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# Generic minimal surfaces

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# GENERIC MINIMAL SURFACES

BY

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## 1. *Introduction.*

In this paper we study generic minimal immersions of a two dimensional Riemannian manifold $M^2$, which we call briefly a surface, into a Riemannian manifold $\tilde{M}^{2+\ell}(c)$ of constant sectional curvature c, $\ell \geq 1$ (superscripts denote dimensions). In general, giving a minimal immersion $f: M^2 \to \tilde{M}^{2+\ell}(c)$, the n-th normal space $N^n(p)$ of the immersion has dimension $\leq 2$, for every point p in M and $n = 1,2,\ldots$ . Following [Ch], we will say that p is a *generic point* of M if:

(a) when $\ell = 2s$ $(2 + \ell = 2s + 2,\ s \geq 1)$ is *even*, then $\dim N^n(p) = 2$ for $n = 1,\ldots,s$;

(b) when $\ell = 2s - 1$ $(2 + \ell = 2s + 1,\ s \geq 1)$ is *odd*, then $\dim N^n(p) = 2$ for $n = 1,\ldots,s-1$ and $\dim N^s(p) = 1$.

Obviously $N^m(p) = \{0\}$ for $m \geq s+1$ in either case and we convention

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that $N^o(p) = T_pM$ = tangent space to M at p. We say that f is a *generic minimal immersion* (or that M is a *generic minimal surface*) if every point of M is generic. Examples of generic minimal two-spheres in the Euclidean sphere $S^{2s+2}(1)$ can be found in [Ca], [B]; examples of generic minimal two tori can be found in [K]. We prove that these are the only (orientable) topological types permitted of compact generic minimal surfaces.

(1.1) *Theorem. Suppose that* $M^2$ *is a compact, connected and oriented surface which admits a generic minimal immersion into some* $\tilde{M}^{2+\ell}(c)$, $\ell \geq 1$. *Then* $M^2$ *must be homeomorphic either to a sphere or to a torus, according to* $\ell = 2s$ *or* $\ell = 2s - 1$, $s \geq 1$.

The even case of the theorem is a consequence of the following more general fact:

(1.2) $\chi(N^nM) = (n+1)\chi(M)$, $n = 1,\ldots,s$

where $N^nM$ is the n-th normal bundle of the immersion with a canonical orientation. To give a complete proof of (1.2) we introduce in §2 an intrinsically defined orientable 2-vector bundle $\sigma^n_o(M)$ over M, whose fiber over p is a suitably chosen subspace of the n-linear symmetric maps of $T_pM$, and prove that

(1.3) $\chi(\sigma^n_o(M)) = (n+1)\chi(M)$, $n = 1,2,\ldots.$

It happens that $\sigma^n_o(M)$ and $N^nM$ are orientably isomorphic (see (4.1)) which, when put together with (1.3), gives (1.2). In §3 we compute the intrinsic curvature $K_n^*$ of $N^nM$ and observe that $K_s^* > 0$ when $\ell = 2s$, hence giving $\chi(M) > 0$. For the odd case, we use the fact that $N^sM$ is one-dimensional when $\ell = 2s - 1$ to construct a nowhere vanishing cross-field on M, giving therefore $\chi(M) = 0$.

*Remarks*. (1) The assumed orientability is not a serious restriction: if M is not orientable, we can use the two-fold oriented covering of M and arrive to similar conclusions.

(2) Through the paper we are assuming, without explicit mention, that all manifolds and maps are differentiable of class $C^\infty$, and immersions are always isometric.

## 2. *A 2-vector bundle of multilinear maps on surfaces*.

Let V be a 2-dimensional real vector space equipped with an inner product $<,>$ and with complex structure J. We say that a subset $\{x,y\}$ of V is isothermal if $|x| = |y|$ and $<x,y> = 0$. Denote by $L_o^n(V)$ the space of n-linear maps $A: V\times\ldots\times V \longrightarrow V$ which have zero trace. For $n \geq 2$, having zero trace means that

$$A(x,x,*,\ldots,*) + A(Jx,Jx,*,\ldots,*) = 0$$

for every x in V. Given A in $L_o^n(V)$ and an orthonormal basis $\{e_1,e_2\}$ of V, we will write, for brevity,

$$A(e_{i_1},\ldots,e_{i_n}) = A_{i_1 \ldots i_n}, \quad i_j = 1,2; \; j = 1,\ldots,n.$$

