nº 13

Group rings whose torsian units form a subgroup II

Cesar Polcino Milies

Fevereiro 1981

GROUP RINGS WHOSE TORSION UNITS FORM A SUBGROUP II

César Polcino Milies*
Instituto de Matemática e Estatística
Universidade de São Paulo
05508 - São Paulo - Brazil

§1 - INTRODUCTION

Let R be a ring with identity. We denote by U(R) the unit group of R and by TU(R) the set of elements of finite order in this group. We shall say, for briefness, that R has the t.p.p. (*torsion product property*) whenever TU(R) is closed under multiplication. We follow the notations in [7].

S.K. Sehgal and H.J. Zassenhaus have determined the classes of all groups G such that U(ℤG) is a nilpotent or FC group in [8] and [9] respectively. If restricted to finite groups, it is easy to see that both classes coincide.

M.M. Parmenter and the author have shown in [2] that, in the finite case, the characterization of such groups G follows from the fact that, in both cases, TU(ℤG) forms a subgroup. It was shown in [3] that some other properties that groups in these classes have in common, when G is infinite, again follow from

* This work was written while the author was on leave at Memorial University of Newfoundland on an exchange program by NSERC (Canada) and CNPq (Brazil).

the fact that $\mathbb{Z}G$ has the t.p.p. in both cases. It also comes that if G is either nilpotent or FC this fact is equivalent to the absence of nilpotent elements.

When working with group rings over a field K it is still the case that groups G should have similar properties when U(KG) is either a nilpotent or an FC group.

In this paper we determine groups G that are either nilpotent or FC and such that, for a field K, the group ring KG has the t.p.p.

We shall discuss separately the cases where char(K) = p>0 and char(K) = 0 and apply the methods of the second case to group rings over a class of rings that includes p-adic integers.

§2 - DIVISION RINGS AND MATRIX RINGS WITH THE T.P.P.

The results in this section will be needed in the sequel.

PROPOSITION 2.1 - Let D be a division ring with char(D) = p>0. Then D has the t.p.p. if and only if all the elements in TU(D) are central in D. In this case $D_1 = TU(D) \cup \{0\}$ is a central subfield of D.

PROOF - Assume that D has the t.p.p. We first claim that if $a, b \in D_1$ then $a+b \in D_1$.

To see this, denote by P the prime subfield of D. Since $a^{-1}b$ is an element of finite order, $P(a^{-1}b)$ is a finite field; hence, there exists a positive interger n such that $(a^{-1}b)^{p^n} = a^{-1}b$.

Then, we also have that $(1+a^{-1}b)^{p^n} = 1+a^{-1}b$, showing that either $1+a^{-1}b = 0$ or $o(1+a^{-1}b)$ is finite. In both cases $1+a^{-1}b \in D_1$.

Consequently, $a+b = a(1+a^{-1}b) \in D_1$.

The argument above shows that D_1 is a subdivision ring of D and it is easy to see that it is invariant under conjugations. The Brauer-Cartan-Hua theorem [5,14.1.3] shows that either D_1 is central in D or $D_1 = D$.

Now, if $D_1 = D$ then D is a division ring whose multiplicative group is torsion and it follows from [5,14.1.6] that D must actually be a field.

In both cases, TU(D) is central and the converse follows trivially. □

PROPOSITION 2.2 - Let D be a division ring with $char(D) = p>0$. For an integer $n > 1$ the full matrix ring $M_n(D)$ has the t.p.p. if and only if D is a field which is an algebraic extension of its prime field P. If this is the case, then all elements in $GL(n,D)$ are of finite order.

PROOF - First assume that $GL(m,D)$ has the t.p.p. and suppose that D contains an element X which is transcendental over P.

Set:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} X+1 & -X \\ X & -X+1 \end{bmatrix}.$$

Then $A, B \in GL(2,P(X))$ and straightforward calculation shows that $A^p = B^p = I$. We shall show that AB is not of finite order.

In fact, we have that.

$$AB = \begin{bmatrix} X+1 & -X \\ -1 & 1 \end{bmatrix}$$

and an easy induction argument shows that $(AB)^m$ has the form:

$$(AB)^m = \begin{bmatrix} f_{11} & -f_{12} \\ -f_{21} & f_{22} \end{bmatrix}$$

where f_{ij} are monic polynomials with $\deg(f_{1j}) = m$ and $\deg(f_{2j}) = m-1$. Hence $(AB)^m \neq I$ for all m.

