Bull Braz Math Soc, New Series
https://doi.org/10.1007/s00574-021-00267-4

Constant Components of the Mertens Function and Its Connections with Tschebyschef's Theory for Counting Prime Numbers

André Pierro de Camargo[1] · Paulo Agozzini Martin[2]

Received: 21 March 2021 / Accepted: 26 June 2021

Abstract
In this note we exhibit some large sets $\Theta_x \subset \{1, 2, \ldots, \lfloor x \rfloor\}$ such that the sum of the Möbius function over Θ_x is small and independent of x. We show that the existence of some of these sets are intimately connected with the existence of the alternating series used by Tschebyschef and Sylvester to bound the prime counter function $\Pi(x)$.

Keywords Meterns function · Möbius function · Tschebyschef theory · Prime number theorem

1 Introduction

Let

$$M(x) = \sum_{j \le x} \mu(j), \ x \ge 1, \tag{1}$$

be the Mertens function, where μ is de Möbius function: $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is a product of k distinct prime factors and $\mu(n) = 0$ if $n > 1$ is not square-free. Estimating the magnitude of the Mertens function is of great interest due to its tight connections with the Prime Number Theorem and the Riemann Hypothesis. In fact, these two statements are known (Diamond 1982; Titchmarsh 1988, p. 370), to be equivalent, respectively, to

$$M(x) = o(x)$$

✉ André Pierro de Camargo
andrecamargo.math@gmail.com

1 Federal University of the ABC Region, Santo André, Brazil

2 Institute of Mathematics and Statistics of University of Sao Paulo, São Paulo, Brazil

Published online: 02 July 2021

and

$$M(x) = O\left(x^{0.5+\epsilon}\right) \ \forall \epsilon > 0$$

(see also Alkan 2012 for other equivalent forms of the Riemann Hypothesis and NG (2004) for some results on the magnitude of $M(x)$ under the Riemann Hypothesis). Therefore, it is of interest to identify large sets $\Theta \subset \{1, 2, \ldots, \lfloor x \rfloor\}$ such that the sum $\sum_{j \in \Theta} \mu(j)$ is small in magnitude for large values of x (some studies have been carried out in the cases where Θ is a truncated semigroup or an arithmetic progression. See Alkan and Haydar (2013) and the references therein).

In this note, we show that the sum of the Möbius function over some unions of the sets

$$\Theta_{x,\ell,n} := \left\{ j \leq x : \left\lfloor \frac{x}{j} \right\rfloor \equiv \ell \ (mod\ n) \right\}, \ \ell < n, \ n \geq 2, \tag{2}$$

are not only small, but are constant (independent of x) for $x \geq n$. We shall call these constant functions the constant components of the Mertens function. The theorems we state here are the form

$$\sum_{j \in \bigcup_{\ell \in L} \Theta_{x,\ell,n}} \mu(j) = -1 \tag{3}$$

for some $L \subset \{0, 1, \ldots, n-1\}$. For instance, we have

$$\sum_{\substack{j \leq x, \\ \left\lfloor \frac{x}{j} \right\rfloor \text{ odd}}} \mu(j) = -1. \tag{4}$$

In view of the disproof of the Mertens conjecture (Odlysco and Riele 1985), namely

$$\limsup_{x \to \infty} \frac{M(x)}{\sqrt{x}} > 1.06 \quad \text{and} \quad \liminf_{x \to \infty} \frac{M(x)}{\sqrt{x}} < -1.009,$$

this tells us that the sums in the splitting

$$M(x) = \sum_{\substack{j \leq x, \\ \left\lfloor \frac{x}{j} \right\rfloor \text{ even}}} \mu(j) + \sum_{\substack{j \leq x, \\ \left\lfloor \frac{x}{j} \right\rfloor \text{ odd}}} \mu(j)$$

have a very distinct behavior regarding cancellation. This can be interpreted as some kind of bias, in connection with Tchebyschef's bias (Knapowski and Turán 1964; Rubinstein and Sarnak 1994) (other types of bias regarding the Mertens and related functions were recently obtained in Alkan 2020a, b).

We show that the existence of constant components of the Mertens function of a special kind is intimately related to the existence of the alternating series used by Tschebyschef and Sylvester to bound the prime counter function $\Pi(x)$.

In the next section, we present some constant components of the Mertens function. In Sect. 3, we present a principle for generating new sets with the aforementioned property from previous known ones. In Sect. 4, we make some remarks on Tschebyschef's theory for counting prime numbers and prove a conjecture of Sylvester. Finally, in Sect. 5, we prove an equivalence theorem that allows us to obtain some necessary existence conditions for constant components of the Mertens function of special kind from Tschebyschef's theory. We also explain why all constant components we found have the value -1.

2 Some Constant Components of the Mertens Function

Lemma 1 *Assume that $n \in \mathbb{N}, n \geq 2$, factorizes as $n = rk$ and let*

$$A_i = \sum_{j \in \left(\bigcup_{ik \leq \ell < (i+1)k} \Theta_{x,\ell,n} \right)} \mu(j), \quad i = 0, 1, \dots, r-1.$$

For $x \geq n$, we have

$$\sum_{i=1}^{r-1} i\, A_i = -(r-1). \tag{5}$$

Proof Our proof is based on the Möbius inversion, (Landau 1958 p. 33):

$$1 = \sum_{j \leq x} \mu(j) \left\lfloor \frac{x}{j} \right\rfloor, \; x \geq 1. \tag{6}$$

For $x \geq n$, we have

$$\begin{aligned} -(n-1) &= \sum_{j \leq x} \mu(j) \left\lfloor \frac{x}{j} \right\rfloor - n \sum_{j \leq x/n} \mu(j) \left\lfloor \frac{x/n}{j} \right\rfloor \\ &= \sum_{j \leq x} \mu(j) \left(\left\lfloor \frac{x}{j} \right\rfloor - n \left\lfloor \frac{x/n}{j} \right\rfloor \right). \end{aligned}$$

In addition,

$$\left(\left\lfloor \frac{x}{j} \right\rfloor - n \left\lfloor \frac{x/n}{j} \right\rfloor \right) = \ell, \text{ for } \left\lfloor \frac{x}{j} \right\rfloor \equiv \ell \; (mod\; n).$$

Therefore,

$$-(n-1) = \sum_{\ell=0}^{n-1} \ell \left(\sum_{j \,\in\, \Theta_{x,\ell,n}} \mu(j) \right). \tag{7}$$