By symmetry and zero trace, we are reduced to $A_{i_1\ldots i_n} = \pm A_{1\ldots 1}$ or to $A_{i_1\ldots i_n} = \pm A_{1\ldots 12}$. If $(e_1,e_2)$ and $(\bar{e}_1,\bar{e}_2)$ are positive (orthonormal) bases of V, with $\bar{e}_1 = \cos t e_1 + \sin t e_2$ (and hence $\bar{e}_2 = J\,\bar{e}_1$), by induction on n we easily get

$$A(\bar{e}_1,\ldots,\bar{e}_1) = \cos nt A_{1\ldots 1} + \sin nt A_{1\ldots 12},$$

(2.1)

$$A(\bar{e}_1,\ldots,\bar{e}_1,\bar{e}_2) = -\sin nt A_{1\ldots 1} + \cos nt A_{1\ldots 12}.$$

Consider now the space $\sigma_o^n(V)$ of all A in $L_o^n(V)$ which satisfy the following property:

(P) either $A \equiv 0$ or there exists a positive $(e_1,e_2)$ such that $(A_{1\ldots 1}, -A_{1\ldots 12})$ is a positive isothermal basis of V.

Clearly $\sigma_o^1(V) = L_o^1(V)$: every A in $L_o^1(V)$ satisfies (P). In what follows we present some expected properties of $\sigma_o^n(V)$; all of them can be deduced from (2.1), or from the previous ones, by elementary linear algebraic arguments.

(2.2) *Properties of* $\sigma_o^n(V)$. (a) If A satisfies (P) for some positive $(e_1,e_2)$, then A satisfies (P) for every positive $(e_1,e_2)$.

(b) For $A \in \sigma_o^n(V)$ and $x \in V$, the vectors $A(x,\ldots,x)$ and $A(x,\ldots,x,Jx)$ have the same lenght depending only on x and A.

(c) For every A, B in $\sigma_o^n(V)$ we have

$$\langle A_{1\ldots 1}, B_{1\ldots 1}\rangle - \langle A_{1\ldots 12}, B_{1\ldots 12}\rangle = 0 =$$

$$\langle A_{1\ldots 1}, B_{1\ldots 12}\rangle + \langle A_{1\ldots 12}, B_{1\ldots 1}\rangle.$$

(Use the oriented angle between $A_{1\ldots 1}$ and $B_{1\ldots 1}$).

(d) If A is nonzero in $\sigma_o^n(V)$, then JoA is nonzero in $\sigma_o^n(V)$ and JoA = -AoJ. The correct meaning of the last equality is $JoA(*,\ldots,*) = -A(J*, *,\ldots,*) = \ldots = -A(*,\ldots,*, J*)$.

(e) $\sigma_o^n(V)$ is a 2-dimensional real vector space (use (c)).

(f) The mapping

$$A,B \in \sigma_o^n(V) \longrightarrow \langle\langle A,B\rangle\rangle \overset{def}{=} \frac{1}{2}(\langle A_{1\ldots 1},B_{1\ldots 1}\rangle + \langle A_{1\ldots 12},B_{1\ldots 12}\rangle$$

independs on the positives $(e_1,e_2)$ and is an inner product in $\sigma_o^n(V)$.

(g) For each nonzero A in $\sigma_o^n(V)$, the pair (A, JoA) is a $\langle\langle,\rangle\rangle$-isothermal basis of $\sigma_o^n(V)$, and the change of two such bases has positive determinant (use (c) and (d)). Therefore $\sigma_o^n(V)$ canonically inherits from (V,J) the orientation determined by the pair (A, JoA).

Now let M be an oriented Riemannian surface with metric $\langle,\rangle$ and complex structure J. For each positive integer n, we consider the 2-vector bundle $\sigma_o^n(M)$ over M, whose fiber over p is $\sigma_o^n(p) = \sigma_o^n(T_pM)$. By (2.2)-(f)-(g), each $\sigma_o^n(p)$ has canonically an inner product $\langle\langle , \rangle\rangle|p$ and an orientation arising from those of $T_pM$. Therefore $\sigma_o^n(M)$ has a canonical structure of oriented Riemannian 2-vector bundle over M. Define, in a standard way, a Riemannian connection in $\sigma_o^n(M)$ by

$$(D_XA)(X_1,\ldots,X_n) = \nabla_X(A(X_1,\ldots,X_n)) - \sum_{j=1}^{n} A(X_1,\ldots,\nabla_X X_j,\ldots,X_n),$$

where A is a section of $\sigma_o^n(M)$, $\nabla$ is the Riemannian connection of M and $X, X_1,\ldots,X_n$ are vector fields tangent to M.