Since $GL(2,P(X))$ can be embedded in $G(n,D)$ for arbitrary $n > 1$ we get a contradiction and necessity follows.

Now assume that D is an algebraic extension of P. Then all elements in D are of finite order and [5,14.1.6] shows again that D is a field. Finally, given any matrix $A \in GL(n,D)$, if we denote by F the finite field that is obtained by adjoining to P all entries in A, we see that $A \in GL(n,F)$ which is a finite group. Hence, sufficiency also follows. □

§3 - LOCALLY FINITE GROUPS

We shall first deal separately with locally finite groups because, in this case, when $char(K) = p > 0$ the nature of the field will have to be taken into consideration.

PROPOSITION 3.1 - Let K be a field with $char(K) = p > 0$ which is an algebraic extension of its prime field P. Then, for any locally finite group G the group ring KG has the t.p.p.

PROOF - Let $u=\sum u(g)g$, $v=\sum v(g)g$ be elements of finite order. Then $H=\langle supp(u)\cup supp(v)\rangle$ is a finite group. Also, if F denotes the extension of the prime field P of K obtained by adjoining to P all elements in the set $\{u(g)\}_{g\in G}\cup\{v(g)\}_{g\in G}$, then F is a finite field.

Hence, FH is a finite ring and $uv\in U(FG)$. Consequently, uv has finite order. □

If K is not algebraic over its prime field, we have to consider separately two different cases, depending on the existence of elements of order p in G.

PROPOSITION 3.2 - Let K be a field with $char(K)=p>0$ which is not algebraic over its prime field K and G a locally finite group containing no elements of order p. Then KG has the t.p.p. if and only if G is abelian.

PROOF - As in the proposition above, we can assume that G is finite. Since $p\nmid |G|$ we can write:

$$KG \cong \bigoplus_{i=1}^{t} M_{n_i}(D_i)$$

where the D_i are division rings containing K, $1\le i\le t$.

Hence: $U(KG) \cong \prod_{i=1}^{t} GL(n_i, D_i)$

and $TU(KG) \cong \prod_{i=1}^{t} T\,GL(n_i, D_i)$.

If KG has the t.p.p., then the sets $T\,GL(n_i,K) \subset T\,GL(n_i,D_i)$ must be subgroups. Proposition 2.2 shows that we must have $n_i = 1$, $1 \le i \le t$.

Since G is finite, we have that

$$G \subset \prod_{i=1}^{t} TU(D_i),$$

and Proposition 2.1 shows that each $TU(D_i)$ is commutative; thus, G is abelian.

The converse is trivial. □

To deal with the case where G contains elemens of order p we shall assume that the p-Sylow subgroup of G is normal. We remark that this is no restriction when G is nilpotent.

PROPOSITION 3.3 - Let K be a field with $char(K) = p > 0$ which is not algebraic over its prime field P and let G be a locally finite group with non trivial normal p-Sylow subgroup N. Then KG has the t.p.p. if and only if $N \supset G'$, the commutator subgroup of G.

PROOF - Again we limit ourselves to consider the case where G is finite. If $N \triangleleft G$ is a p-group, then the ideal $\Delta_K(G,N)$ is nilpotent and $1+\Delta_K(G,N)$ is a p-group with an exponent [1, theorem 1].

Hence, the natural homomorphism $\psi: KG \longrightarrow KG/N$ induces by restriction an epimorphism $\psi^*: U(KG) \longrightarrow U(KG/N)$ whose kernel is precisely $1+\Delta_K(G,N)$. Since every element in $\ker(\psi^*)$ is of finite order, ψ^* is also epic when restricted to the respective sets of torsion elements.

Thus, KG has the t.p.p. if and only if KG/N does. Since G/N contains no elements of order p, Proposition 3.2 shows that this is the case if and only if G/N is abelian or, equivalently, $N \supset G'$.

□

§4 - NILPOTENT OR FC GROUPS IN CHARACTERISTIC p

We shall first make a few comments. As in the previous section, we can restrict ourselves to work with finitely generated groups. Since torsion groups which are nilpotent or FC and finitely generated are locally finite and we have just dealt with those, we shall assume from now on that the groups considered are not torsion.

