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Vector Fields in  $\mathbb{R}^2$  with Maximal Index

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# Vector fields in $\mathbb{R}^2$ with maximal index

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## Abstract

We consider the Poincaré's method to investigate the local index of vector fields in the plane. If  $m$  is the degree of the first non zero jet,  $X_m$ , of the vector field  $X$  at an isolated zero, we explore the geometry of the pencil generated by the coordinate functions of  $X_m$  when the absolute value of the index of  $X$ ,  $|\text{ind}(X)|$ , is  $m$ . We also find necessary and sufficient conditions for  $|\text{ind}(X)|$  to be  $m$ .

## Resumo

Usando o método de Poincaré investigamos o índice local de campos de vetores no plano. Se  $m$  é o grau do primeiro jato não nulo,  $X_m$ , do campo de vetor  $X$  em um zero isolado, exploramos a geometria do pencil gerado pelas funções coordenadas de  $X_m$  quando o valor absoluto do índice de  $X$ ,  $|\text{ind}(X)|$ , é  $m$ . Também encontramos condições necessárias e suficientes para  $|\text{ind}(X)|$  ser igual a  $m$ .

AMS Mathematical Subject Classifications: 58K45, 58K40.

## 1 Introduction

To any natural number  $m$ , there exist examples of vector fields  $X$  in  $\mathbb{R}^2$ , such that the absolute value of the index of  $X$ , at an isolated zero, is equal to  $m$ .

The aim of this paper is to give conditions to the index of a generic vector field, with  $m$ -jet distinct from zero, to be  $\pm m$ . Using the Poincaré's method we describe the geometry of these vector fields and also interpret this geometry in terms of the pencil generated by the  $m$ -jet of their coordinate functions.

Our first result, Theorem 3.1, was motivated by [1], in which A. Cima, A. Gasull and J. Torregrosa construct examples of vector fields in the plane with given multiplicities and indices. Let  $X = (P, Q)$  be a vector field in the plane with isolated zero at the origin,  $j^\infty X$  its Taylor series at the origin,  $j^\infty X = X_m + X_{m+1} + \dots$  where  $X_i = (P_i, Q_i)$  is the homogeneous part of degree  $i$  of  $X$  with  $X_m^{-1}(0) = \{0\}$ . We give necessary and sufficient conditions for the absolute value of the index of  $X$ ,  $|\text{ind}(X)|$ , to be  $m$ .

In our second result we study vector fields  $X$  for which  $X_m^{-1}\{0\} \neq \{0\}$ . Under mild genericity hypothesis on the pencil generated by  $X_m$ , we obtain necessary conditions for  $|\text{ind}(X)|$  to be  $m$ .

Our method is useful to classify vector fields with a fixed index and we explore this approach in the last section to obtain the classification of families of vector fields with maximal or minimal index, that is  $|\text{ind}(X)| = m$ .

## 2 Basic results

From the point of view of this paper, we can identify the germ of a  $C^\infty$  vector field in the plane with the  $C^\infty$  map germ  $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ . In this context, Mather's  $\mathcal{K}$ -equivalence is an useful equivalence relation because it preserves the absolute value of the index of  $f$ .

We say two map germs  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  are topologically  $\mathcal{K}$ -equivalent (or  $C^0$ - $\mathcal{K}$ -equivalent) if there exist germs of homeomorphisms  $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and  $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p, 0)$  with the property  $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$  such that the following diagram commutes:

$$\begin{array}{ccccc} (\mathbb{R}^n, 0) & \xrightarrow{(id., f)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\ \downarrow h & & \downarrow H & & \downarrow h \\ (\mathbb{R}^n, 0) & \xrightarrow{(id., g)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \end{array}$$

where  $id.$  means the identity mapping of  $\mathbb{R}^n$  and  $\pi_n$  is the canonical projection. This equivalence relation is a topological version of Mather's  $\mathcal{K}$ -equivalence [4]. A  $C^\infty$  map germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  is said to be finitely  $C^0$ - $\mathcal{K}$ -determined if there exists a positive number  $k$  such that for any  $C^\infty$  map germ  $g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  with  $j^k f(0) = j^k g(0)$ ,  $f$  is  $C^0$ - $\mathcal{K}$ -equivalent to  $g$ . Some characterizations of finite  $C^0$ - $\mathcal{K}$ -determinacy are already known (see [7] p. 518). Specially in the case  $n = p$ , if a  $C^\infty$  map germ  $X$  is finitely  $C^0$ - $\mathcal{K}$ -determined, then  $X^{-1}(0) = \{0\}$  as germs; therefore we can define the mapping degree  $\text{ind}(X)$  for a finitely  $C^0$ - $\mathcal{K}$ -determined endomorphism germ  $X : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ .

Nishimura (see [5]) proves the following result.

