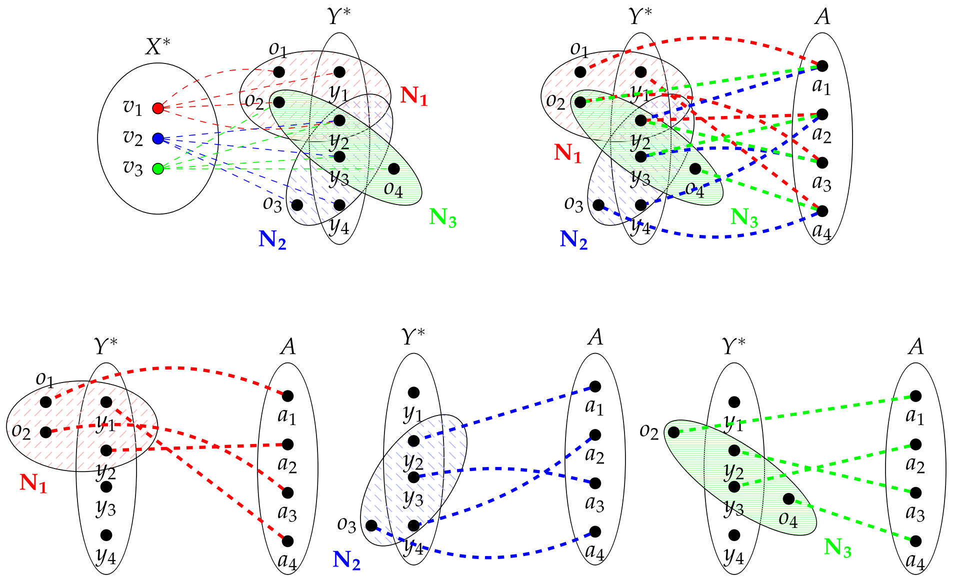

Fig. 7. The setup for an example of Claim 15.

Claim 15. *For each $i \in \{1, 2, \dots, |X^*|\}$, there is a subgraph $K(v_i)$ which contains an immersion of K_{v_i,Y^*} and satisfies that:*

(i) each path of $K(v_i)$ with an endvertex in v_i has length at most 2;
(ii) for each path $v_i z y_j$ in $K(v_i)$ we have $z \in \overline{Y}_C$; and
(iii) if $i \neq j$ and $uw \in E(K(v_i)) \cap E(K(v_j))$, then there is no r with $r \neq i$ and $r \neq j$ such that $uw \in E(K(v_r))$. Moreover, if $v_i z_{i,k} y_k$ and $v_j z_{j,k'} y_{k'}$ are the paths in, respectively, $K(v_i)$ and $K(v_j)$ that contain uv, then $\{u, v\} = \{y_k, y_{k'}\}$.

Proof. Note that, since $X^* \subseteq \overline{X}_C$, Claim 13 assures that for each $i \in \{1, 2, \dots, |X^*|\}$, we have $|N(v_i) \cap \overline{Y}_C| > \lceil n/2 \rceil - |X| = |Y^*|$. In order to use Lemma 8, we define, for each such i, a set $N_i \subset N(v_i) \cap \overline{Y}_C$ with $|N_i| = |Y^*|$, and also a set of auxiliary vertices $A = \{a_1, a_2, \dots, a_{|Y^*|}\}$ with $N(a_j) = \overline{Y}_C$. By (4), we can apply Lemma 8 to $N_1, \dots, N_{|X^*|}$ together with A to obtain disjoint matchings $M_1, M_2, \dots, M_{|X^*|}$ such that M_i matches A to N_i, for $i \in \{1, \dots, |X^*|\}$. Let $M_i = \{z_{i,1}a_1, \dots, z_{i,|Y^*|}a_{|Y^*|}\}$ where $z_{i,j} \in N_i$ for every i, j (see Fig. 7).