(2.3) Theorem. *Let* M *be a compact, connected oriented surface. Then*

$$\chi(\sigma_o^n(M)) = (n+1)\chi(M), \quad n = 1,2,\ldots \; .$$

*Proof*. Given $p \in M$, we associate to each $X \in T_pM$ an element $A(X)$ of $\sigma_o^n(p)$ as follows: if $X = 0$, then $A(X) \equiv 0$; if $X \neq 0$, take the positive $(e_1 = X/|X|,\ e_2 = Je_1)$ in $T_pM$ and let $A(X)$ be the unique n-linear symmetric map of $T_pM$ with zero trace such that

$$A(X)(e_1,\ldots,e_1) = |X|^n X, \quad A(X)(e_1,\ldots,e_1,e_2) = -|X|^n JX.$$

Clearly $A(X) \in \sigma_o^n(p)$ and depends only on X. Moreover $\|A(X)\| = |X|^{n+1}$ and hence $\|A(X)\| = 1$ iff $|X| = 1$. Therefore $(E = A(e_1),\ F = J \circ A(e_1))$ is a positive orthonormal basis of $\sigma_o^n(p)$ if $X \neq 0$, $e_1 = X/|X|$.

Now let $X: M \to TM$ be a generic section of the tangent bundle of M with singularities, say, $p_1, \ldots, p_r$. Then the mapping $A(X): M \to \sigma_o^n(M)$, where $A(X)(p) = A(X(p))$, gives a section of $\sigma_o^n(M)$ with the same singularities of X. To conclude the proof it suffices to show that

$$(2.4)\quad \mathrm{Index}(A(X),\, p_j) = (n+1)\mathrm{Index}(X, p_j),\ j = 1, \ldots, r.$$

For this we consider on $W = M - \{p_1, \ldots, p_r\}$ the tangent frame $(e_1, e_2)$ and on $\sigma_o^n(M)|W$ the frame $(E,F)$, both pointwisely defined as above. Let $w$ and $\theta$ be the 1-forms on W defined by

$$w = \langle \nabla e_1, e_2 \rangle, \quad \theta = \langle\langle DE, F \rangle\rangle.$$

Then (see [Li, p.276] for instance)

$$\mathrm{Index}(X, p_j) = \frac{1}{2\pi}\int_{\partial D_j} w, \quad \mathrm{Index}(A(X), p_j) = \frac{1}{2\pi}\int_{\partial D_j} \theta .$$

Here $D_j$ is a small disk about $p_j$ oriented to agree with M and $\partial D_j$ takes its orientation from $D_j$. We claim that

$$(2.5)\qquad \theta = (n+1)w,$$

which obviously will prove (2.4). In fact, observe firstly that

$$E_{1\ldots 1} = e_1,\ E_{1\ldots 12} = -e_2,\ F_{1\ldots 1} = e_2,\ F_{1\ldots 12} = e_1.$$

Hence for an arbitrary Y tangent to M we have

$$\theta(Y) = \langle\langle D_Y E, F\rangle\rangle$$

$$= \frac{1}{2}\left(\langle (D_Y E)_{1\ldots 1}, F_{1\ldots 1}\rangle + \langle (D_Y E)_{1\ldots 12}, F_{1\ldots 12}\rangle\right)$$

$$= \frac{1}{2}\left(\langle \nabla_Y e_1 + n\langle \nabla_Y e_1, e_2\rangle e_2, e_2\rangle + \langle -\nabla_Y e_2 + n\langle \nabla_Y e_1, e_2\rangle e_1, e_1\rangle\right)$$

$$= (n+1)w(Y).$$

This proves our claim and completes the proof of the theorem.

*Remarks.* (1) The above argument can be slightly modified to prove the following more general fact: the intrinsic curvature of $(\sigma_o^n(M), \langle\langle,\rangle\rangle)$ is, at each point, n+1 times the Gaussian curvature of M, no matter if M is compact or not.

(2) A proof that $\chi(\sigma_o^1(M)) = 2\chi(M)$ (Theorem (2.3) for n=1) is contained in the proof of Theorem 1 of [AFR]. Compare also Proposition (4.1) below with the proposition in [AFR,p.110].