Also, we remark that a nilpotent or FC group which is not torsion contains a central element of infinite order (see [4, Theorem 2.24] and [5,15.1.17]).

Hence, in both cases we can assume that G contains a central element of infinite order. We shall use this fact in the proofs of our next results.

THEOREM 4.1 - Let K be a field of characteristic $p > 0$ and let G be a nilpotent or FC group, which is not a torsion group and contains no element of order p. Then KG has the t.p.p. if and only if T, the torsion subgroup of G, is abelian and every idempotent of KT is central in KG.

PROOF - Assume that KG has t.p.p. and set $a,b \in T$. Then $\langle a,b\rangle$ is a finite group, and $p \nmid |\langle a,b\rangle|$ so:

$$K\langle a,b\rangle = \bigoplus_{i=1}^{t} M_{n_i}(D_i).$$

Where $M_{n_i}(D_i)$ are full matrix rings over division rings D_i which contain K on their centers, $1 \le i \le t$.

Also, if x denotes a central element of infinite order in G, we have that:

$$K\langle x,a,b\rangle = K(\langle x\rangle \times \langle a,b\rangle) \cong K\langle x\rangle \otimes_K K\langle a,b\rangle \cong \oplus (K\langle x\rangle \otimes_K M_{n_i}(D_i)).$$

Now, for each index i, we get:

$$K\langle x\rangle \otimes_K M_{n_i}(D_i) \cong K\langle x\rangle \otimes_K M_{n_i}(K) \otimes_K D_i \cong M_{n_i}(K\langle x\rangle) \otimes_K D_i.$$

Hence, KG contains a copy of $M_{n_i}(K\langle x\rangle)$ and Proposition 2.2 shows that if KG has the t.p.p. then $n_i = 1$, $1 \le i \le t$.

Now $K\langle x\rangle \otimes_K D_i \cong D_i\langle x\rangle$ hence:

$$K\langle x,a,b\rangle \cong \bigoplus_{i=1}^{t} D_i\langle x\rangle.$$

Since KG has the t.p.p. every D_i also has the t.p.p. so Proposition 2.1 shows that $TU(D_i) \subset Z(D_i)$, the center of D_i, $1 \le i \le t$.

Also, we know from [8, VI.1.6] that $U(D_i\langle x\rangle) = U(D_i)$ thus:

$$\langle a,b\rangle\ TU(K\langle x,b\rangle) = \prod_{i=1}^{t} TU(D_i).$$

Consequently; $\langle a,b\rangle$ is abelian.

Now, assume that there exists an idempotent $e \in KT$ which is not central in KG. Then, it is shown in [7,VI.3.12] that KG contains a copy of a full ring of matrices $M_n(S)$ where $S = eKGe$ and $n > 1$.

Since $\langle supp(e)\rangle \cap \langle x\rangle = \{1\}$ it follows that

$$K\langle x\rangle \cong K\langle x\rangle e \text{ and } K\langle x\rangle e = eK\langle x\rangle e \subset eKGe = S.$$

Hence, KG contains a copy of $M_n(K\langle x\rangle)$ with $n > 1$, a contradiction. This completes the proof of the "only if" part of our statement.

To prove the converse, assume that R is abelian and that all the idempotents in KT are central in KG. We may suppose that G is finitely generated and therefore T finite. Thus

$$KT \cong \bigoplus_{i=1}^{t} K_i,$$

where each K_i is a field, $1 \le i \le t$.

Also, KG can be written as a crossed product:

$$KG = KT(G/T,\rho,\sigma),$$

and, since all idempotents in KT are central, we get:

$$KG \cong \bigoplus_{i=1}^{t} K_i(G/T,\rho,\sigma),$$

thus:

$$TU(KG) \cong \prod_{i=1}^{t} TU(K_i(G/T,\rho,\sigma)).$$

Finally, G/T can be ordered so [8,VI.1.6] shows again that the units in each summand are only the trivial ones, hence:

$$TU(KG) = \prod_{i=1}^{t} TU(K_i).$$

Consequently, KG has the t.p.p. □

THEOREM 4.4 - Let K be a field with char(K) = p>0 and let G be a nilpotent or FC group, which is not torsion, with a non trivial normal p-Sylow subgroup N. Then KG has the t.p.p. if and only if $N \supset T'$, the commutator of the torsion subgroup T of G and every idempotent in KT is central in KG modulo $\Delta_K(G,N)$.