For each $v_i \in X^*$, we obtain $K(v_i)$ by using y_j whenever a_j is used in a matching. In other words, for every $1 \le j \le |Y^*|$, if $z_{i,j}a_j \in M_i$, then we use the path $v_i z_{i,j} y_j$. Notice that $z_{i,j}$ could be y_j itself. When that is the case, we use the path $v_i y_j$. Formally, we define

$$P(i,j) = \begin{cases} v_i z_{i,j} y_j & \text{if } y_j \neq z_{i,j} \\ v_i y_j & \text{if } y_j = z_{i,j}\,. \end{cases}$$

Notice that $P(i,j)$ may not be edge-disjoint from $P(i,k)$ if $k \neq j$, but this can only happen when $P(i,j) = v_i y_k y_j$ and $P(i,k) = v_i y_j y_k$. If that is the case, we redefine $P(i,j)$ as $v_i y_j$ and $P(i,k)$ as $v_i y_k$. Thus, after doing all the necessary changes, we can assume that $P(i,j)$ is disjoint from $P(i,k)$ whenever $j \neq k$ (see Fig. 8). Finally we define $K(v_i) = \bigcup_{j=1}^{|Y^*|} P(i,j)$ and since the $P(i,j)$'s are edge disjoint each $K(v_i)$ contains an immersion of K_{v_i,Y^*} and clearly satisfies items (*i*) and (*ii*). Furthermore, as $M_1, \dots, M_{|X^*|}$ are disjoint matchings, if $uw \in E(K(v_i)) \cap E(K(v_j))$ for some pair $i \neq j$, then it must be that $u, w \in Y^*$. Indeed, by construction $K(v_i)$ (resp. $K(v_j)$) is obtained from a star centered in v_i (resp. v_j) and with leaves in $\overline{Y}_C$ by adding edges joining its leaves to the vertices

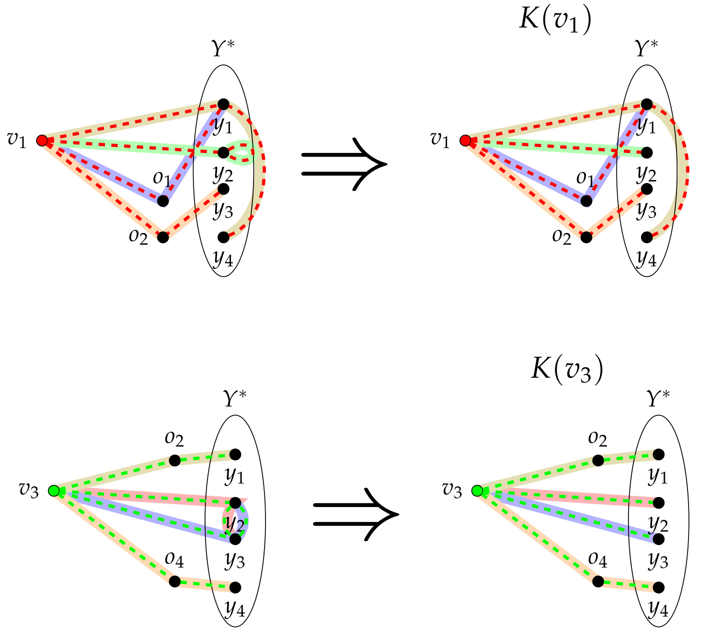

Fig. 8. Defining $K(v_1)$ and $K(v_3)$ for the graph in Fig. 7.

in Y^*. Then we have $u, w \notin \{v_i, v_j\}$, and hence $u, w \in \overline{Y}_C$, and uw is the "second" edge of those paths, i.e., uv joins a vertex z_{i,k_i} to $y_{k_i} \in Y^*$ and a vertex z_{j,k_j} to $y_{k_j} \in Y^*$. Assume without loss of generality that $u = z_{i,k_i}$ and $w = y_{k_i}$. Then $w \in Y^*$. Moreover, this implies that $z_{i,k_i}a_{k_i} \in M_i$. If $u = y_{k_j}$, then $u \in Y^*$ as desired. Otherwise we must have $u = z_{j,k_j} = z_{i,k_i}$ and $w = y_{k_j} = y_{k_i}$. But this implies that $z_{i,k_i}a_{k_i} \in M_j$, and hence $M_i \cap M_j \neq \emptyset$, a contradiction.