## 3. *Generic minimal immersions.*

Consider an isometric immersion $f: M \longrightarrow \tilde{M}$ of a surface M into a constantly curved Riemannian manifold $\tilde{M} = \tilde{M}^{2+\ell}(c)$. Let $N^n(p) \subset T_p\tilde{M}$ denote the n-th normal space of M at $p \in M$. The (n+1)-th fundamental form of M is the (n+1)-linear tensor

$$B^n : T_pM\times\ldots\times T_pM \longrightarrow N^n(p)$$

inductively defined by

$$(3.1) \qquad B^n(X_1,\ldots,X_{n+1}) = T^n((\tilde{\nabla}_{\tilde{X}_{n+1}}\ldots\tilde{\nabla}_{\tilde{X}_2}\tilde{X}_1)(p)),$$

where $\tilde{\nabla}$ is the Riemannian connection of $\tilde{M}$, $X_j$ are vector fields tangent to M around p, $\tilde{X}_j$ are local fields on $\tilde{M}$ which extend $X_j$ and $T^n$ denotes projection onto $[T_pM \oplus N^1(p) \oplus \ldots \oplus N^{n-1}(p)]^\perp$. It is well known that $N^n(p)$ is spanned by the image of $B^n$ and, since $\tilde{M}$ has constant curvature, $B^n$ is symmetric for n = 1,2,... . For details see [S., pp.240-244]. In terms of a local frame $(e_1,\ldots,e_{2+\ell})$ on $\tilde{M}$ adapted to the immersion, we will write

$$B^n(e_{i_1},\ldots,e_{i_{n+1}}) = B^n_{i_1\ldots i_{n+1}}, \qquad 1 \le i_1,\ldots,i_{n+1} \le 2 \quad ;$$

$$h^\alpha_{i_1\ldots i_{n+1}} = \langle B^n_{i_1\ldots i_{n+1}}, e_\alpha\rangle, \qquad 3 \le \alpha \le \ell+2.$$

The square of the lenght of $B^n$ is, by definition,

$$\|B^n\|^2 = \sum_{i_1,\ldots,i_{n+1}} |B^n_{i_1\ldots i_{n+1}}|^2 = \sum_{i_1,\ldots,i_{n+1},\alpha} (h^\alpha_{i_1\ldots i_{n+1}})^2,$$

which independs on the frame.

From now on suppose M connected, oriented with complex structure J, and f minimal. Then $B^n$ has zero trace for $n = 1,2,\ldots$ and hence

$$(3.2) \qquad B^n_{i_1\ldots i_{n+1}} = \pm B^n_{1\ldots 1} \quad \text{or} \quad \pm B^n_{1\ldots 12} .$$

Thus $\dim N^n(p) \leq 2$ for every p in M. If $X = |X|(\cos t e_1 + \sin t e_2)$ is tangent to M, by induction on n we obtain

$$(3.3) \qquad \begin{aligned} B^n(X,\ldots,X) &= |X|^{n+1}(\cos(n+1)t\, B^n_{1\ldots 1} + \sin(n+1)t\, B^n_{1\ldots 12}), \\ B^n(X,\ldots,X,JX) &= |X|^{n+1}(-\sin(n+1)t\, B^n_{1\ldots 1} + \cos(n+1)t\, B^n_{1\ldots 12}). \end{aligned}$$

Fix an oriented $(e_1,e_2)$ and for a unit $X = \cos t\, e_1 + \operatorname{sen} t\, e_2$ write $B^n(X,\ldots,X) = B^n(t)$. The following proposition is straightforward from (3.3) (see propositions 1.1 and 1.2 of [C]).

(3.4) *Proposition.* (a) *For every* p *in* M, *the set*

$$\varepsilon^n(p) = \{B^n(X,\ldots,X) \mid X \in T_pM,\ |X| = 1\} \subset N^n(p)\}$$

*is an ellipse with center at the origin of* $N^n(p)$.

(b) $B^n(t + 2k\pi/(n+1)) = B^n(t) = -B^n(t + (2k+1)\pi/(n+1))$ *for every integer* k.

(c) *The tangent line to* $\varepsilon^n(p)$ *by the point* $B^n(t + \pi/2(n+1))$ *is parallel to the vector* $B^n(t)$.

(d) $\varepsilon^n(p)$ *is a circle if and only if for every isothermal subset* $\{X,Y\}$ *of* $T_pM$, $\{B(X,\dots,X), B(X,\dots,X,Y)\}$ *is an isothermal subset of* $N^n(p)$.