**Theorem 2.1** *Let  $X, Y : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  be two finitely  $C^0$ - $\mathcal{K}$ -determined map germs and  $n \neq 4$ . Then the following are equivalent:*

- (1)  $X$  and  $Y$  are topologically  $\mathcal{K}$ -equivalent,
- (2)  $|\text{ind}(X)| = |\text{ind}(Y)|$ .

To compute the index of a vector field in  $\mathbb{R}^2$ , one can use Poincaré's method. Let  $C$  be a simple closed curve in  $\mathbb{R}^2$ , let  $X$  be a vector field defined on a simply

connected open region of  $\mathbb{R}^2$  which contains the curve  $C$  and let  $r$  be some straight line in the  $(x, y)$ -plane. Suppose that there exist only finitely many points  $M_k$  (for  $k = 1, 2, \dots, n$ ) on  $C$  at which the vector  $X(M_k)$  is parallel to  $r$ . Let  $M$  be a point describing the curve in the counterclockwise sense, and let  $p$  (respectively  $q$ ) be the number of points of  $M_k$  at which the vector  $X(M)$  passes through the direction of  $r$  in the counterclockwise (respectively clockwise) sense. Points  $M_k$  at which the vector field  $X(M)$  assumes the direction of  $r$  while moving, say, in the clockwise sense and then begins to move in the opposite direction (or vice versa) are not counted. Then the index of  $C$ ,  $i(C)$ , is defined by  $i(C) = (p - q)/2$ . If we have a zero  $M$  of  $X$ , we define the index  $ind_M(X)$  of  $X$  at  $M$  by  $ind_M(X) = i(C)$ , where  $C$  is a simple closed curve on which there are no zeros of  $X$  and which is such that it surrounds only the point  $M$ .

We also use the following result of Gutierrez and Ruas, which shows that the index of a Newton non-degenerate vector field, in the sense of [2], is determined by its principal part (see [2]).

**Theorem 2.2** *Let  $X = (P, Q) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  be the germ of a smooth vector field in the plane. Let  $X_\Gamma = (P|_{\Gamma(f)}, Q|_{\Gamma(g)})$  be the principal part of the vector field  $X$ . If  $X$  is Newton non-degenerate, then, there exists a neighborhood  $U$  of  $(0, 0)$  such that,  $\forall (x, y) \in U \setminus \{(0, 0)\}$  and  $\forall s \in [0, 1]$ ,*

$$0 < |sX(x, y) + (1 - s)X_\Gamma(x, y)|$$

*In particular  $(0, 0)$  is an isolated singularity of both  $X$  and  $X_\Gamma$  and*

$$\text{ind}(X) = \text{ind}(X_\Gamma)$$

### 3 Main results

Let  $X = (P, Q)$  be the germ of a  $C^\infty$  vector field at the origin such  $X$  is finitely  $C^0$ - $\mathcal{K}$ -determined,  $j^\infty X$  its Taylor series at the origin,  $j^\infty X = X_m + X_{m+1} + \dots$  where  $X_i = (P_i, Q_i)$  is the homogeneous part of degree  $i$  of the Taylor expansion of  $X$ . We want to find conditions on  $X$  such that  $|\text{ind}(X)| = m$ , based on the geometry of the pencil generated by the homogeneous polynomials  $P_m$  and  $Q_m$ .

Let  $H^m(2, 1)$  be the space of the real homogeneous polynomials  $a_0x^m + a_1x^{m-1}y + \dots + a_my^m$ . We denote by  $\Omega_m$  the open set of  $H^m(2, 1)$ , consisting of polynomials with  $m$  distinct (real or complex) roots. Then  $\Omega_m = \Omega_{\{m,0\}} \cup \Omega_{\{m,1\}} \cup \dots \cup \Omega_{\{m,m\}}$  where  $\Omega_{\{m,i\}}$  is the set of the elements in  $\Omega_m$  with  $i$  distinct real roots. We denote the  $m$ -jet of  $X$  by  $X_m$ , which we think as a pencil  $\mathcal{L} = \{\alpha P_m + \beta Q_m, \alpha, \beta \in \mathbb{R}\}$  in  $\mathcal{P}^m$ , the  $m - 1$ -real projective space, or alternatively its complexification,  $\mathcal{L}_\mathbb{C} =$

$\{\alpha P_m + \beta Q_m, \alpha, \beta \in \mathbb{C}\}$  in the complex projective space  $\mathcal{P}_{\mathbb{C}}^m$ . The non-degenerate pencil  $\mathcal{L}_{\mathbb{C}}$  is thought as a line in  $\mathcal{P}_{\mathbb{C}}^m$ . The discriminant variety of forms with repeated roots is a hypersurface  $\Delta_{\mathbb{C}}$  with singular locus that we denote by  $C_{\mathbb{C}}$  (we denote the corresponding sets over the real by  $\Delta$  and  $C$ ). The obvious invariant of the pencil is the way in which the line  $\mathcal{L}_{\mathbb{C}}$  meets  $\Delta_{\mathbb{C}}$  and  $C_{\mathbb{C}}$ .