Let $u = y_h$ and $w = y_k$. Then either $z_{i,h} = y_k$ or $z_{i,k} = y_h$. Assume, without loss of generality, that $z_{i,h} = y_k$. This means that $z_{i,h}a_h = y_ka_h \in M_i$. Thus $y_ka_h \notin M_r$ for $r \neq i$. This, in turn, implies that $z_{j,k} = y_h$, which means that $y_ha_k \in M_j$ and $y_ha_k \notin M_r$ for $r \neq j$. This proves (*iii*). □

From now on, let $K(v_1), \ldots, K(v_{|X^*|})$ be the subgraphs given by Claim 15. We would like the v_i, Y^*-paths on these subgraphs to be the X^*, Y^*-paths in our immersion. Yet, if $i \neq j$, $K(v_i)$ might not be edge disjoint from $K(v_j)$. Fortunately, by Claim 15 (*iii*) these intersections are restricted, and, in what follows, we fix them.

Let H be the (multi)graph with vertex set $V(G)$, and whose edge set is the disjoint union of $E(K(v_1)), \ldots, E(K(v_{|X^*|}))$, where the multiplicity of an edge e is $|\{i : e \in E(K(v_i))\}|$. Notice that by Claim 15 (*iii*), the multiplicity of every edge of H is at most 2, and the only edges with multiplicity 2 are between vertices of Y^*. To fix these repetitions, we take detours through other vertices in Y. We now need to divide Case 2 into two subcases, depending on whether $|\overline{Y}_C| \geq \ell$.

Case 2.1, $\gamma \geq \ell$. Recall that in this case, we have $|\overline{X}_C|, |\overline{Y}_C| \geq \ell$, and that we assume, without loss of generality, that

$$|Y \setminus \overline{Y}_a| = |Y_a| \geq 2(\ell - |\overline{X}_a|)\,. \tag{5}$$

Let $u \in Y^*$. By Claim 15 (*iii*) we know that if u is incident to an edge of multiplicity 2, say uw, then there is a path in some $K(v_i)$ that contains uw and ends at u. Recall that $m_H(e)$ is the multiplicity of e in H, and, for a given vertex $u \in Y^*$, let p_u be defined as

$$p_u = \sum_{uv \in E(H), v \in Y^*} (m_H(uv) - 1)\,,$$

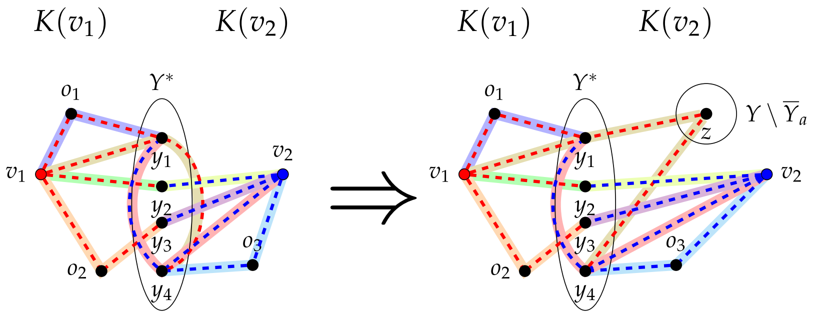

Fig. 9. Removing a multiple edge through $Y \setminus \overline{Y}_a$.