We call $\varepsilon^n(p)$ the *n-th curvature ellipse* at p. From (3.2) we have dim $N^n(p) = 2$ iff $B^n_{1\dots 1}$, $B^n_{1\dots 12}$ are linearly independent and from (3.4) this happens iff $\varepsilon^n(p)$ is non-degenerate. In case dim $N^n(p) = 2$, $(B^n_{1\dots 1}, B^n_{1\dots 12})$ is a basis of $N^n(p)$ and (if we maintain the $(e_1,e_2)$ positives) the change of two such bases has positive determinant, by (3.3). Therefore the orientation of $T_pM$ determines an orientation in $N^n(p)$ if dim $N^n(p) = 2$. This orientation agrees with the orientation determined by the direction in which $B^n(t)$ traverses $\varepsilon^n(p)$. Define the *n-th normal curvature* at p to be

$$K_n(p) = \frac{2}{\pi} \text{ area } \varepsilon^n(p).$$

Hence $K_n(p) \geq 0$ and $K_n(p) = 0$ iff $\varepsilon^n(p)$ is degenerate, iff dim $N^n(p) < 2$. Let $\lambda_n \geq \mu_n \geq 0$ denote the lenght of the semi-axes of $\varepsilon^n$. Then $K_n = \lambda_n \mu_n$.

<u>*Remark.*</u> One can always choose a positive $(e_1,e_2)$ in $T_pM$ in a way that $B^n_{1\dots 1}$ and $B^n_{1\dots 12}$ give a semimajor and a semiminor axis of $\varepsilon^n(p)$, respectively; this follows from (3.4)-(b),

(c). In this case, $\lambda_n = |B^n_{1\ldots1}|$, $\mu_n = |B^n_{1\ldots12}|$ and $\|B\|^n = 2^n(\lambda_n^2 + \mu_n^2)$.

Suppose f generic in addition to minimal; see §1. In this case we can consider the n-th normal bundle of the immersion, which is the vector bundle $N^nM$ whose fiber over p in M is $N^n(p)$, $n = 1,\ldots,s$. When $1 \leq n \leq s$ in the even case $\ell = 2s$, or when $n \leq s-1$ in the odd case $\ell = 2s-1$, $N^nM$ is an oriented 2-vector bundle over M and the curvature function $K_n$ is positive. Every point p of M has an open neighborhood $U \subset M$ on which we can define a frame $(e_1,\ldots,e_{2+\ell})$ such that

(3.5) for $0 \leq n \leq s$ in the even case, or for $0 \leq n \leq s-1$ in the odd case, $(e_{2n+1}, e_{2n+2})$ spans the fibers of $N^nM\,|\,U$ and is positively oriented, where $N^oM = TM$; in the odd case, $e_{2s+1} = e_{2+\ell}$ spans $N^sM\,|\,U$, which is 1-dimensional in this case.

Now extend this frame to a neighborhood of p in $\tilde{M}$ and define 1-forms $w_A(e_B) = \delta_{AB}$, $w_{A,B} = \langle\bar{\nabla} e_A, e_B\rangle$, $1 \leq A,B \leq 2+\ell$. These are the dual and the connection forms of $\tilde{M}$ associated to the frame. When we restrict $w_{A,B}$ to M and exterior differentiate, we obtain (by the choice of the frame) that $dw_{2n+1,2n+2}$ are the curvature forms of $N^nM\,|\,U$, in case $N^nM$ is 2-dimensional. These forms are globally defined on M. The Gaussian curvature K of M is given by $dw_{1,2} = -K\, w_1 \wedge w_2$; the *intrinsic curvature* $K^*_n$

of $N^nM$ is given by

$$dw_{2n+1,2n+2} = -K^*_n \, w_1 \wedge w_2, \qquad n \geq 1.$$

In a frame as in (3.5) we also have

$$\|B^n\|^2 = 2^n((h^{2n+1}_{1\ldots1})^2 + (h^{2n+2}_{1\ldots1})^2 + (h^{2n+1}_{1\ldots12})^2 + (h^{2n+2}_{1\ldots12})^2),$$

(3.6)

$$K_n = 2(h^{2n+1}_{1\ldots1}\, h^{2n+2}_{1\ldots12} - h^{2n+1}_{1\ldots12}\, h^{2n+2}_{1\ldots1}),$$

$n = 1,\ldots,s$ where $h^{2s+2}_{1\ldots1} = 0 = h^{2s+2}_{1\ldots12}$ in the odd case.