For $s < k$, we also have

$$\Theta_{x,s,k} \stackrel{(2)}{=} \Theta_{x,s,n} \,\cup\, \Theta_{x,k+s,n} \,\cup\, \Theta_{x,2k+s,n} \,\ldots \,\cup\, \Theta_{x,(r-1)k+s,n}.$$

Hence, $-(n-1) \;=$

$$\begin{aligned}
&\sum_{i=0}^{r-1} \left[\sum_{s=0}^{k-1} (ik+s) \left(\sum_{j \,\in\, \Theta_{x,ik+s,n}} \mu(j) \right) \right] \\
&= \sum_{s=0}^{k-1} s \left(\sum_{j \,\in \bigcup\limits_{0 \le i \le (r-1)} \Theta_{x,ik+s,n}} \mu(j) \right) + k \sum_{i=0}^{r-1} i \sum_{s=0}^{k-1} \left(\sum_{j \,\in\, \Theta_{x,ik+s,n}} \mu(j) \right) \\
&= \sum_{s=0}^{k-1} s \left(\sum_{j \,\in\, \Theta_{x,s,k}} \mu(j) \right) + k \sum_{i=1}^{r-1} i \; A_i.
\end{aligned}$$

However, using (7) for k in the place of n, we find that the first sum in the right-hand side of the equation above is $-(k-1)$, that is

$$-(r-1)k \;=\; -(n-1) + (k-1) \;=\; k \sum_{i=1}^{r-1} i \; A_i.$$

□

For $r = 2$ and $k = n/2$, n even, Lemma 1 yields

Theorem 1 *For n even, $x \geq n$ and $\Theta(x,n) := \bigcup\limits_{n/2 \le \ell < n} \Theta_{x,\ell,n}$,*

$$\sum_{j \,\in\, \Theta(x,n)} \mu(j) \;=\; -1.$$

Corollary 1 *Let n be an even number. By Theorem 1, we can express the Mertens function (1) as*

$$M(x) \;=\; -1 \;+ \sum_{j \,\in \bigcup\limits_{\ell < n/2} \Theta_{x,\ell,n}} \mu(j).$$

In particular, for $n = 2$*, we obtain* (4) *and*

$$M(x) = -1 + \sum_{\substack{j \le x, \\ \left\lfloor \frac{x}{j} \right\rfloor \, even}} \mu(j).$$

Example 1 For $n = 2$ and $x = 20$, we have

$j \in Supp(\mu)$	1	2	3	5	6	7	10	11	13	14	15	17	19
$\mu(j)$	1	-1	-1	-1	1	-1	1	-1	-1	1	1	-1	-1
$\left\lfloor \frac{x}{j} \right\rfloor$	20	10	6	4	3	2	2	1	1	1	1	1	1
$j \in \Theta_{x,1,n}$					✓			✓	✓	✓	✓	✓	✓

Therefore, $\sum_{j \in \Theta(x,n)} \mu(j) = 1 - 1 - 1 + 1 + 1 - 1 - 1 = -1.$

Example 2 For $n = 4$ and $x = 20$, we have

$j \in Supp(\mu)$	1	2	3	5	6	7	10	11	13	14	15	17	19
$\mu(j)$	1	-1	-1	-1	1	-1	1	-1	-1	1	1	-1	-1
$\left\lfloor \frac{x}{j} \right\rfloor$	20	10	6	4	3	2	2	1	1	1	1	1	1
$j \in \Theta_{x,2,n}$		✓	✓			✓	✓						
$j \in \Theta_{x,3,n}$					✓								

Therefore, $\sum_{j \in \Theta(x,n)} \mu(j) = (-1 - 1 - 1 + 1) + 1 = -1.$

The values in the next table shows that the cardinality of $\Theta(x, n)$ is large for large x.

Table 1 The ratio $r = \frac{\#\Theta(x,n) \cap Supp(\mu)}{\#\{1,2,\ldots,\lfloor x \rfloor\} \cap Supp(\mu)}$ for some values of x and n

$n \backslash x$	10000	24622	41711	60628	81032	102706	125495	149285
2	0.69028	0.69363	0.69376	0.69306	0.69301	0.69273	0.69328	0.69350
4	0.34605	0.34692	0.34710	0.34714	0.34637	0.34650	0.34665	0.34651
6	0.23031	0.23081	0.23139	0.23064	0.23134	0.23088	0.23094	0.23137
8	0.17475	0.17363	0.17404	0.17350	0.17316	0.17345	0.17329	0.17315
10	0.13760	0.13875	0.13886	0.13861	0.13864	0.13851	0.13882	0.13875
12	0.11721	0.11631	0.11575	0.11555	0.11550	0.11551	0.11554	0.11542

Theorem 2 *If* $n = 2qk$ *and* $x \geq n$*,*

$$\sum_{j \in \bigcup_{1 \leq i \leq q} \left(\bigcup_{(2i-1)k \leq \ell < (2i)k} \Theta_{x,\ell,n} \right)} \mu(j) = -1.$$

Proof Let

$$B_i = \sum_{j \in \left(\bigcup_{ik \leq \ell < (i+1)k} \Theta_{x,\ell,n} \right)} \mu(j), \quad i = 0, 1, \ldots, 2q-1.$$

By Lemma 1,

$$B_1 + 2B_2 + 3B_3 + 4B_4 + 5B_5 + 6B6 + \cdots + (2q-1)B_{2q-1} = -(2q-1).$$

By Lemma 1, for $k' = 2k$ and $r' = q$,

$$(B_2 + B_3) + 2(B_4 + B_5) + \cdots + (q-1)(B_{2q-2} + B_{2q-1}) = -(q-1).$$

Hence,

$$B_1 + B_3 + B_5 + \cdots + B_{2q-1} = -(2q-1) + 2(q-1) = -1.$$

□

Corollary 2 *If* $n = 4k$ *and* $x \geq n$*,*

$$\sum_{j \in \left(\bigcup_{k \leq \ell < 2k} \Theta_{x,\ell,n} \right)} \mu(j) = \sum_{j \in \left(\bigcup_{2k \leq \ell < 3k} \Theta_{x,\ell,n} \right)} \mu(j).$$

Proof By Theorems 1 and 2,

$$-1 = \sum_{j \in \left(\bigcup_{2k \leq \ell < 4k} \Theta_{x,\ell,n} \right)} \mu(j) = \sum_{j \in \left(\bigcup_{k \leq \ell < 2k \text{ or } 3k \leq \ell < 4k} \Theta_{x,\ell,n} \right)} \mu(j).$$