and let $q_u = |N_H(u) \cap (Y \setminus Y^*)|$. Let us see that $q_u + p_u \leq |X^*| = \ell - |\overline{X}_a|$. Indeed, there are precisely $|X^*|$ paths in H that have u as an endpoint, where p_u of these paths use edges with multiplicity 2, and q_u use edges that join u to vertices in $Y \setminus Y^*$. Since edges with multiplicity 2 are incident only to vertices in Y^*, none of these paths were counted twice, but there could be some of the $|X^*|$ paths that are not counted above. Now we show that if uw is an edge with multiplicity 2 in H, then

$$|Y \setminus Y^*| > q_u + q_w \,. \tag{6}$$

As uw has multiplicity 2, we have $p_u, p_w \geq 1$. So using (5) and that $Y^* \subseteq \overline{Y}_a$, we indeed have

$$\begin{aligned} q_u + q_w &\leq \ell - |\overline{X}_a| - p_u + \ell - |\overline{X}_a| - p_w \\ &\leq \ell - |\overline{X}_a| - 1 + \ell - |\overline{X}_a| - 1 \\ &= 2(\ell - |\overline{X}_a| - 1) \\ &\leq |Y \setminus \overline{Y}_a| - 2 \\ &\leq |Y \setminus Y^*| - 1 \,. \end{aligned}$$

We now describe a process that at each step picks an edge uw with multiplicity 2 of H, reduces p_u and p_w by 1, increases q_u and q_w by 1, and leaves p_t and q_t unchanged for every vertex $t \in Y^* \setminus \{u, w\}$. In other words, the function $I : Y^* \to \mathbb{N}$ such that $I(u) = p_u + q_u$ is an invariant of this process, while the number of edges with multiplicity 2 decreases.

Let $v_1, v_2 \in X^*$ be such that $uw \in E(K(v_1)) \cap E(K(v_2))$. By (6), there is a vertex $z \in Y \setminus Y^*$ such that z is neither adjacent to u nor w in H. Modify H by replacing the path v_1uw in $K(v_1)$ with the path v_1uzw, and leave v_2wu unchanged (see Fig. 9). Then this replacement does not increase the multiplicity of any edge, reduces the multiplicity of edge uv by 1, and increases the sizes of $N_H(u) \cap Y \setminus Y^*$ and $N_H(w) \cap Y \setminus Y^*$ by 1, because after switching we have $z \in N_H(u) \cap N_H(w)$. Furthermore, as $z \in Y \setminus Y^*$, it does not change the values of p_t and q_t for any vertex $t \in Y^* \setminus \{u, w\}$. Hence after each step we still have $q_u + p_u \leq |X^*| = \ell - |\overline{X}_a|$ for every $u \in Y^*$, and thus we can continue to assume (6). Repeating this process until H becomes a simple graph ends up giving us that H contains an immersion of K_{X^*,Y^*} and therefore G contains an immersion of $K_{\ell,\lceil n/2\rceil-\ell}$.

Case 2.2, $\gamma < \ell$. Recall that in this case we assume $\gamma = |\overline{Y}_C| < \ell$. In this case, we modify the subgraphs given by Claim 15 so that the set of edges of each $K(v_i)$ joining two vertices in Y^* induces a matching. For each $1 \leq i \leq |X^*|$ let $Q(v_i)$ be the subgraph of $K(v_i)$ induced by the vertices of Y^*. Notice that for each vertex $y_j \in Y^*$, the degree of y_j in $Q(v_i)$ is the number of times y_j is in the middle of some path $v_i z_{i,k} y_k$ (i.e., $y_j = z_{i,k}$), which is at most 1 because $v_i y_j$ can be used only once, plus the number of times y_j is the end of some path $v_i z_{i,k} y_k$, which is 1 because there is precisely one path in $K(v_i)$ joining v_i to y_j. Thus we have $d_{Q(v_i)}(y_j) \leq 2$, and so each component of $Q(v_i)$ is either a path or a cycle.