(3.7) *Proposition. Let* $f\colon M^2 \to \tilde{M}^{2+\ell}(c)$ *be a generic minimal immersion of a connected oriented surface, with either* $\ell = 2s$ *or* $\ell = 2s-1$, $s \geq 1$. *If* $n$ *is such that* $N^nM$ *is 2-dimensional, then*

$$\text{(a)}\quad K^*_1 = K_1 - \frac{\|B^2\|^2}{2K_1} = 2\lambda_1\mu_1 - \frac{\lambda_2^2 + \mu_2^2}{\lambda_1\mu_1} \qquad (n = 1);$$

$$\text{(b)}\quad K^*_n = \frac{2K_n}{K^2_{n-1}} \cdot \frac{\|B^{n-1}\|^2}{2^{n-1}} - \frac{2}{K_n} \cdot \frac{\|B^{n+1}\|^2}{2^{n+1}}$$

$$= \frac{\lambda_n\mu_n}{\lambda^2_{n-1}\,\mu^2_{n-1}} \cdot (\lambda^2_{n-1} + \mu^2_{n-1}) - \frac{\lambda^2_{n+1} + \mu^2_{n+1}}{\lambda_n\mu_n} \qquad (n \geq 2).$$

*In particular,* $K^*_s > 0$ *in the even case* $\ell = 2s$.

*Proof*. Fix n satisfying the hypothesis. Then we can split M into two disjoint subsets:

$$E(n) = \{p \in M \mid \varepsilon^n(p) \text{ is not a circle}\},$$

$$C(n) = M - E(n) = \{p \in M \mid \varepsilon^n(p) \text{ is a circle}\}.$$

Obviously E(n) is open and C(n) is closed in M. For each p in E(n) there exists an open subset $U \subset E(n)$ around p on which we can choose a frame $(e_1,\dots,e_{2+\ell})$ as in (3.5). By the Remark above, we can specialise the choice of $(e_1,e_2)$ and $(e_{2n+1}, e_{2n+2})$ in this frame to obtain, in U,

$$(3.8) \qquad B^n_{1\dots1} = \lambda_n e_{2n+1}, \quad B^n_{1\dots12} = \mu_n e_{2n+2} \; ;$$

where $\lambda_n > \mu_n > 0$. On the other hand, if $\varepsilon^n$ is a circle in a neighborhood of a point p in C(n), we can start with any oriented $(e_{2n+1}, e_{2n+2})$ and choose $(e_1,e_2)$ in a way to obtain (3.8) again, with $\lambda_n = \mu_n > 0$ in this case.

Let then p be a point in E(n), or in IntC(n), and let $U \subset E(n)$, or $U \subset \text{IntC}(n)$, be a neighborhood of p on which we have chosen a frame as in (3.5), satisfying also (3.8) for the fixed n. We claim that

$$(3.9) \qquad w_{\alpha,\gamma} \equiv 0 \quad \text{if } \alpha = 2n+1,\ 2n+2 \quad \text{and} \quad \gamma \leq 2n-2,\ \gamma \geq 2n+5.$$

In fact, since the $e_\alpha$ are sections of $N^nM|U$, we can apply Lemma 69 of [S, p.247] to conclude that

$$(\tilde{\nabla}_X e_\alpha)(q) \in V(q) \overset{\text{def}}{=} N^{n-1}(q) \oplus N^n(q) \oplus N^{n+1}(q)$$

for every q in U and X tangent to M. The spaces V(q), $q \in U$, are spanned by $e_{2n-1},\dots,e_{2n+4}$ evaluated at q. But these vectors are orthogonal to the $e_\gamma(q)$, which proves our claim. From (3.9) and the equations of structure of E. Cartan, we get

$$\begin{aligned}
(3.10)\quad K_n^* &= - dw_{2n+1,2n+2}(e_1,e_2) \\
&= (w_{2n-1,2n+1} \wedge w_{2n-1,2n+2} + w_{2n,2n+1} \wedge w_{2n,2n+2} \\
&\quad + w_{2n+1,2n+3} \wedge w_{2n+2,2n+3} + w_{2n+1,2n+4} \wedge w_{2n+2,2n+4})(e_1,e_2).
\end{aligned}$$

Suppose n > 1 firstly. Using (3.6) and (3.8), we can write

$$e_{2n-1} = \frac{2}{K_{n-1}} (h^{2n}_{1\dots12}\, B^{n-1}_{1\dots1} - h^{2n}_{1\dots1}\, B^{n-1}_{1\dots12}),$$

$$e_{2n} = \frac{2}{K_{n-1}} (h^{2n-1}_{1\dots1}\, B^{n-1}_{1\dots1} - h^{2n-1}_{1\dots1}\, B^{n-1}_{1\dots12}),$$

$$e_{2n+1} = \frac{1}{\lambda_n} B^n_{1\dots1},\quad e_{2n+2} = \frac{1}{\mu_n} B^n_{1\dots12}.$$