PROOF - As in proposition 3.3, we consider the natural homomorphism ψ: KG ⟶ KG/N which induces, by restriction, an epimorphism on torsion units ψ^*: TU(KG) ⟶ TU(KG/N), also in this case.

Thus, we have shown that KG has the t.p.p. if and only if KG/N does.

Our statement now follows from theorem 4.1 and the fact that idempotents can be lifted modulo $\Delta_K(G,N)$. □

It may be interesting to note that, considering the theorem above and Proposition 4.1 and 4.5 in [6] and also [7,VI,4.15], the following results are immediate.

COROLLARY 4.3 - Let K be a field with char(K) = p>0, $p \neq 2,3$, and G a nilpotent or FC group which is not torsion. Then KG has the t.p.p. if and only if U(KG) is locally solvable.

COROLLARY 4.4 - Let K be a prime field with char(K) = p>0 and G a nilpotent or FC group, containing no elements of order p, which is not torsion. If KG has the t.p.p. then it contains no nilpotent elements.

Of course, there will be no analogue to corollary 4.4 when G contains on element t of order p, since KG will always contain a non-zero nilpotent element, namely,

$$\sum_{t=0}^{p-1} t^i .$$

§5 - FIELDS OF CHARACTERISTIC 0 AND p-ADIC INTEGERS

The methods we use in the sequel will also allow us to study group rings over rings of p-adic integers.

LEMMA 5.1 - Let $\mathbb{Z}_{(p)}$ be a localization of the ring of rational integers $\mathbb{Z}$ at a prime ideal (p). If G is any group such that $\mathbb{Z}_{(p)}G$ has the t.p.p. then the torsion subgroup T of G is abelian and, for any $t \in T$, $x \in G$ we have that $x^{-1}tx = t^i$, where $i = i(x)$ locally.

PROOF - Since $U(\mathbb{Z}G) \subset U(\mathbb{Z}_{(p)}G)$ it follows that $\mathbb{Z}G$ also has the t.p.p. and it was shown in [3] that either T is as in our statement or $T = K_8 \times A$ where $K_8 = \langle a,b \mid a^4 = 1,\ a^2 = b^2,\ bab^3 = a^2 \rangle$ is the quaternion group of order 8 and A is an elementary abelian 2-group.

To complete the proof, it will suffice to show that $\mathbb{Z}_{(p)}K_8$ does not have the t.p.p.

Set $\alpha = x+ya$ with $x,y\in\mathbb{Z}$ such that $p\nmid x$, $p|y$. Then α is a unit in $\mathbb{Z}_{(p)}K_8$ and it was shown in [2, theorem 2] that $[b,\alpha] = (b^{-1})^{\alpha}$ is not an element of finite order.

THEOREM 5.2 - Let K be a field of characteristic 0 and G a nilpotent or FC group. Then KG has the t.p.p. if and only if T is abelian and, for any $t\in T$, $x\in G$ we have that $x^{-1}tx = t^i$, where $i = i(x)$ locally.

PROOF - Since, $\mathbb{Z}_{(p)}G\subset KG$, necessity follows from the previous lemma.

Now, assume that T is as in the statement of the theorem. As in theorem 4.1 we may consider, without loss of generality. that T is finite.

Writing

$$KT = \bigoplus_{i=1}^{t} K_i,$$

a direct sum of fields, we know from [7,VI.3.22] that every unit u in KG can be written as:

$$u = \sum_i f_i g_i \text{ with } f_i\in K_i,\ g_i\in G,\ 1\le i\le t.$$

Since $T\triangleleft G$ and [7,VI.1.16] shows that the idempotents of

KT are central in KG we have that $g_i f_i = f_i' g_i$, with $f_i' \in K_i$. Hence:

$$u^m = \sum_i \bar{f}_i g_i^m \text{ where } \bar{f}_i \in K_i, \quad 1 \le i \le t.$$

Thus, $u \in TU(KG)$ if and only if, for some $n, g_i^m = 1$, $1 \le i \le t$, i.e, if and only if $u \in U(KT)$.