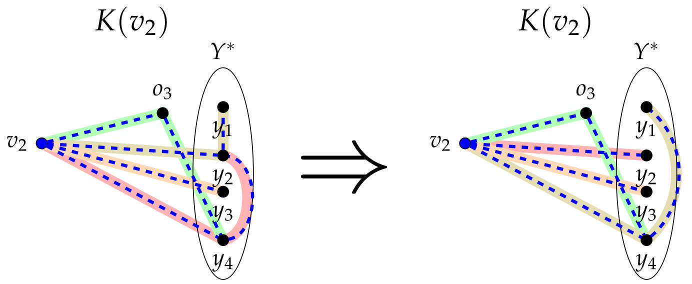

Fig. 10. Removing a path in Y^*.

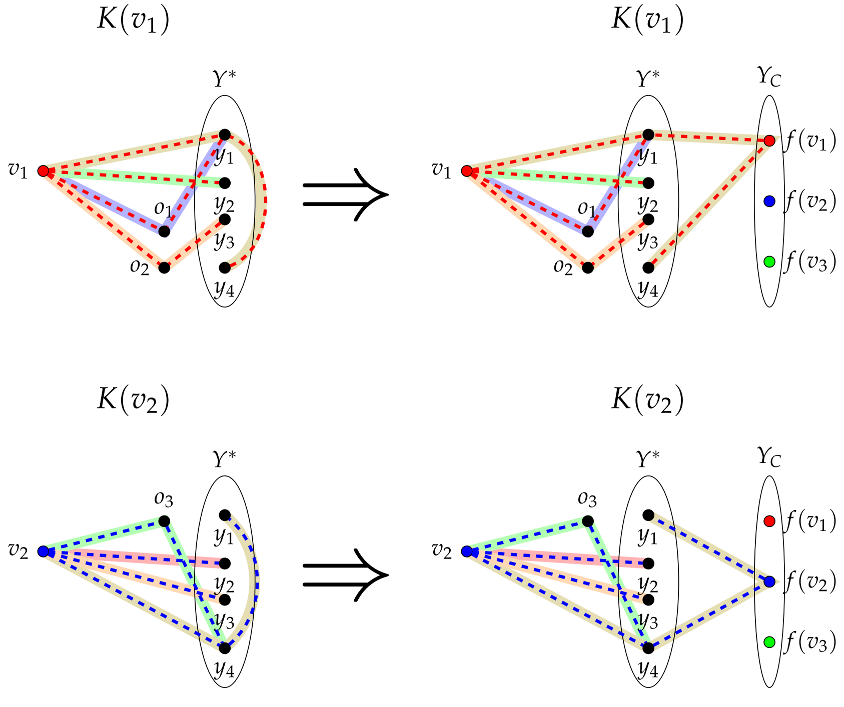

Fig. 11. Removing edges in Y^* using Y_C.

If $t_1t_2\cdots t_st_1$ is a cycle in $Q(v_i)$, then we replace, in $K(v_i)$, the path $v_it_jt_{j+1}$ with the path v_it_{j+1}, for every $1 \leq j \leq s$, where $t_{s+1} = t_1$. If $t_1t_2\cdots t_s$ is a component of $Q(v_i)$ that is a path, then we replace, in $K(v_i)$, the path $v_it_jt_{j+1}$ with the path v_it_{j+1}, for each $1 \leq j \leq s-2$, and replace the path $v_it_{s-1}t_s$ with the path $v_it_1t_s$ (see Fig. 10). After these operations $E(Q(v_i))$ is a matching.

Now, Claim 14 implies $|Y_C| > \ell - |\overline{X}_a| = |X^*|$. Thus, there is an injection $f : X^* \to Y_C$, and we can replace in H every path $vuw \in K(v)$ for which $uw \in Y^*$ with the path $vuf(v)w$ (see Fig. 11). Notice that no edge is used more than once because $E(Q(v))$ is a matching in Y^*. Applying this replacement for every $v \in X^*$ yields that H contains an immersion of K_{X^*,Y^*} and G contains the desired immersion of $K_{\ell,\lceil n/2\rceil-\ell}$. This concludes the proof of Theorem 3.