These relations plus direct calculations using (3.1), (3.9) and the definition of $w_{A,B}$, give

$$w_{2n-1,2n+1}(e_1) = 2\lambda_n h^{2n}_{1\ldots1}/K_{n-1},\ w_{2n-1,2n+2}(e_1) = -2\mu_n h^{2n}_{1\ldots1}/K_{n-1},$$

$$w_{2n-1,2n+1}(e_2) = 2\lambda_n h^{2n}_{1\ldots12}/K_{n-1},\ w_{2n-1,2n+2}(e_2) = 2\mu_n h^{2n}_{1\ldots12}/K_{n-1},$$

$$w_{2n,2n+1}(e_1) = -2\lambda_n h^{2n-1}_{1\ldots12}/K_{n-1},\ w_{2n,2n+2}(e_1) = 2\mu_n h^{2n-1}_{1\ldots1}/K_{n-1},$$

$$w_{2n,2n+1}(e_2) = -2\lambda_n h^{2n-1}_{1\ldots1}/K_{n-1},\ w_{2n,2n+2}(e_2) = -2\mu_n h^{2n-1}_{1\ldots12}/K_{n-1},$$

$$w_{2n+1,2n+3}(e_1) = h^{2n+3}_{1\ldots1}/\lambda_n,\ w_{2n+2,2n+3}(e_1) = h^{2n+3}_{1\ldots12}/\mu_n,$$

$$w_{2n+1,2n+3}(e_2) = h^{2n+3}_{1\ldots12}/\lambda_n,\ w_{2n+2,2n+3}(e_2) = -h^{2n+3}_{1\ldots1}/\mu_n,$$

$$w_{2n+1,2n+4}(e_1) = h^{2n+4}_{1\ldots1}/\lambda_n,\ w_{2n+2,2n+4}(e_1) = h^{2n+4}_{1\ldots12}/\mu_n,$$

$$w_{2n+1,2n+4}(e_2) = h^{2n+4}_{1\ldots12}/\lambda_n,\ w_{2n+2,2n+4}(e_2) = -h^{2n+4}_{1\ldots1}/\mu_n,$$

By bringing all this into (3.10) we easily obtain (b) for points in E(n) or in IntC(n), $n > 1$. The case $n = 1$ follows the same lines but is easier and is left as an exercise. This will prove also (a) for points in E(n) or in IntC(n). These formulas extend to the commom boundary of the two sets by continuity. Finally, if $\ell = 2s$ is even, then $\lambda_n \geq \mu_n > 0$, $n = 1,\ldots,s$ and $\lambda_{s+1} = \mu_{s+1} = 0$.

Therefore $K_s^* > 0$ in this case and the proposition is proved.

*Remark*. For a two-sphere $S^2$ fully minimally immersed in $\tilde{M}^{2+\ell}(c)$ we have $c > 0$ and $\ell$ must be even, say $\ell = 2s$. Moreover (see [Ch] ) or §2 of [E])

$$(3.11)\quad |B^n_{1\ldots1}| = |B^n_{1\ldots12}| = r_n \geq 0, \quad \langle B^n_{1\ldots1}, B^n_{1\ldots12}\rangle = 0,$$

$$n = 1,\ldots,s.$$

Then the n-th curvature ellipse is everywhere a circle with ratius $r_n$ $(= \lambda_n = \mu_n)$, and the non-generic points (= the singular points of $r_1,\ldots,r_s$) are isolated. Noticing that the local invariants $k_n$ introduced in [Ch] satisfy $k_n = r_n/r_{n-1}$ $(r_o = 1)$, then the formulae of [Ch, p.38] can be easily interpreted in terms of the $r_n$. In particular, at the generic points we have

$$(3.12)\quad \Delta\log(r_1\ldots r_n) = \frac{(n+1)(n+2)}{2}K - c + \frac{2r^2_{n+1}}{r^2_n}, \quad n = 1,\ldots,s.$$

## 4. *Proof of Theorem (1.1)*.

Through this section we assume the hypotheses of Theorem (1.1). Let $n \geq 1$ be such that $N^nM$ is a 2-vector bundle. We shall firstly prove that, in this case, $N^nM$ is isomorphic with the intrinsically defined bundle $\sigma^n_o(M)$ of §2.