Under generic conditions, the pencil intercepts  $\Omega_{\{m,m\}}$ . In fact, if  $\mathcal{L} \cap \Omega_m = \emptyset$  then  $\mathcal{L} \subset \Delta$  and the pencil is singular. We will prove in Corollary 4.3 that if  $\mathcal{L} \cap \Omega_{\{m,j\}} \neq \emptyset$ , then  $|\text{ind}(X)| \leq j$ . Since we are interested in vector fields  $X$  such that  $|\text{ind}(X)| = m$  then, the generic situation  $\mathcal{L} \cap \Omega_m \neq \emptyset$  implies that

$$\mathcal{L} \cap \Omega_{\{m,m\}} \neq \emptyset \quad (*)$$

In this work, we give necessary and sufficient conditions for  $|\text{ind}(X)|$  to be  $m$ , assuming that condition  $(*)$  holds. The main results are:

**Theorem 3.1** *Let us assume that  $X_m^{-1}(0) = \{0\}$ . Then the following conditions are equivalent*

- (i)  $|\text{ind}(X)| = m$
- (ii)  $\mathcal{L} \subset \Omega_{\{m,m\}}$
- (iii) *the roots of any linearly independent pair  $P_m$  and  $Q_m$ , of elements of  $\mathcal{L}$ , define  $2m$  distinct alternating branches.*

If  $X_m^{-1}(0) \neq \{0\}$  then  $P_m, Q_m$  have a common factor  $c(x, y)$ , that is  $X_m = c(x, y).(p, q)$ ,  $\text{degree}(c(x, y)) = d$  with  $1 \leq d \leq m$ . If  $d = m$ ,  $\mathcal{L}$  is degenerate that is  $\mathcal{L}$  reduces to a point. Otherwise,  $1 \leq d \leq m - 1$ , and the hypothesis  $|\text{ind}(X)| = m$  imposes geometric conditions on the geometry of the pencil  $\mathcal{L}$ . In this case we prove that:

**Theorem 3.2** *Let  $X$  be finitely  $C^0$ - $\mathcal{K}$ -determined,  $X_m = c(x, y).(p, q)$  with  $p^{-1}(0) \cap q^{-1}(0) = \{0\}$ ,  $\text{degree}(c(x, y)) = d$ . If  $\mathcal{L} \cap \Omega_m \neq \emptyset$  and  $|\text{ind}(X)| = m$  then  $\mathcal{L}$  meets  $\Delta$  at exactly  $d$  points and  $\mathcal{L} \cap C = \emptyset$ .*

The condition  $|\text{ind}(X)| = m$  imposes restrictions on the maximum degree  $d$  of  $c(x, y)$  when  $d \geq 3$  (see Proposition 5.6).

We give in Propositions 5.3 and 5.4 a complete characterization of vector fields with  $|\text{ind}(X)| = m$ , when  $d = 1$  and  $d = 2$ .

**Remark 3.1** *The hypothesis  $\mathcal{L} \cap \Omega_{\{m,m\}} \neq \emptyset$  is not a necessary condition for  $|\text{ind}(X)| = m$ . For instance, let  $X = (x^2 - xy^8 - xy^4 + y^{12}, x^2 - xy^3 - xy^6 - y^9)$ . The pencil associated to  $X_2$  is degenerate, but the index of  $X$  is equal to  $-2$ .*

## 4 Proof of the main results

The following Lemma is a direct consequence of the Poincaré's method.

**Lemma 4.1** *Let  $X = (P, Q)$  be a finitely  $C^0$ - $\mathcal{K}$ -determined germ at the origin,  $j^\infty X = X_m + X_{m+1} + \dots$  with  $X_m \neq 0$ . The absolute value of the index of  $X$  is maximal,  $|\text{ind}(X)| = m$ , if and only if the following holds:*

- (i) *each of the zero sets of  $P(x, y)$  and  $Q(x, y)$  decomposes into  $2m$  semi-algebraic branches, and the complement of the set  $P^{-1}(0) \cup Q^{-1}(0)$  in a sufficiently small ball  $B_\epsilon(0)$  centered at 0, is a union of  $4m$  sectors,*
- (ii) *the boundary of each sector in  $B_\epsilon(0)$  is the union of one half-branch of  $P$  and one half-branch of  $Q$ ,*
- (iii) *the sign of  $P \cdot Q$  alternates on each pair of consecutive sectors.*

**Corollary 4.2** *Let  $X : \mathbb{R}^2, 0 \rightarrow \mathbb{R}^2, 0$  be finitely  $C^0$ - $\mathcal{K}$ -determined at the origin, then*