□

Example 3 For $n = 12$, $q = 2$, $k = 3$ and $x = 62$, we have

$j \in Supp(\mu)$	1	2	3	5	6	7	10	11	13	14	15	17	19
$\mu(j)$	1	−1	−1	−1	1	−1	1	−1	−1	1	1	−1	−1
$\left\lfloor \frac{x}{j} \right\rfloor$	62	31	20	12	10	8	6	5	4	4	4	3	3
$\left\lfloor \frac{x}{j} \right\rfloor \pmod{n}$	2	7	8	0	10	8	6	5	4	4	4	3	3
$j \in U(*)$					✓			✓	✓	✓	✓	✓	✓
$j \in Supp(\mu)$	21	22	23	26	29	30	31	33	34	35	37	38	39
$\mu(j)$	1	1	−1	1	−1	−1	−1	1	1	1	−1	1	1
$\left\lfloor \frac{x}{j} \right\rfloor$	2	2	2	2	2	2	2	1	1	1	1	1	1
$\left\lfloor \frac{x}{j} \right\rfloor \pmod{n}$	2	2	2	2	2	2	2	1	1	1	1	1	1
$j \in U(*)$													
$j \in Supp(\mu)$	41	42	43	46	47	51	53	55	57	58	59	61	62
$\mu(j)$	−1	−1	−1	1	−1	1	−1	1	1	1	−1	−1	1
$\left\lfloor \frac{x}{j} \right\rfloor$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\left\lfloor \frac{x}{j} \right\rfloor \pmod{n}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$j \in U(*)$													

$$(*)\ U = \left(\bigcup_{k \le \ell < 2k} \Theta_{x,\ell,n} \right) \cup \left(\bigcup_{3k \le \ell < 4k} \Theta_{x,\ell,n} \right).$$

Therefore, $\sum_{j \in U} \mu(j) = 1 - 1 - 1 + 1 + 1 - 1 - 1 = -1$.

Theorem 3 *If $n = 6k$ and $x \geq n$,*

$$\sum_{j \in \left(\bigcup_{k \le \ell < 3k} \Theta_{x,\ell,n} \right) \cup \left(\bigcup_{5k \le \ell < 6k} \Theta_{x,\ell,n} \right)} \mu(j) = -1$$

and

$$\sum_{j \in \left(\bigcup_{2k \le \ell < 3k} \Theta_{x,\ell,n} \right) \cup \left(\bigcup_{4k \le \ell < 6k} \Theta_{x,\ell,n} \right)} \mu(j) = -1.$$

Proof For $r = 6$ and $k = n/6$, let

$$B_i = \sum_{j \in \left(\bigcup_{ik \le \ell < (i+1)k} \Theta_{x,\ell,n} \right)} \mu(j), \quad i = 0, 1, \ldots, 5.$$

By Lemma 1,

$$B_1 + 2B_2 + 3B_3 + 4B_4 + 5B_5 \;=\; -5. \tag{8}$$

By lemma 1 for $k'' = 3k$ and $r'' = 2$, we also have

$$B_3 + B_4 + B_5 \;=\; -1. \tag{9}$$

In addition, by Theorem 2,

$$B_1 + B_3 + B_5 \;=\; -1. \tag{10}$$

Hence, by (8), (9) and (10),

$$\begin{aligned}
2B_1 + 2B_2 + 2B_5 &= B_1 + 2B_2 + 3B_3 + 4B_4 + 5B_5 \\
&\quad + B_1 + B_3 + B_5 \\
&\quad + -4(B_3 + B_4 + B_5) \\
&= -5 - 1 + 4 \;=\; -2, \\
B_2 + B_4 + B_5 &= B_3 + B_4 + B_5 \\
&\quad + (B_1 + B_2 + B_5) \\
&\quad - (B_1 + B_3 + B_5) \\
&= -1 + 1 - 1 \;=\; -1.
\end{aligned}$$

□

Corollary 3 *If $n = 6k$ and $x \geq n$,*

$$\sum_{j \in \left(\bigcup_{k \leq \ell < 2k} \Theta_{x,\ell,n}\right)} \mu(j) \;=\; \sum_{j \in \left(\bigcup_{4k \leq \ell < 5k} \Theta_{x,\ell,n}\right)} \mu(j).$$

Proof In the proof of Theorem 3, we have $-1 = B1 + B_2 + B_5 \;=\; B_2 + B_4 + B_5$. □

Example 4 For $n = 18$, $q = 2$, $k = 3$, $x = 40$ and

$$\begin{aligned}
U &= \left(\bigcup_{k \leq \ell < 3k} \Theta_{x,\ell,n}\right) \cup \left(\bigcup_{5k \leq \ell < 6k} \Theta_{x,\ell,n}\right), \\
V &= \left(\bigcup_{2k \leq \ell < 3k} \Theta_{x,\ell,n}\right) \cup \left(\bigcup_{4k \leq \ell < 6k} \Theta_{x,\ell,n}\right),
\end{aligned}$$

we have

$j \in Supp(\mu)$	1	2	3	5	6	7	10	11	13	14	15	17	19
$\mu(j)$	1	−1	−1	−1	1	−1	1	−1	−1	1	1	−1	−1
$\left\lfloor \frac{x}{j} \right\rfloor$	40	20	13	8	6	5	4	3	3	2	2	2	2
$\left\lfloor \frac{x}{j} \right\rfloor \ (mod\ n)$	4	2	13	8	6	5	4	3	3	2	2	2	2
$j \in U$	✓			✓	✓	✓	✓	✓	✓				
$j \in V$			✓	✓	✓								
$j \in Supp(\mu)$	21	22	23	26	29	30	31	33	34	35	37	38	39
$\mu(j)$	1	1	−1	1	−1	−1	−1	1	1	1	−1	1	1
$\left\lfloor \frac{x}{j} \right\rfloor$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\left\lfloor \frac{x}{j} \right\rfloor \ (mod\ n)$	1	1	1	1	1	1	1	1	1	1	1	1	1
$j \in U$													
$j \in V$													

Therefore, $\sum_{j \in U} \mu(j) = 1 - 1 + 1 - 1 + 1 - 1 - 1 = -1$, $\sum_{j \in V} \mu(j) = -1 - 1 + 1 = -1$.

3 An Extension Principle

In this section, we prove a theorem that tells us how to obtain new theorems like those of the previous section from simpler results of the same kind. Roughly speaking, it states that Theorems 1, 2, and 3 can be obtained automatically from the cases $k = 1$ of the same statements.