4. Evidence for Conjecture 5

In this section we show that Theorem 3 implies Corollary 4 (see Corollary 17), and prove Proposition 6. For the first proof, we use arguments of Plummer et al. [26]. We need the following theorem and, for that, a definition: a graph G is said to be *k-critical* if $\chi(G) = k$ and $\chi(G - v) < k$, for every $v \in V(G)$.

Theorem 16 (*Stehlík [31]*). *Let G be a k-critical graph such that its complement is connected. For every $v \in V(G)$, $G - v$ has a $(k - 1)$-coloring in which every color class contains at least two vertices.*

Corollary 17. *Let k, ℓ be positive integers with $k \geq \ell + 1$, and let G be a k-chromatic graph with independence number 2. Then G contains an immersion of $K_{\ell,k-\ell}$.*

Proof. Assume that G is a counterexample that minimizes the number of vertices. It suffices to show that it is also a counterexample to Theorem 3. In order to apply Theorem 16, we show that G is k-critical and that its complement is connected. If $k = \ell+1$, we use that any graph with chromatic number k has $K_{1,k-1} = K_{k-\ell,\ell}$ as a subgraph. Thus, we can assume that $k \geq \ell + 2$.

Let us first show that G is k-critical. Indeed, if G is not k-critical, then there would be some $v \in V(G)$ such that $\chi(G - v) = k$. Then $G - v$ has at least k vertices and is either a complete graph or has independence number 2, which, by choice of G, means it contains an immersion of $K_{\ell,k-\ell}$, a contradiction.

Next, we show that the complement of G is connected. If it were not, $V(G)$ could be partitioned into two sets V_1, V_2 such that $uv \in E(G)$ whenever $u \in V_1, v \in V_2$. Let k_1, k_2 be the chromatic numbers of $G[V_1]$ and $G[V_2]$, respectively, and note that $k = k_1 + k_2$. As $k \geq \ell + 2$, let ℓ_1, ℓ_2 be such that $\ell_1 + \ell_2 = \ell$, $k_1 \geq \ell_1 + 1$, and $k_2 \geq \ell_2 + 1$. Then, $G[V_1]$ and $G[V_2]$ must contain an immersion of $K_{\ell_1,k_1-\ell_1}$ and an immersion of $K_{\ell_2,k_2-\ell_2}$, respectively, because G is a counterexample that minimizes the number of vertices. But every vertex in V_1 is adjacent in G to every vertex in V_2. Thus, G must contain an immersion of $K_{\ell_1,k_1-\ell_1,\ell_2,k_2-\ell_2}$, which in turn implies an immersion of $K_{\ell_1+\ell_2,k_1-\ell_1+k_2-\ell_2} = K_{\ell,k-\ell}$. This contradicts the fact that G is a counterexample. Therefore, the complement of G is connected.

Finally, by Theorem 16, $G - v$ has a $(k - 1)$-coloring such that each color class has at least two vertices. But since G has independence number 2, each color class has size exactly 2. Thus we have $|V(G)| = 2k-1$. In particular, $\left\lceil \frac{|V(G)|}{2} \right\rceil = \frac{|V(G)|+1}{2} = k$, meaning that G does not contain an immersion of $K_{\ell,\lceil |V(G)|/2 \rceil-\ell}$ and is a counterexample to Theorem 3, as desired. □

Finally, we give a proof of Proposition 6, which claims that every graph with chromatic number at least 3 contains an immersion of $K_{1,1,\chi(G)-2}$.

Proof of Proposition 6. Assume $\chi(G) = s + 2 \geq 3$, and let $f : V(G) \to [s + 2]$ be a proper coloring of G that minimizes the number of vertices colored with $s + 2$. Let v be a vertex with $f(v) = s + 2$. By the minimality condition, v has at least one neighbor of each color. Furthermore, assume that f minimizes the number of vertices in $N(v)$ colored with $s + 1$.