- (i) *if  $j^m X \neq (0, 0)$  and  $j^l X = (0, 0)$  for all  $l < m$  then  $|\text{ind}(X)| \leq m$ ;*
- (ii) *if  $|\text{ind}(X)| = m$  then  $j^l X = 0$  for all  $l < m$ .*

**Corollary 4.3** *Let  $\mathcal{L}$  be the pencil  $\alpha P_m + \beta Q_m$ , with  $\alpha, \beta \in \mathbb{R}$ . If there exist  $j$  such that  $\mathcal{L} \cap \Omega_{\{m, j\}} \neq \emptyset$ ,  $0 \leq j \leq m$ , then  $|\text{ind}(X)| \leq j$ .*

**Proof:** If  $\mathcal{L} \cap \Omega_{\{m, j\}} \neq \emptyset$  then  $X$  is  $\mathcal{K}$ -equivalent to  $(P_m, Q_m + R(x, y))$  where  $P_m \in \Omega_{\{m, j\}}$  and  $j^m R(0, 0) = 0$ . Now the result follows from the Lemma 4.1.

Let  $Y(x, y) = (\pm x \prod_{i=1}^{m-1} (y - p_i x), \prod_{i=1}^m (y - q_i x))$  where  $p_1 < p_2 < \dots < p_{m-1}$ ,  $q_1 < q_2 < \dots < q_m$ ,  $q_i \neq p_j \forall i, j$ .

The proof of Theorem 3.1 follows from the following two results. We observe that (iii)  $\Rightarrow$  (i) in Theorem 3.1 was previously given in [3].

**Lemma 4.4** *The absolute value of the index of  $Y$  is  $m$  if and only if  $q_1 < p_1 < q_2 < p_2 < \dots < p_{m-1} < q_m$ .*

**Proof:** The result follows from Lemma 4.1.

**Lemma 4.5** *Let us suppose  $X_m^{-1}(0) = \{0\}$ . Then  $|\text{ind}(X)| = m$  implies that  $X$  is  $C^0$ - $\mathcal{K}$ -equivalent to  $Y$ .*

**Proof:** If  $X_m^{-1}(0) = \{0\}$  then  $\text{ind}(X) = \text{ind}(X_m)$  ( see [1] or [2]) and by Nishimura's result  $X$  is  $C^0$ - $\mathcal{K}$ -equivalent to  $X_m$ . How  $|\text{ind}(X)| = m$  then  $X_m$  is equivalent to  $Y$  by Lemma 4.1.

**Proposition 4.6** *Let us suppose  $X_m^{-1}(0) = \{0\}$ . If there exists a linearly independent pair of elements of the pencil,  $P_m$  and  $Q_m \in \Omega_{\{m, m\}}$ , such that their  $2m$  roots, defining  $2m$  distinct branches, do not alternate, then  $\mathcal{L} \cap \Delta \neq \emptyset$ .*

**Proof:** With a convenient choice of affine coordinates, we can write the pencil in the form  $(y - \lambda_1(t)x) \cdots (y - \lambda_m(t)x)$ , with  $t \in \mathbb{R}$  and  $\lambda_i(t)$  continuous for any  $i$ . If  $X_m^{-1}(0) = \{0\}$ , then

- (i)  $\forall i = 1, \dots, m$ ,  $\lambda_i(t)$  is injective, consequently it is monotonic;
- (ii)  $Im(\lambda_i) \cap Im(\lambda_j) = \emptyset, \forall i \neq j$ .

Let  $m = 2$ . If the roots  $p_i = \lambda_i(0)$  and  $q_i = \lambda_i(1)$ ,  $i = 1, 2$ , do not alternate, then there is  $t_0$  such that  $\lambda_1(t_0) = \lambda_2(t_0)$ , that is  $Im(\lambda_1) \cap Im(\lambda_2) \neq \emptyset$ . This means that the pencil is not contained in  $\Omega_2$ .

For any  $m$ , if the pencil generated by  $P_m$  and  $Q_m$  is contained in  $\Omega_m$  and the  $2m$  roots of  $P_m$  and  $Q_m$  do not alternate, then there exist  $\lambda_i(t)$  and  $\lambda_{i+1}(t)$  such that the pencil  $(y - \lambda_i(t)x)(y - \lambda_{i+1}(t)x)$  is not contained in  $\Omega_2$ , and this is a contradiction.

The following result follows as a consequence of the Theorems 3.1 and 2.1.