Following (Sylvester 1912, p. 704), a couple

$$r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m,\quad r_1 \leq r_2 \leq \ldots r_q,\ s_1 \leq s_2 \leq \ldots s_m, \tag{11}$$

of sequences of positive integers satisfying

$$\sum_{\ell=1}^{q} \frac{1}{r_\ell} - \sum_{\ell=1}^{m} \frac{1}{s_\ell} = 0 \tag{12}$$

will be called a *harmonic scheme*. MacLeod (1967)) and others (see Cohen et al. (2007) and the re-ferences therein) used harmonic schemes to bound $\frac{M(x)}{x}$. In particular, MacLeod considered the harmonic scheme

$$1, 30; 2, 3, 5 \tag{13}$$

and the function

$$f(x) = \lfloor x \rfloor - \left\lfloor \frac{x}{2} \right\rfloor - \left\lfloor \frac{x}{3} \right\rfloor - \left\lfloor \frac{x}{5} \right\rfloor + \left\lfloor \frac{x}{30} \right\rfloor \tag{14}$$

that satisfies $f(x)=0$ or $f(x)=1$ for $x \geq 1$. He proved that, for $x \geq 30$,

$$\sum_{j \leq x} \mu(j) f(x/j) = \sum_{j \in U_x} \mu(j) = -1, \quad \text{for } U_x := \{j \leq x : f(x/j) = 1\}.$$

We note that

$$U_x = V_x := \bigcup_{\ell \in \{1,2,3,4,5,7,8,9,11,13,14,17,19,23,29\}} \Theta_{x,\ell,30}.$$

In other words, the following theorem is implicit in the work of MacLeod

Theorem 4 *For $x \geq 30$,*

$$\sum_{j \in V_x} \mu(j) = -1.$$

For an harmonic scheme (11), let

$$f[r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m](x) = \sum_{\ell=1}^{q} \left\lfloor \frac{x}{r_\ell} \right\rfloor - \sum_{\ell=1}^{m} \left\lfloor \frac{x}{s_\ell} \right\rfloor, \quad x \geq 1. \tag{15}$$

Lemma 2 *Let $f := f[r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m]$ be given by (15) and let η be any integer multiple of*

$$l.c.m(r_1, r_2, \ldots r_q,\ s_1, s_2, \ldots, s_m)$$

($l.c.m$ stands for least common multiple). We have

$$Im(f) = \{\tau_1, \tau_2, \ldots, \tau_\xi\} \text{ is finite} \tag{16}$$

and, for $x \geq \eta$,

$$q - m = \sum_{j \leq x} \mu(j) f(x/j) = \sum_{i=1}^{\xi} \tau_i \left(\sum_{j \in \Omega_i} \mu(j) \right), \tag{17}$$

where $\Omega_i = \bigcup_{\substack{0 \leq u < \eta \\ f(u) = \tau_i}} \Theta_{x,u,\eta}$.

Proof For $x \geq \eta$, we have

$$
\begin{aligned}
\sum_{j \leq x} \mu(j) f(x/j) &= \sum_{j \leq x} \mu(j) \left(\sum_{\ell=1}^{q} \left\lfloor \frac{x/j}{r_\ell} \right\rfloor - \sum_{\ell=1}^{m} \left\lfloor \frac{x/j}{s_\ell} \right\rfloor \right) \\
&= \sum_{\ell=1}^{q} \left(\sum_{j \leq x} \mu(j) \left\lfloor \frac{x/j}{r_\ell} \right\rfloor \right) - \sum_{\ell=1}^{m} \left(\sum_{j \leq x} \mu(j) \left\lfloor \frac{x/j}{s_\ell} \right\rfloor \right) \\
&= \sum_{\ell=1}^{q} \left(\sum_{j \leq x/r_\ell} \mu(j) \left\lfloor \frac{x/r_\ell}{j} \right\rfloor \right) - \sum_{\ell=1}^{m} \left(\sum_{j \leq x/s_\ell} \mu(j) \left\lfloor \frac{x/s_\ell}{j} \right\rfloor \right) \\
&\overset{(6)}{=} \sum_{\ell=1}^{q} 1 - \sum_{\ell=1}^{m} 1 \; = \; q - m.
\end{aligned}
$$

To prove the second equality of (17), we note that f is periodic with period $T = \eta$ and this proves (16). Therefore, we can split

$$
\sum_{j \leq x} \mu(j) f(x/j) = \sum_{i=1}^{\xi} \sum_{\substack{j \leq x \\ f(x/j) = \tau_i}} \mu(j) \, \tau_i . \tag{18}
$$

For fixed $x \geq \eta$ and $j \leq x$, write

$$
\frac{x}{j} = a\,\eta + u + \delta, \quad \text{with } a, u \in \mathbb{N}, \quad 0 \leq u < \eta \text{ and } 0 \leq \delta < 1.
$$

Note that

$$
\begin{aligned}
f(x/j) \; &= \; \sum_{\ell=1}^{q} \left\lfloor \tfrac{x/j}{r_\ell} \right\rfloor - \sum_{\ell=1}^{m} \left\lfloor \tfrac{x/j}{s_\ell} \right\rfloor \\
&= \sum_{\ell=1}^{q} \left(a \tfrac{\eta}{r_\ell} + \left\lfloor \tfrac{u+\delta}{r_\ell} \right\rfloor \right) - \sum_{\ell=1}^{m} \left(a \tfrac{\eta}{s_\ell} + \left\lfloor \tfrac{u+\delta}{s_\ell} \right\rfloor \right) \\
&\overset{(12)}{=} \sum_{\ell=1}^{q} \left\lfloor \tfrac{u}{r_\ell} \right\rfloor - \sum_{\ell=1}^{m} \left\lfloor \tfrac{u}{s_\ell} \right\rfloor \; = \; f(u).
\end{aligned}
$$

Therefore, $f(x/j) = \tau_i$ if and only if $f(u) = \tau_i, \;\; u = \left\lfloor \frac{x}{j} \right\rfloor \pmod{\eta}$. In other words,

$$
\{ j \leq x : f(x/j) = \tau_i \} \; = \bigcup_{\substack{0 \leq u \leq \eta \\ f(u) = \tau_i}} \Theta_{x,u,\eta}.
$$

This and (18) complete the proof. □

Corollary 4 *Let* $f := f[r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m]$ *be given by* (15) *and let* η *be any integer multiple of*

$$l.c.m(r_1, r_2, \ldots r_q,\ s_1, s_2, \ldots, s_m).$$

If

$$Im(f) = \{0, 1\},$$

then, for $x \geq \eta$,

$$q - m = \sum_{j \in \Omega} \mu(j),$$

where $\Omega = \bigcup_{\substack{0 \leq u < \eta \\ f(u)=1}} \Theta_{x,u,\eta}$.