Since T is abelian, it readilly follows that KG has the t.p.p. □

The next result will include group rings over rings of p-adic integers.

COROLLARY 5.3 - Let R be an integral domain containing $\mathbb{Z}_{(p)}$ and G a nilpotent or FC group. Then RG has the t.p.p. if and only if T is abelian and, for any $t \in T$, $x \in G$, we have that $x^{-1} t x = t^i$, where $i = i(x)$ locally.

PROOF - Let K denote the field of quotients of R. Then

$$U(\mathbb{Z}_{(p)} G) \subset U(RG) \subset U(KG)$$

and the result follows. □

Again, comparing the results above with [7,VI.1.24] and [7,VI.4.12] we obtain:

COROLLARY 5.4 - Let G be a nilpotent of FC group. Then QG has the t.p.p. if and only if U(QG) is solvable.

COROLLARY 5.5 - Let G be a nilpotent of FC group. If QG has the t.p.p. then QG contains no nilpotent elements.

REFERENCES

[1] - J.M.BATEMAN, On the solvability of the unit groups of group algebras, *Trans. Amer. Math. Soc.* *157* (1971), 73-86.

[2] - M.M.PARMENTER & C.POLCINO MILIES, Group rings whose units form a nil-potent or FC group, *Proc. Amer. Math. Soc.* *68* (1978), 247-248.

[3] - C.POLCINO MILIES, Group rings whose torsion units form a subgroup I, *Proc. Amer. Math. Soc.* (to appear).

[4] - D.J.S.ROBINSON, *Finiteness Conditions and Generalized Solubre Groups*, Part I, Springer Verlag, Berlin - Heidelberg - New York, 1972.

[5] - W.R.SCOTT, *Group Theory*, Prentice Hall, Englewood Cliffs, N.J., 1964.

[6] - S.K.SEHGAL, Nilpotent elements in group rings, *Manuscripta Math.*, *15* (1975), 65-80.

[7] - S.K.SEHGAL, *Topics in Group Rings*, Marcel Dekker, New York, 1978.

[8] - S.K.SEHGAL & H.J.ZASSENHAUS, Integral group rings with nilpotent unit groups, *Comm. Algebra* 5 (1977), 101-111.

[9] - S.K.SEHGAL & H.J.ZASSENHAUS, Group rings whose units form an FC group, *Math. Z.* *153* (1977), 29-35.

TRABALHOS

DO

DEPARTAMENTO DE MATEMÁTICA

TITULOS PUBLICADOS

8001 - PLETCH, A. Local freeness of profinite groups. São Paulo, IME-USP, 1980. 10p.

8002 - PLETCH, A. Strong completeness in profinite groups. São Paulo, IME-USP, 1980. 8p.

8003 - CARNIELLI, W.A. & ALCANTARA, L.P. de. Transfinite induction on ordinal configurations. São Paulo, IME-USP, 1980. 22p.

8004 - JONES RODRIGUES, A.R. Integral representations of cyclic p-groups. São Paulo, IME-USP, 1980. 13p.

8005 - CARRADA, M. & ALCANTARA, L.P. de. Notes on many-sorted systems. São Paulo, IME-USP, 1980. /25/p.

8006 - POLCINO MILIES, F.C. & SEHGAL, S.K. FC-elements in a group ring. São Paulo, IME-USP, 1980. /10/p.

8007 - CHEN, C.C. On the Ricci condition and minimal surfaces with constantly curved Gauss map. São Paulo, IME-USP, 1980. 10p.

8008 - CHEN, C.C. Total curvature and topological structure of complete minimal surfaces. São Paulo, IME-USP, 1980. 21p.

8009 - CHEN, C.C. On the image of the generalized Gauss map of a complete minimal surface in R^4. São Paulo, IME-USP, 1980. 8p.

8110 - JONE RODRIGUES, A.R. Units of ZCp^n. São Paulo, IME-USP, 1980. 7p.

8111 - KOTAS, J. & COSTA, N.C.A. da. Problems of modal and discussive logics. São Paulo, IME-USP, 1981. 35p.

8112 - BRITO, FABIANO B. & GONÇALVES, DACIBERG L. *Algebras não associativas, sistemas diferenciais polinomiais homogêneas e classes características*. São Paulo, IME-USP, 1981. 7p.