**Corollary 4.7** *The vector fields  $X$  such that  $j^\infty X = X_m + X_{m+1} + \cdots$ ,  $X_m$  has isolated zero and  $\mathcal{L} \subset \Omega_{\{m,m\}}$ , form an unique  $C^0 - \mathcal{K}$ -orbit.*

We now assume that  $X$  is finitely  $C^0$ - $\mathcal{K}$ -determined,  $X_m^{-1}(0) \neq \{0\}$  and the pencil  $\mathcal{L}$ ,  $\alpha P_m + \beta Q_m$ , does not reduce to a point. Then  $P_m, Q_m$  have a common factor  $c(x, y)$ , that is  $X_m = c(x, y) \cdot (p, q)$  with  $p^{-1}(0) \cap q^{-1}(0) = \{0\}$  (in others words, the pencil  $\alpha P_m + \beta Q_m$  is of the form  $c(x, y)(\alpha p(x, y) + \beta q(x, y))$ ). The condition (\*) implies that  $c(x, y)$  is non-degenerate, in the sense that it is a homogeneous polynomial of degree  $d$ , with  $1 \leq d \leq m-1$ , having  $d$  distinct real roots and  $degree(p) = degree(q) = m - d$ .

The proof of the Theorem 3.2 follows from Lemmas 4.8, 4.9 and 4.10 bellow.

**Lemma 4.8** *If the pencil  $\mathcal{L}$  is of the form  $\alpha P_m + \beta Q_m = c(x, y)(\alpha p(x, y) + \beta q(x, y))$ ,  $c(x, y)$  has at least one linear factor and  $degree(c) = d$ ,  $1 \leq d \leq m-1$ , then  $\mathcal{L} \cap \Delta \neq \emptyset$ .*

**Proof:** We can change coordinates to obtain  $c(x, y) = x\bar{c}(x, y)$ ,  $p(x, y) = a_0x^{m-d} + \cdots + a_{m-d}y^{m-d}$  and  $q(x, y) = b_0x^{m-d} + \cdots + b_{m-d}y^{m-d}$ . If  $a_{m-d} = 0$ , then  $x$  is a factor of multiplicity 2 of  $P_m = cp$ , hence the result follows. If  $a_{m-d} \neq 0$ , by a change of coordinates we can eliminate the term  $c(x, y)b_{m-d}y^{m-d}$  of the second coordinate function, and this implies that  $x$  is a factor of multiplicity 2 of  $Q_m$ .

Notice that the Lemma 4.8 applies to any linear factor of  $c(x, y)$ , then we can state the following:

**Lemma 4.9** *If the pencil is of the form  $c(x, y)(\alpha p(x, y) + \beta q(x, y))$ , where  $c(x, y) \in \Omega_{\{d,d\}}$ ,  $1 \leq d \leq m-1$ , then  $\mathcal{L}$  intercepts  $\Delta$  in at least  $d$  points.*

Let  $X_m^{-1}(0) \neq \{0\}$ ,  $|\text{ind}(X)| = m$  and the pencil intercepts  $\Omega_{\{m,m\}}$  at  $P_m$ . Then  $X$  is  $C^0 - \mathcal{K}$ -equivalent to  $(P_m, Q_m + R(x, y))$ ,  $j^m R(0, 0) = 0$  and  $\mathcal{L} \cap \Delta \neq \emptyset$ . That is  $X_m$  is  $C^0 - \mathcal{K}$ -equivalent to  $c(x, y)(p(x, y), q(x, y))$ ,  $\text{degree}(c(x, y)) = d$ , where one of the roots of  $q$  may coincide with some root of  $c(x, y)$  by the proof of Lemma 4.8. With the following Lemma we conclude that the pencil intercepts  $\Delta$  on exactly  $d$  points and  $\forall \alpha, \beta, \alpha P_m + \beta Q_m \notin \mathcal{C}$ .

**Lemma 4.10** *With the above conditions  $\forall \alpha, \beta, \alpha p(x, y) + \beta q(x, y) \in \Omega_{\{m-d, m-d\}}$ .*

**Proof:** Since  $|\text{ind}(X)| = m$ , the result is a consequence of the Poincaré's method.

## 5 Classification

In this section we classify finitely  $C^0$ - $\mathcal{K}$ -determined vector fields with  $|\text{ind}(X)| = m$  under the hypothesis that  $\mathcal{L} \cap \Omega_m \neq \emptyset$  and  $X_m^{-1}(0) \neq \{0\}$ . This means that  $\mathcal{L} \cap \Omega_{\{m,m\}} \neq \emptyset$  and  $\alpha P_m + \beta Q_m = c(x, y)(\alpha p(x, y) + \beta q(x, y))$ , with  $\text{degree}(c(x, y)) = d$ ,  $p^{-1}(0) \cap q^{-1}(0) = \{0\}$ ,  $1 \leq d \leq m$ .

As in the previous section, the hypothesis  $\mathcal{L} \cap \Omega_{\{m,m\}} \neq \emptyset$  implies that the vector field  $X = (P, Q)$  is equivalent to  $(P_m, Q_m + R)$ , where  $P_m \in \Omega_{\{m,m\}}$  and  $j^m R(0, 0) = 0$ . In what follows, we use the symbol  $\simeq$  to indicate the  $C^0 - \mathcal{K}$ -equivalence.

We have the following as pre-normal forms for the vector fields  $X$  which satisfy these conditions.