Example 5 Theorem 4 is the special case of Corollary 4 for the harmonic scheme (13) used by Macleod and f given in (14). In fact, we have

$$\begin{cases} f(u) = 1\ , u \in \{1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 17, 19, 23, 29\}, \\ f(u) = 0\ , u \in \{0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28\}. \end{cases}$$

Corollary 4 provides a framework for the computational search of constant components of Mertens function via finding harmonic schemes (11) such that the image of the associated function $f[r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m]$ is $\{0, 1\}$. In Table 2 we exhibit a few more constant components of the Mertens function we found using a computer program. Nevertheless, our main interest in Corollary 4 is a recipe for obtaining new theorems (regarding constant components of the Mertens function) from known ones. We have

Theorem 5 (Extension principle) *Let* $f := f[r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m]$ *be given by* (15) *and let* η *be any integer multiple of*

$$l.c.m(r_1, r_2, \ldots r_q,\ s_1, s_2, \ldots, s_m).$$

Assume that

$$Im(f) = \{0, 1\}.$$

By Corollary 4, *let* $\Omega = \{\ell_1, \ell_2, \ldots, \ell_\nu\}$ *be such that, for* $x \geq \eta$,

$$q - m = \sum_{j \in \bigcup_{i=1}^{\nu} \Theta(x, \ell_i, \eta)} \mu(j). \tag{19}$$

Table 2 Harmonic schemes and the corresponding constant components of the Mertens function $\sum_{j \in \bigcup_{\ell \in L} \Theta_{x,\ell,n}} \mu(j) = -1$

Scheme	n	L
6; 10 15	30	6, 7, 8, 9, 12, 13, 14, 18, 19, 24, 25, 26, 27, 28, 29
5, 110; 10, 11, 55	110	5, 6, 7, 8, 9, 10, 15, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 29, 30, 31, 32, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 65, 75, 76, 85, 86, 87, 95, 96, 97, 98, 105, 106, 107, 108, 109
7, 112; 14, 16, 56	112	7, 8, 9, 10, 11, 12, 13, 14, 15, 21, 22, 23, 24, 25, 26 27, 28, 29, 30, 31, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44 45, 46, 47, 49, 50, 51, 52, 53, 54, 55, 63, 77, 78, 79, 91 92, 93, 94, 95, 105, 106, 107, 108, 109, 110, 111
5, 45; 10, 15, 18	90	5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 25, 26, 27, 28, 29, 35, 45, 46, 47, 48, 49, 50, 51, 52, 53, 55, 56 57, 58, 59, 65, 66, 67, 68, 69, 70, 71, 85, 86, 87, 88, 89
1, 12; 2, 3, 4	12	1, 2, 3, 5, 7, 11
2, 36; 4, 6, 9	36	2, 3, 4, 5, 6, 7, 8, 10, 11, 14, 15, 16, 17, 22, 23 26, 34, 35
5, 120; 10, 15, 24	120	5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 20, 21, 22, 23, 25, 26, 27, 28, 29, 35, 36, 37, 38, 39, 40 41, 42, 43, 44, 45, 46, 47, 55, 56, 57, 58, 59, 65, 66, 67 68, 69, 70, 71, 85, 86, 87, 88, 89, 95, 115, 116, 117, 118, 119

Let $k \geq 1$, $n = k\eta$ and $\Omega^{(k)} := \bigcup_{i=1}^{\nu}\{k\ell_i, k\ell_i + 1, \ldots, k(\ell_i + 1) - 1\}$. Then, for $x \geq n$,

$$q - m = \sum_{j \in \left(\bigcup_{\ell \in \Omega^{(k)}} \Theta(x,\ell,n)\right)} \mu(j). \tag{20}$$

We prove Theorem 5 at the end of this section. By now, it is more convenient to include some examples of this somewhat technical result to understand it better.

Example 6 The case $n = 2$ of Theorem 1 can be obtained by Corollary 4 with

$$f[1; 2, 2](x) = \lfloor x \rfloor - 2\left\lfloor \frac{x}{2} \right\rfloor.$$

In this case, $\Omega = \{1\}$ is unitary. For $k \geq 1$ and $n' = 2k$, we have $\Omega^{(k)} = \{k, k + 1, \ldots, 2k - 1\}$. Hence, by Theorem 5,

$$-1 = 2 - 3 = \sum_{j \in \bigcup_{k \leq \ell < 2k} \Theta(x,\ell,n')} \mu(j), \quad x \geq n',$$

and this is exactly the general form of Theorem 1.

Example 7 The first case of Theorem 3 for $n = 6$ can be obtained by Corollary 4 with

$$f[1, 6; 2, 3, 3](x) \;=\; \lfloor x \rfloor - \left\lfloor \frac{x}{2} \right\rfloor - 2\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{6} \right\rfloor .$$

In this case, $\Omega = \{1, 2, 5\}$. For $k \geq 1$ and $n' = 6k$, we have

$$\Omega^{(k')} \;=\; \{k, k+1, \ldots, 2k-1\} \cup \{2k, 2k+1, \ldots, 3k-1\} \cup \{5k, 5k+1, \ldots, 6k-1\}.$$

Hence, by Theorem 5,

$$-1 = q - m \;=\; \sum_{j \,\in\, \left(\bigcup_{k \leq \ell < 3k} \Theta_{x,\ell,n}\right) \cup \left(\bigcup_{5k \leq \ell < 6k-1} \Theta_{x,\ell,n'}\right)} \mu(j) \;=\; -1,$$

what is exactly the general form of the first case of Theorem 3.

In summary, Theorem 5 acts over constant components of the Mertens function of special kind by extending (or dilating) the original subset of $\{0, 1, \ldots, \eta - 1\}$ of the summation index to get another constant component whose summation index lies in $\{0, 1, \ldots, k(\eta - 1)\}$.

The extended version of Theorem 4 is

Corollary 5 *Let* $k \geq 1$, $Y = \{1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 17, 19, 23, 29\}$ *and*

$$W_x := \bigcup_{y \,\in\, Y} \left(\bigcup_{k\, y \,\leq\, \ell \,<\, (k+1)\, y} \Theta_{x,\ell,30k} \right).$$

For $x \geq 30k$,

$$\sum_{j \,\in\, W_x} \mu(j) \;=\; -1.$$

Remark 1 We found that Theorems 1, 2 and 3 can be obtained by means of Theorem 5. However, it is not much clear whether every constant component of the Mertens function can be obtained by Corollary 4 and Theorem 5 by means of a suitable harmonic scheme.