(4.1) *Proposition. The map* $\phi: N^nM \to \sigma_o^n(M)$ *defined by*

$$<\phi(p,u)(X_1,\dots,X_n),\ X>(p) = <B^n(X_1,\dots,X_n,X),\ u>(p)$$

*for each* $(p,u)$ *in* $N^nM$ *and sections* $X_1,\dots,X_n,X$ *of* TM, *is an orientation preserving bundle isomorphism. Consequently,*

$$\chi(N^nM) = (n+1)\chi(M).$$

*Proof.* Let u be in $N^n(p)$ and fix a positive frame $(e_1,e_2)$ around p. Then

$$\phi(p,u)_{1\dots1} = a(p,u)e_1 + b(p,u)e_2,\ -\phi(p,u)_{1\dots12} = -b(p,u)e_1 + a(p,u)e_2$$

where $a(p,u) = <B^n_{1\dots1},u>(p)$, $b(p,u) = <B^n_{1\dots12},u>(p)$, with n and n+1 lower indexes for $\phi(p,u)$ and $B^n$, respectively. These relations and the linear independence of $B^n_{1\dots1}$ and $B^n_{1\dots12}$ show that: if $u = 0$, then $\phi(p,u) \equiv 0$; if $u \neq 0$ then $(\phi(p,u)_{1\dots1}, -\phi(p,u)_{1\dots12})$ is a positive isothermal basis of $T_pM$. Thus $\phi(p,u) \in \sigma_o^n(p)$ and $\phi(p,\cdot)$ is injective. Therefore $\phi$ is a bundle isomorphism. Now write $(u,v) = (B^n_{1\dots1}, B^n_{1\dots12})$ and observe that $(u,v)$ determines the orientation of $N^nM$ and, deleting the points in M, $(\phi(u), J\circ\phi(u))$ determines the orientation of $\sigma_o^n(M)$. We put $\phi(v) = c\phi(u) + dJ\circ\phi(u))$ so that the change $(\phi(u), J\circ\phi(u)) \to$

$(\phi(u),\phi(v))$ has d as determinant. But $d = |u \wedge v|^2(|u|^4+\langle u,v\rangle^2)^{-1} > 0$, as it is easy to see. Hence $\phi$ is orientation preserving, as we wished. Moreover, $\chi(N^nM) = \chi(\sigma_o^n(M))$ by the compactness of M. Then $\chi(N^nM) = (n+1)\chi(M)$ by Theorem (2.3) and the proof of (4.1) is complete.

Now we proceed to the proof of Theorem (1.1). If $\ell = 2s$ is even, we use Proposition (4.1) to obtain that

$$(n+1)\chi(M) = \frac{1}{2\pi}\int_M K_n^* \, dM, \quad n = 1,\dots,s.$$

where dM is the area element of M. But $K_s^* > 0$ on M by (3.7). So $\chi(M) > 0$ and M is homeomorphic to a sphere, in this case. On the other hand, if $\ell = 2s-1$ is odd, then there exists a (at least) locally defined unit field $e_{2s+1}$ normal to M which generates $N^sM$; see (3.5). By (3.4)-(b), the equation $B^n(X,\dots,X) = \pm\lambda_s e_{2s+1}$ determines a unique, modulo $\pi/(s+1)$, unit vector in $T_pM$, for each p in M. Therefore there exists a unique set $\{X_1,\dots,X_{2s+2}\}$ of 2s+2 pairwise linearly independent unit vectors in $T_pM$ such that their tips form a regular polygon. By letting p vary on M we produce an everywhere defined (2s+2)-cross field tangent to M. Therefore $\chi(M) = 0$ by Proposition 1.7 of [Li] and hence M is homeomorphic to a torus, in this case. This completes the proof of Theorem (1.1).

*Remarks.* (1) Under the hypotheses of Theorem (1.1) we can also conclude, in the even case $\ell = 2s$, that (c > 0 and) area $(M^2) = 2\pi(s+1)(s+2)/c$. In fact, since we already know that $M^2 \approx S^2$, we also know by (3.12) and genericity that the equality $\Delta\log(r_1 \ldots r_s) = (s+1)(s+2)K/2 - c$ holds everywhere on M. Integration plus Stokes and Gauss-Bonnet theorems yield our claim.

(2) A non singular holomorphic curve in $\mathbf{C}P^n$ (or in $\mathbf{C}^n$) satisfies (3.11) and is, in particular, a minimal surface; see [La]. We call attention to the similarity between Theorem (1.1), even case, and Corollary 1 of [La, p.54].

(3) The bundle isomorphism and the (2s+2)- cross field above exist independently of the compactness of M; only genericity was used in their construction. This observation may be useful in other situations, such as in the study of complete minimal surfaces in $\mathbf{R}^n$.

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