(Notice that if  $m = 1$ ,  $X_m^{-1}(0) \neq \{0\}$  implies that  $m = d = 1$ .)

**Lemma 5.1** *Assume  $|\text{ind}(X)| = m$ .*

(1) *Let  $m \geq 2$  and  $d \leq m - 1$ :*

(i) *If  $d = 1$ , then  $X \simeq (xp(x, y), x^2q(x, y) + \epsilon y^k)$ ,  $\text{degree } p = m - 1$ ,  $\text{degree } q = m - 2$ , with  $k > m$ .*

(ii) *If  $d = 2$ , then  $X \simeq (xyp(x, y), x^2yq(x, y) + \epsilon_1 x^l + \epsilon_2 y^k)$ ,  $\text{degree } p = m - 2$ ,  $\text{degree } q = m - 3$ , with  $\epsilon_1 \epsilon_2 \neq 0$  where  $l > m$  and  $k > m$ .*

(iii) *If  $d \geq 3$ , then there are two possibilities:*

$$1. X \simeq (xy \prod_{i=1}^{d-2} (y - r_i x) p(x, y), x^2 y \prod_{i=1}^{d-2} (y - r_i x) q(x, y) + \epsilon_1 x^l + \epsilon_2 y^k), \quad l > m, \\ k > m, \text{ or}$$

$$2. X \simeq (xy \prod_{i=1}^{d-2} (y - r_i x) p(x, y), x^2 y \prod_{i=1}^{d-2} (y - r_i x) q(x, y) + x^2 y \alpha(x, y) + \epsilon_1 x^l + \\ \epsilon_2 y^k), \quad \text{order}(x^2 y \alpha(x, y)) = n > m, \quad n \leq l \text{ and } n \leq k.$$

where  $q(x, y)$  has  $m - d - 1$  simple roots distinct from  $x$  and  $y$ , and  $\text{degree}(q(x, y)) = m - d - 1$ .

(2) If  $d = m \geq 1$ , the normal forms are as above, taking  $q(x, y) = 0$ . We denote them respectively by (i'), (ii') and (iii')1. In this case, (iii')1. does not occur.

**Proof:** When  $m \geq 2$ ,  $d \leq m - 1$ , by the proof of the Lemma 4.8, we can take  $X_m = (xp(x, y), x^2\bar{q}(x, y))$ ,  $X \simeq X_m + (0, R(x, y))$ . Let  $\Gamma_+(X)$  be the Newton polyhedra of  $X$  (see [2]). In the cases (i) and (ii), we can see that the principal part  $X_\Gamma$  of  $X$ , with respect to the Newton filtration coincides with the given normal forms. Furthermore,  $X$  is Newton non-degenerate in the sense of [2]. Hence it follows from the Theorem 2.2 of [2] that  $\text{ind}(X) = \text{ind}(X_\Gamma)$ . In case (iii),  $X$  is always Newton degenerate. With simple coordinate changes we can reduce  $R(x, y)$  to the following form:  $R(x, y) = \epsilon_1 x^l + \epsilon_2 y^k + x^2 y \alpha(x, y)$ ,  $k, l > m$ . There are two cases to be considered:

1.  $\text{order}(x^2 y \alpha) > \min\{l, k\}$ ,
2.  $\text{order}(x^2 y \alpha) \leq \min\{l, k\}$ .

Using Lemma 4.1 and Nishimura's result, we obtain that conditions 1 and 2 lead respectively, to the normal form 1. and 2. of (iii).

In case  $d = m$ , we can take  $X_m = (xp(x, y), xp(x, y))$ , then  $X \simeq (xp(x, y), R(x, y))$ ,  $j^m R(0, 0) = 0$ . If  $d = 1$  ( $d = 2$ ) by a change of coordinates we can eliminate the term  $x\psi(x, y)$  (resp.  $xy\psi(x, y)$ ) of  $R(x, y)$ . If  $d \geq 3$  the proof is analogous as (1).(iii) but we observe that since  $(P_m, Q_m + R)$  is equivalent to  $(P_m, R)$ , the condition  $d \geq 3$  does not imply that  $X$  is always Newton degenerate, this can occur only when  $l = k$  and  $n \leq l$ . The normal form (iii')1. does not occur, since we can use the Lemma 4.1 to show that the absolute value of the index is strictly less than  $m$ .

**Remark 5.1** 1. Given a vector field of types (i), (ii) and (iii)2., for all  $d \leq m - 1$ , we can choose  $l, k, \epsilon_1, \epsilon_2$  and  $\alpha$  such that  $|\text{ind}(X)| = m$ . This does not follow for vector fields of type (iii)1. In fact, in this case there are values of  $d$  for which  $|\text{ind}(X)| < m, \forall \epsilon_1, \epsilon_2, l, m$  (see Proposition 5.6).