3.1 Proof of Theorem 5

Denote by g the function $f[r_1, r_2, \ldots r_q;\; s_1, s_2, \ldots, s_m]$. By Corollary 4, the set Ω in the statement of Theorem 5 is,

$$\Omega \;= \{\ell_1, \ell_2, \ldots, \ell_\nu\} \;=\; \bigcup_{\substack{0 \leq u < \eta \\ g(u) = 1}} \Theta_{x,u,\eta}. \tag{21}$$

Let f^* be the function (15)

$$f^*(x) := f[k\,r_1, k\,r_2, \dots k\,r_q;\ k\,s_1, k\,s_2, \dots, k\,s_m](x)$$

associated to the harmonic scheme obtained by multiplying all the terms of the original harmonic scheme by k. Clearly

$$f^*(x) \ = \ g(x/k). \tag{22}$$

Hence, f^* also satisfies the hypothesis of Corollary 4 and we obtain

$$q - m \ = \ \left(\sum_{j \ \in \ \Omega^*} \mu(j) \right),$$

where $\Omega^* \ = \bigcup_{\substack{0 \ \leq \ u \ < \ k\eta \\ f^*(u) = 1}} \Theta_{x,u,k\eta}$. Nevertheless, (21) and (22) tells us that

$$\Omega^* \ = \bigcup_{\ell \ \in \bigcup_{i=1}^{\nu} \{k\ell_i, k\ell_i+1, \dots, (k+1)\ell_i - 1\}} \Theta_{x,\ell,k\eta}.$$

This completes the proof.

4 Tschebyschef's Theory for Counting Prime Numbers

In the remarkable work (Tschebyschef 1852), Tschebyschef used the harmonic scheme (13) to obtain lower and upper bounds for the function

$$\psi(x) \ = \sum_{\substack{p^r \ \leq x \\ p \text{ prime}}} \log(p).$$

He noted that

$$T(x) \ := \ \log(\lfloor x \rfloor !) \ = \sum_{j \geq 1} \psi(x/j) \tag{23}$$

and that $T(x) - T(x/2) - T(x/3) - T(x/5) + T(x/30) \ =$

$$\psi(x) - \psi(x/6) + \psi(x/7) - \psi(x/10) + \psi(x/11) - \psi(x/12) + \psi(x/13) + \dots \tag{24}$$

is an alternating series whose non-vanishing coefficients have absolute value equals to 1. The left-hand side of (24) is, by Stirling approximation, asymptotic to

$$A\,x, \text{ with } A := \frac{\log(2)}{2} + \frac{\log(3)}{3} + \frac{\log(5)}{5} - \frac{\log(30)}{30} \approx 0.921292.$$

This allowed him to prove that

$$A \le \liminf_{x\to\infty} \frac{\psi(x)}{x} \le \limsup_{x\to\infty} \frac{\psi(x)}{x} \le \frac{6}{5}A,$$

and, consequently (Diamond 1982),

$$A \le \liminf_{x\to\infty} \frac{\Pi(x)}{x/log(x)} \le \limsup_{x\to\infty} \frac{\Pi(x)}{x/log(x)} \le \frac{6}{5}A \tag{25}$$

(another elementary method for estimating $\Pi(x)$ was obtained by Diamond and Erdös (1980)).

For an harmonic scheme (11), let

$$f_\psi[r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m](x) = \sum_{\ell=1}^{q} T\left(\frac{x}{r_\ell}\right) - \sum_{\ell=1}^{m} T\left(\frac{x}{s_\ell}\right), \tag{26}$$

$x \ge 1$, with T given by (23). We have

$$f_\psi[r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m](x) = \sum_{j\ge 1} b_j \psi(x/j), \tag{27}$$

with $b_1, b_2, \ldots$ being integers specified by $r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m$. Sylveste (1881, 1912 pp. 704–706), noted that any harmonic scheme such that the right-hand side of (27) is an alternating series with $|b_j| \le 1\ \forall j$ leads to bounds of the form (25). More precisely, if

$$f_\psi[r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m](x) = \sum_{k\ge 1} \psi(x/n_{2k-1}) - \psi(x/n_{2k}), \tag{28}$$

with $n_j > n_i,\ j > i$, for every $x \ge 1$, then

$$n_1\tilde{A} \le \liminf_{x\to\infty} \frac{\Pi(x)}{x/\log(x)} \le \limsup_{x\to\infty} \frac{\Pi(x)}{x/\log(x)} \le \frac{n_1 n_2}{n_2 - n_1}\tilde{A}, \tag{29}$$

with

$$\tilde{A} := -\sum_{\ell=1}^{q} \frac{\log(r_\ell)}{r_\ell} + \sum_{\ell=1}^{m} \frac{\log(s_\ell)}{s_\ell}.$$

We shall call (28) the Tchebyschef's condition.

Note that (29) would prove the Prime Number Theorem,

$$\lim_{x\to\infty} \frac{\Pi(x)}{x/\log(x)} = 1,$$

if one could exhibit an infinite number of harmonic schemes satisfying (28) with arbitrarily large n_2/n_1. Sylvester also remarked that (28) is not necessary to bound $\Pi(x)$. This advance was possibly motivated by Sylvester's concerns in finding useful harmonic schemes satisfying the Tschebyshef's condition (Sylvester 1912, p. 707):

It would, I believe, be perfectly futile to seek for stigmatic schemes, involving higher prime numbers than 5, that should give rise to stigmatic series of sum-sums in which the successive coefficients should be alternately positive and negative unity ...

We confirm Sylvester's suspicion in the following sense

Theorem 6 *There is no harmonic scheme that satisfies* (28) *with* $n_1 = 1$ *and* $n_2 \geq 7$.