Let $vw \in E(G)$ with $f(w) = s + 1$. For $1 \leq i \leq s$, consider the subgraph induced by vertices colored with i or $s + 1$, and let K_{wi} be the connected component containing vertex w. If K_{wi} does not contain neighbors of v colored with i, then we can switch the colors of K_{wi} (from $s + 1$ to i and vice versa), obtaining a new coloring of G with less neighbors of v colored with $s + 1$. Hence K_{wi} must contain at least one neighbor of v colored with i. Label one such vertex w_i.

For $1 \leq i \leq s$, let P_i be a path in K_{wi} joining w to w_i. Notice that if $1 \leq i \neq j \leq s$, then the path P_i is edge-disjoint from the path P_j, since P_i uses edges with endpoints colored i and $s + 1$ while P_j uses edges with endpoints colored j and $s + 1$. Further notice that these paths do not use any edge with v as one of its endpoints. Hence

$$F = \bigcup_{i=1}^{s} E(P_i) \cup \{vw_i : 1 \leq i \leq s + 1\}$$

induces an immersion of $K_{1,1,s}$. □

5. Immersions of matchings

In this section, we discuss a related result that we obtained when proving Theorem 3, and that we believe could have further applications. Let G be a graph with a matching M, and let $A, B \subseteq V(G)$ be disjoint sets of vertices of G. We say that a matching M is an *(A, B)-matching* if $|M| = |A| = |B|$ and every edge in M has a vertex in A and another vertex in B. We prove the following.

Theorem 18. *Let k and s be positive integers with $s \leq k/2+1$, let $G = K_{2k}$, and let $(A_1, B_1), \ldots, (A_s, B_s)$ be pairs of sets of vertices of G such that for every $i \in \{1, \ldots, s\}$ we have*

(a) $k \geq |A_i| = |B_i| \geq 2(i-1)$; and
(b) $A_i \cap B_i = \emptyset$.

Then G contains edge-disjoint matchings $M_1, \ldots, M_s$ such that M_i is an (A_i, B_i)-matching for $i \in \{1, \ldots, s\}$.

Proof. Assume $M_1, \ldots, M_{i-1}$ were found. Let $U = M_1 \cup \cdots \cup M_{i-1}$ be the set of used edges and let $F = E(G[A_i \cup B_i]) \setminus U$ be the set of free edges. We use Hall's Theorem. For that, let $X \subseteq A_i$. We prove that $|N_F(X) \cap B_i| \geq |X|$. Suppose, for a contradiction, that $|N_F(X) \cap B_i| < |X|$. Since $d_U(u) \leq i-1$ for every $u \in V(G)$, we have $|N_F(u) \cap B_i| \geq |B_i| - (i-1)$. By choosing $u \in X$ we obtain

$$|X| > |N_F(X) \cap B_i| \geq |N_F(u) \cap B_i| \geq |B_i| - (i-1)\,. \tag{7}$$

On the other hand, if $u \in B_i \setminus N_F(X)$, then

$$|A_i| - |X| \geq d_F(u) \geq |A_i| - (i-1)\,,$$

from where we get $|X| \leq i-1$, which, together with (7), gives $|B_i| < 2(i-1)$, a contradiction. □

As we conclude this paper, we raise the following question, which if true, would be a generalization of Theorem 18.

Question. Let k and s be positive integers with $s \leq k$, let $G = K_{2k}$, and let $(A_1, B_1), \ldots, (A_s, B_s)$ be pairs of sets of vertices of G such that for every $i \in \{1, \ldots, s\}$ we have $|A_i| = |B_i| \leq k$ and $A_i \cap B_i = \emptyset$. Does G contain edge-disjoint immersions $M_1, \ldots, M_s$ such that M_i is an immersion of an (A_i, B_i)–matching for $i \in \{1, \ldots, s\}$?

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