2. If  $m = d$ , the normal forms are (i'), (ii') and (iii')2. and for any  $m$  we can choose  $l, k, \epsilon_1, \epsilon_2$  and  $\alpha$  such that  $|\text{ind}(X)| = m$ .

In what follows we obtain necessary and sufficient conditions for  $|\text{ind}(X)| = m$  in the cases  $d$  equal to 1 or 2. Our method applies to the cases  $d \geq 3$ , but the statement of the general results become too technical. In Lemma 5.5 and Proposition 5.6 we give some partial information for this case.

## 5.1 Case $d = 1$

In this case, if  $|\text{ind}(X)| = m$ , the normal forms of  $X$  are given by (i) and (i') of the Lemma 5.1 and we have the following Proposition that extends a result of A. Cima, A. Gasull and J. Torregrosa, [1], and give a method to construct vector fields with index equal to  $\pm m$ .

**Proposition 5.2** (1) *If  $X \simeq (xp(x, y), x^2q(x, y) + \epsilon y^k)$  with  $k > m \geq 2$  then  $|\text{ind}(X)| = m$  if and only if the following holds:*

- (i)  $p(x, y)$  has  $m - 1$  distinct real branches, that is  $p(x, y) = \prod_{i=1}^{m-1} (y - p_i x)$ ,  $p_1 < p_2 < \dots < p_{m-1}$ ,
- (ii)  $q(x, y)$  has  $m - 2$  distinct real branches, that is  $q(x, y) = \prod_{i=1}^{m-2} (y - q_i x)$ ,  $q_1 < q_2 < \dots < q_{m-2}$ ,
- (iii)  $m \equiv k \pmod{2}$ ,  $\epsilon < 0$  and  $p_1 < q_1 < p_2 < \dots < p_{m-2} < q_{m-2} < p_{m-1}$ .

(2) *When  $m = d = 1$ ,  $X \simeq (x, \epsilon y^k)$ ,  $k > 1$  and  $|\text{ind}(X)| = 1$  if and only if  $k \equiv 1 \pmod{2}$ .*

Let  $B_\epsilon$  be a small open ball centered at 0. We denote by  $S_1$  (respectively  $S_2$ ) the sector in  $B_\epsilon$  bounded by the two consecutive half branches of  $p$ , containing one of the half branches  $\bar{c}_1$  (resp.  $\bar{c}_2$ ) of  $c(x, y) = 0$ , see Figure 1. Denote by  $S_{c_q}^i$  the closed subsector of  $S_i$  bounded by  $\bar{c}_i$  and the unique half branch of  $q$  inside  $S_i$ , ( we can change coordinates, as in Lemma 4.8, in such way that the branches  $q$  and  $c$  coincide. In the normal forms in Lemma 5.1 both are taken to be equal to  $x$  and in this case  $S_{c_q}^i$  is the half branch  $\bar{c}_i$ ,  $i = 1, 2$ ). The notation  $F|_{\bar{c}_i}$  means  $F(x, y)$  restricted to  $\bar{c}_i$  and  $S^i \setminus_{c_q}$  indicates the complement of  $S_{c_q}^i$  in  $S_i$ .

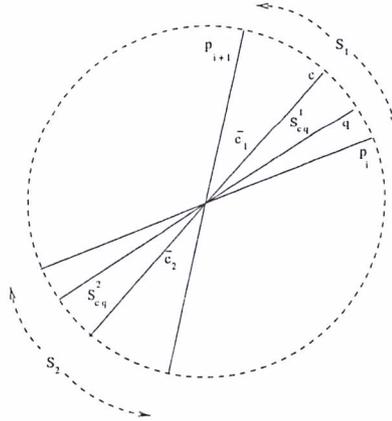


Figure 1: Sectors and subsectors

The Proposition 5.3 follows from the Proposition 5.2 and Lemma 4.1.

**Proposition 5.3** (1) If  $X \simeq (P_m, Q)$ ,  $m \geq 2$ ,  $P_m = cp$ ,  $Q_m = cq$ ,  $c(x, y)$  linear,  $Q|_{c(x,y)} = R(x, y)$  with  $j^m R(0, 0) = 0$ . Then  $|\text{ind}(X)| = m$  if and only if  $\alpha p(x, y) + \beta q(x, y) \in \Omega_{\{m-1, m-1\}}$ ,  $\forall \alpha, \beta$ , and the sign of  $Q_m$  in  $S^i \setminus_{c_q}$  is the opposite of the sign of  $R(x, y)|_{\bar{c}_i}$ , for each  $i = 1, 2$ .

(2) In case  $m=1$ ,  $X \simeq (c, R)$  then  $|\text{ind}(X)| = m$  if and only if the signs of  $R(x, y)|_{\bar{c}_1}$  and  $R(x, y)|_{\bar{c}_2}$  are opposite.

## 5.2 Cases $d = 2$ and $d \geq 3$

In this case, if  $|\text{ind}(X)| = m$ , the normal forms of  $X$  are given by (ii), (iii), (ii') and (iii')2. of the Lemma 5.1. The results are similar to that in 5.1.