Proof Let be given an harmonic scheme (11). The precise definition of the coefficients b_j in (27) is

$$b_j = \sum_{\substack{1 \leq i \leq q \\ r_i | j}} 1 - \sum_{\substack{1 \leq i \leq m \\ s_i | j}} 1. \tag{30}$$

Assume, by contradiction, that there is an harmonic scheme

$$r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m,$$

that satisfies (28) with $n_1 = 1$ and $n_2 \geq 7$, that is

$$b_1 = 1 \text{ and } b_2 = b_3 = b_4 = b_5 = b_6 = 0. \tag{31}$$

By (30) and (31), we must have

$$r_1 = 1, r_2 = 6 \text{ and } s_1 = 2, s_2 = 3, s_3 = 5.$$

We now analyze the coefficients b_j, b_{j+1} and b_{j+2} for j of the form

$$j = \kappa\sigma + 17,\quad \sigma = 2 \times 3 \times 5 \times p_1 \times p_2 \times \cdots \times p_\nu,$$

where $2,\ 3,\ 5,\ p_1,\ p_2, \ldots p_\nu$ are all the distinct prime factors of $\left(\prod_{i=1}^{q} r_i\right)\left(\prod_{i=1}^{m} s_i\right)$, with (possibly) the exception of 17 and 19. Note that, because

$$(\kappa_2 - \kappa_1)\sigma = [\kappa_1\sigma + 17] - [\kappa_2\sigma + 17]$$

and

$$(\kappa_2 - \kappa_1)\sigma \;=\; [\kappa_1\sigma + 19] - [\kappa_2\sigma + 19],$$

for $\kappa_1, \kappa_2 \;\in \mathbb{N}$, there is at least one number $\kappa^* \;\in\; \{1, 2, 4\}$ such that

$$j_1^* \;:=\; \kappa^*\sigma + 17$$

is not a multiple of 19 and

$$j_2^* \;:=\; \kappa^*\sigma + 19$$

is not a multiple of 17 (we must check this in the case that 17 or 19 are in the harmonic scheme in question). It turns out that j_1^* and j_2^* are not divisible by any of the numbers $r_2, r_3, ...r_q, s_1, s_2, \ldots, s_m$. Therefore, by (30),

$$b_{j_1^*} \;=\; 1 \;\text{ and }\; b_{j_2^*} \;=\; b_{j_1^*+2} \;=1.$$

In addition, because the non-vanishing coefficients b_j must alternate in sign, we must have

$$b_{j_1^*+1} \;=\; -1. \tag{32}$$

However, the only numbers among $r_1, r_2, \ldots, r_q, s_1, s_2, \ldots s_m$ that divide

$$j_1^* + 1 \;=\; \kappa^*\sigma + 18$$

are 1, 2, 3, and 6 and this and (30) imply that $b_{j_1^*+1} \;=\; 0$. This contradicts (32).

□

Theorem 7 *If the harmonic scheme* $r_1, r_2, \ldots r_q; \;\; s_1, s_2, \ldots, s_m,$ *satisfy the Tschebyschef's condition, then* $m = q + 1$.

Proof Let

$$j^* = \left(\prod_{i=1}^{q} r_i\right)\left(\prod_{i=1}^{m} s_i\right).$$

Clearly, for (28) to hold, we must have

$$r_1 \;<\; \min\{r_2, \ldots, r_q, s_1, s_2, \ldots, s_m\}, \;\; b_{r_1} = 1$$

and r_1 is the smallest j such that b_j is non-vanishing (for our purposes, we can assume that the r-list and the s-list are disjoint). Therefore, the only possible values of j in the range

$$j^* - r_1, \;\; j^* - r_1 + \; 1, \ldots, \;\; j^* - 1, \;\; j^*, \;\; j^* + 1, \;\; \ldots, \;\; j^* + r_1$$

such that b_j may be non-vanishing are

$$j = j^* - r_1; \quad j = j^*; \quad j = j^* + r_1.$$

Because r_1 is the only number of the sequence $r_1, r_2, \ldots, r_q, s_1, s_2, \ldots s_m$ that divides $j^* - r_1$ and $j^* + r_1$, (30) tells us that:

$$b_{j^*-r_1} = 1 \text{ and } b_{j^*+r_1} = 1.$$

Thus, the unique possibility of having b_j alternating in sign is to have $b_{j^*} = -1$. In other words, we must have

$$-1 = b_{j^*} \overset{(30)}{=} \sum_{\substack{1 \le i \le q \\ r_i | j^*}} 1 - \sum_{\substack{1 \le i \le m \\ s_i | j^*}} 1 = q - m.$$

□

5 Connections with the Constant Components of The Mertens Function

The next result establishes a strong connection between the constant components of the Mertens function that can be derived by Corollary 4 and the harmonic schemes that satisfy the Tschebyschef's condition.

Theorem 8 *For an harmonic scheme* (11)*, the following are equivalent*

- *The image of* $f[r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m]$ *is* $\{0, 1\}$.
- $f_\psi[r_1, r_2, \ldots r_q;\ s_1, s_2, \ldots, s_m]$ *satisfies* (28).

Proof For every fixed positive integer n, let us consider the partial sums

$$\begin{cases} \tilde{T}\left(\frac{x}{r_1}\right) = \psi(x/r_1) + \psi(x/(2r_1)) + \cdots + \psi(x/(k_1 r_1)), \\ \quad \vdots \quad \vdots \\ \tilde{T}\left(\frac{x}{r_q}\right) = \psi(x/r_q) + \psi(x/(2r_q)) + \cdots + \psi(x/(k_q r_q)), \\ -\tilde{T}\left(\frac{x}{s_1}\right) = -\psi(x/s_1) - \psi(x/(2s_1)) - \cdots - \psi(x/(k'_1 s_1)), \\ \quad \vdots \quad \vdots \\ -\tilde{T}\left(\frac{x}{s_m}\right) = -\psi(x/s_m) + \psi(x/(2s_m)) - \cdots - \psi(x/(k'_q s_m)), \end{cases} \tag{33}$$

with $k_i := \lfloor n/r_i \rfloor, i = 1, 2, \ldots, q$ and $k'_i := \lfloor n/s_i \rfloor, i = 1, 2, \ldots, m$. Note that the sum of the coefficients of the ψs in the sum of the right-hand sides of (33) is

$$\sum_{\ell=1}^{q} \left\lfloor \frac{n}{r_i} \right\rfloor - \sum_{\ell=1}^{m} \left\lfloor \frac{n}{s_i} \right\rfloor. \tag{34}$$

However, this sum is also $\sum_{j=1}^{n} b_j$ for b_j defined in (27). In other words, we have

$$\sum_{j=1}^{n} b_j \;=\; f[r_1, r_2, \dots r_q;\; s_1, s_2, \dots, s_m](n). \tag{35}$$

This shows that the coefficients b_1, b_2, b_3, ... only can alternate in sign and have all magnitude 1 (with the first non-vanishing b equals to 1) if and only if $f[r_1, r_2, \dots r_q;$ $s_1, s_2, \dots, s_m](n) \;\in\; \{0, 1\}$ for all n. □

The equivalence given in Theorem 8 allows obtaining interesting information about constant components of the Mertens function in terms of the harmonic schemes that satisfy the Tschebyschef's condition (28). In fact, we have

Corollary 6 *No constant component of the Mertens function that can be derived by Corollary* (4) *is of the form*

$$\sum_{j \,\in\, \bigcup_{\ell \,\in\, \Omega} \Theta_{x,\ell,n}} \mu(j), \tag{36}$$

with $\{1, 2, 3, 4, 5, 6\} \;\subset \Omega$.

Proof Assume, by contradiction, that the result is false and let $r_1, r_2, \dots, r_q;$ $s_1, s_2, \dots, s_m$ be an harmonic scheme such that

$$\Omega \;=\; \{0 \le u < \eta : f(u) = 1\}, \tag{37}$$

with $f := f[r_1, r_2, \dots, r_q; s_1, s_2, \dots, s_m]$ and η be any multiple of

$$l.c.m.(r_1, r_2, \dots r_q, s_1, s_2, \dots, s_m).$$

By Theorem 8, the sequence of the b_js in the left-hand side of (27) must alternate in sign with $|b_j| \le 1$ for all $j \ge 1$. In addition, (35) and the assumption about Ω tells us that

$$\sum_{j=1}^{n} b_j \;=\; 1$$

for $n = 1, 2, 3, 4, 5, 6$. However, this is only possible if $b_1 = 1$ and b_2, b_3, b_4, b_5 and b_6 are all vanishing. This contradicts Theorem 6. □

So far, all the constant components of the Mertens function we found have value -1, that is, they are of the form

$$\sum_{j \,\in\, \bigcup_{\ell \,\in\, L} \Theta_{x,\ell,n}} \mu(j) \;=\; -1.$$

Our remarks about Tschebyschef's theory allows us to state that this is not accidental:

Corollary 7 *If*

$$q - m = \sum_{j \in \bigcup_{\ell \in \Omega} \Theta_{x,\ell,n}} \mu(j)$$

is a constant component of the Mertens function derived by Corollary (4)*, then* $q - m = -1$.

Proof By Theorem 8, the harmonic scheme $r_1, r_2, \dots r_q, s_1, s_2, \dots, s_m$ in the statement of Corollary 4 must satisfies the Tschebyschef's condition (28). Theorem 7 tells us that this is only possible if $m = q + 1$. □

We close this section by listing (see Table 3) the harmonic schemes of Table 2, together with the corresponding bounds for $\frac{\Pi(x)}{x/\log(x)}$ given in (29).

6 Final Remarks

In this note we presented, for arbitrary values of n, some sets $L \subset \{0, 1, \dots, n-1\}$ such that (3) holds for $x \geq n$. For the constant components derived by Corollary 4 and Theorem 5, we showed that the unique possible value for the constant is minus one (see Corollary 7). A natural question is whether there would be n and L such that the expression in the left-hand side of (3) is constant for $x \geq n$ with some other value rather than -1. Another interesting question is determining whether there would be n odd and L such that the expression in the left-hand side of (3) is constant for $x \geq n$. For $n = 3, 5, 7, \dots, 17$, we computationally checked that (3) is not constant for x in the range [30, 100] for every subset L of $\{0, 1, \dots, n-1\}$. These are interesting topics for future research.

Table 3 Harnomic schemes and the lower and upper bounds given in (29)

Scheme	Bounds for large x
6; 10 15	$0.673011 \leq \frac{\Pi(x)}{x/\log(x)} \leq 1.682530$
5, 110; 10, 11, 55	$0.782451 \leq \frac{\Pi(x)}{x/\log(x)} \leq 1.434491$
7, 112; 14, 16, 56	$0.794888 \leq \frac{\Pi(x)}{x/\log(x)} \leq 1.413136$
5, 45; 10, 15, 18	$0.824456 \leq \frac{\Pi(x)}{x/\log(x)} \leq 1.141556$
1, 12; 2, 3, 4	$0.852275 \leq \frac{\Pi(x)}{x/\log(x)} \leq 1.136368$
2, 36; 4, 6, 9	$0.886440 \leq \frac{\Pi(x)}{x/\log(x)} \leq 1.139710$
5, 120; 10, 15, 24	$0.907153 \leq \frac{\Pi(x)}{x/\log(x)} \leq 1.145879$

References

Alkan, E.: Ramanujan sums and the Burgess zeta function. Int. J. Number Theory **8**, 2069–2092 (2012)

Alkan, E.: Biased behavior of weighted Mertens sums On sums over the Möbius function and discrepancy of fractions. Int. J. Number Theory **16**(3), 547–577 (2020a)

Alkan, E.: Inequalities between sums over prime numbers in progressions. Res. Number Theory **6**(3), 36 (2020b)

Alkan, E., Haydar, G.: On sums over the Möbius function and discrepancy of fractions. J. Number Theory **133**, 2217–2239 (2013)

Cohen, H., Dress, F., Marraki, M.E.: Explicit estimates for summatory functions linked to the Möbius μ function. Funct. Approx. Comment. Math. **39**(1), 51–63 (2007)

Diamond, H.G., Erdös, P.: On sharp elementary prime number estimates. Enseign. Math. **26**, 313–321 (1980)

Diamond, H.G.: Elementary methods in the study of the distribution of prime numbers. Bull. Am. Math. Soc. **7**(3), 553–589 (1982)

Knapowski, S., Turán, P.: Further developments in comparative number theory I. Acta Arith. **IX**, 23–40 (1964)

Landau, E.: Elementary Number Theory. Chelsea Publishing Company, New York (1958)

MacLeod, R.A.: A new estimate for the sum $M(x) = \sum_{n \leq x} \mu(n)$. Acta Arith. **XIII**, 49–59 (1967)

Ng, N.: The distribution of the summatory function of the Möbius function. Proc. Lond. Math. Soc. **89**(2004), 361–389 (2004)

Odlysco, A.M., Riele, H.T.: Disproof of the Mertens conjecture. J. Reine Angew. Math. **357**, 138–160 (1985)

Rubinstein, M., Sarnak, P.: Chebyshev's bias. Experiment. Math. **3**(3), 173–197 (1994)

Sylvester, J.J.: On Tchebycheff's theory of the totality of the prime numbers comprised within given limits. Am. J. Math. **IV**, 230–247 (1881)

Sylvester, J.J.: The Collected Mathematical Papers of James Joseph Sylvester, vol. IV. Cambridge University Press, London (1912)

Titchmarsh, E.C.: The Theory of the Riemann Zeta Function. Oxford University Press, New York (1988)

Tschebyschef, P.L.: Mémoire sur les nombres premiers. J. Math. Pures Appl. **17**, 366–390 (1852)