Let  $c(x, y) = c_1(x, y)c_2(x, y)$ . Denote respectively by  $\bar{c}_{1i}$  and  $\bar{c}_{2i}$  the half branches given by  $c_1(x, y) = 0$  and  $c_2(x, y) = 0$ . They define four sectors  $C_{1i}$  and  $C_{2i}$ ,  $i = 1, 2$ , such that  $C_j = C_{j1} \cup C_{j2}$  has the boundary given by  $c_1$  and  $c_2$ . Let  $m - 2 = m_1 + m_2$  with  $m_1$  and  $m_2$  being the number of branches of  $p(x, y)$ , respectively, in  $C_1$  and  $C_2$ . As in the case  $d = 1$ , we denote  $S_{ij}$  the sector containing the half branch  $\bar{c}_{ij}$ , and  $S^{ij} \setminus_{c_i q}$  the complement of  $S_{c_i q}^{ij}$  in  $S_{ij}$ ,  $i, j \in \{1, 2\}$ . We have the following result:

**Proposition 5.4** (1) If  $d = 2$ ,  $m > 2$ ,  $X \simeq (P_m, Q) = (c(x, y)p(x, y), c(x, y)q(x, y) + R(x, y))$ ,  $c = c_1 c_2$ ,  $c_i$  linear factors and  $j^m R(0, 0) = (0, 0)$ . The necessary and sufficient conditions for  $|\text{ind}(X)|$  to be  $m$  are

(i)  $\alpha p(x, y) + \beta q(x, y) \in \Omega_{\{m-2, m-2\}}$ ,  $\forall \alpha, \beta$ ;

(ii) the sign of  $R(x, y)|_{\bar{c}_{1i}}$  is the opposite of the sign of  $Q_m$  in the sector  $S^{1i} \setminus_{c_1 q}$ , for each  $i = 1, 2$ ;

(iii) If  $m_i$  is even (odd) then the sign of  $R(x, y)|_{\bar{c}_{2j}}$  is the opposite (same) as the sign of  $R(x, y)|_{\bar{c}_{1k}}$ , where  $\bar{c}_{2j}$  and  $\bar{c}_{1k}$ ,  $j, k \in \{1, 2\}$ , are boundary of  $C_{il}$ , for each  $l = 1, 2$ ,  $i = 1, 2$ .

(2) When  $m = d = 2$ ,  $X \simeq (c, R)$ ,  $j^m R(0, 0) = 0$  and  $|\text{ind}(X)| = 2$  if and only if the sign of  $R(x, y)|_{\bar{c}_{1i}}$  and  $R(x, y)|_{\bar{c}_{2j}}$ , for each  $i, j \in \{1, 2\}$ , are opposite.

The next result holds for any  $d \geq 2$ .

**Lemma 5.5** Let  $X$  be a vector field as in Lemma 5.1, with  $d \geq 2$ . If  $|\text{ind}(X)| = m$  then  $l, k$  and  $m$  are congruent module 2.

We can always describe necessary and sufficient conditions to construct  $X$  with  $j^\infty X = X_m + \dots$  and  $|\text{ind}(X)| = m$  if the condition (\*) holds. When  $\text{degree}(c) = d$  with  $3 \leq d \leq m$ , the conditions are very technical, but the following result is useful

to get a better version of the Theorem 3.2 when the normal form is (iii)1. of Lemma 5.1.

**Proposition 5.6** *Let  $X$  be the pre-normal form (iii) 1. of the Lemma 5.1,  $3 \leq d \leq m - 1$ . Then  $|\text{ind}(X)| < m$  when one of the following conditions hold:*

(i)  $l, k, m$  even,  $\epsilon_1 \epsilon_2 < 0$ , and  $d > \frac{m+2}{2}$ ; or

(ii)  $l, k, m$  even,  $\epsilon_1 \epsilon_2 > 0$ , and  $d > \frac{m}{2}$ ; or

(iii)  $l, k, m$  odd, and  $d > \frac{m+1}{2}$ ;

*Otherwise, taking into account the conditions in Lemma 5.5, to all values of  $d$  not satisfying the above condition, we can construct vector fields  $X$  with  $|\text{ind}(X)| = m$ .*

For the other normal forms of the Lemma 5.1, given  $d$  with  $1 \leq d \leq m - 1$ , we can always construct vector fields  $X$  such that  $|\text{ind}(X)| = m$  where  $\sharp(\mathcal{L} \cap \Delta) = d$ . When  $d = m \geq 3$ ,  $\mathcal{L}$  reduces to a point (degenerated pencil), the normal form of  $X$  is (iii')2. and with appropriated choice of  $\alpha(x, y)$  we can always get  $|\text{ind}(X)| = m$ . Again, our method applies in this case but the conditions are very technical.

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